Chapter 3: Elementary Functions

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The Degree of the Polynomial and Rational Functions

- The degree of the polynomial which has the form of
$$p_n(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$
is n if the complex constant a_n is nonzero
- The rational function which has the form of
$$R_{m,n}(z) = \frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_m z^m}{b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n}$$
has numerator degree m and denominator degree n, if $a_m \neq 0$ and $b_n \neq 0$
- We will begin our study with these two simple types of functions

Deflation of Polynomial Functions

- You can always divide a "dividend" polynomial by a "divisor" polynomial to obtain a "quotient" polynomial and a "reminder" polynomial whose degree is less than that of the divisor
$$dividend = divisor \times quotient + remainder$$
- If z_1 is any arbitrary complex number, then division of $p_n(z)$ by the degree-one polynomial $z - z_1$ must result in a remainder of lower degree: in other words, a constant,
$$p_n(z) = (z - z_1) p_{n-1}(z) + constant \qquad (1)$$
where the quotient polynomial $p_{n-1}(z)$ has degree $n-1$

Deflation of Polynomial Functions (Cont'd)

- If z_1 happens to be a zero of $p_n(z)$, we deduce that the remainder is zero. Thus (1) shows how $z - z_1$ has been factored out from $p_n(z)$. We say $p_n(z)$ has been "deflated"
- If z_2 is a zero of the quotient $p_{n-1}(z)$, we can deflate further by factoring out $z - z_2$, and so on, until we run out of zeros, leaving us with the factorization

$$p_n(z) = (z - z_1)(z - z_2)\cdots(z - z_k)p_{n-k}(z) \qquad (2)$$

- Example on page 99-100 gives us an explicit explanation of how this procedure works

Zeros of Polynomial Functions

- In order to deflate a polynomial function, we must find the zeros first. Hence the two questions arise: 1) How to find a zero of $p_n(z)$; 2) How do we know $p_n(z)$ has any zeros?
- Gauss helped us answer the second question in his doctoral dissertation of 1799: Every nonconstant polynomial with complex coefficients has at least one zero in C
- We immediately conclude that a polynomial of degree n has n zeros, since we can continue to factor out zeros in the deflation process until we reach the final, constant, quotient.
- Repeated zeros are counted according to their multiplicities

Zeros-Characterization of Polynomials

- With the issue of existence of zeros for the quotients settled we have a complete factorization of any polynomial as follows

$$p_n(z) = a_n(z - z_1)(z - z_2)\cdots(z - z_n)$$

- This equation demonstrates that a polynomial of degree n has n zeros and $p_n(z)$ is completed determined by its zeros, up to a constant multiple $\{a_n\}$
- The Fundamental Theorem only tell us there are zeros, it doesn't tell us how to find them
- The cases of degree one the two are simple, but the higher degree is very difficult or unsolvable

Taylor Form of the Polynomials

- Any polynomial function $p_n(z)$ can be expressed in the form of Taylor form centered at z_0 as follows

$$\begin{aligned} p_n(z) &= \frac{p_n(z_0)}{0!} + \frac{p_n'(z_0)}{1!}(z - z_0)^1 + \cdots + \frac{p_n^{(n)}(z_0)}{n!}(z - z_0)^n \\ &= \sum_{k=0}^{n} \frac{p_n^{(k)}(z_0)}{k!}(z - z_0)^k \end{aligned}$$

- We use the nomenclature Maclaurin Form for the Taylor form centered at $z_0 = 0$

Factored Form of Rational Functions

- Since the rational functions are ratios of polynomials, all the previous conclusions can be applied to their numerators and denominators separately
- Probably the most enlightening display comes from the factored from
$$R_{m,n}(z) = \frac{a_m(z-z_1)(z-z_2)\cdots(z-z_m)}{b_n(z-\xi_1)(z-\xi_2)\cdots(z-\xi_n)}$$
where $\{z_k\}$ designates the zeros of the numerator and $\{\xi_k\}$ designates those of the denominator (We assume the common zeros have been canceled)

Factored Form of Rational Functions (Cont'd)

- The zeros of the numerator are, of course, zeros of $R_{m,n}(z)$; zeros of the denominator are called poles of $R_{m,n}(z)$
- Zeros and poles can, of course, be multiple
- Clearly, the magnitude of $R_{m,n}(z)$ grows without bounds as z approaches a poles
- With the knowledge of poles, we can express $R_{m,n}(z)$ in terms of partial fractions which will be discussed subsequently

Partial Fractional Decomposition

- If $R_{m,n} = \frac{a_0+a_1z+a_2z^2+\ldots+a_mz^m}{b_n(z-\xi_1)^{d_1}(z-\xi_2)^{d_2}\cdots(z-\xi_r)^{d_r}}$ is a rational function whose denominator degree $n = d_1 + d_2 + \cdots + d_r$ exceeds its numerator degree m, then $R_{m,n}$ has a partial fraction decomposition of the from
$$\begin{aligned} R_{m,n} &= \frac{A_0^{(1)}}{(z-\xi_1)^{d_1}} + \frac{A_1^{(1)}}{(z-\xi_1)^{d_1-1}} + \cdots + \frac{A_{d_1-1}^{(1)}}{(z-\xi_1)} \\ &+ \frac{A_0^{(2)}}{(z-\xi_2)^{d_2}} + \cdots + \frac{A_{d_2-1}^{(1)}}{(z-\xi_2)} \\ &+ \cdots + \frac{A_0^{(r)}}{(z-\xi_r)^{d_r}} + \cdots + \frac{A_{d_r-1}^{(r)}}{(z-\xi_r)} \end{aligned} \tag{3}$$
where the $\{A_s^{(j)}\}$ are constants (The ξ_k's are assumed distinct)

How to Find the Coefficients $\{A_s^{(j)}\}$

- The brute-force procedure consists in rearranging the proposed form (3) over a common denominator and comparing the resulting numerator, term be term, with the original numerator of $R_{m,n}$. But this will result in solving a group of linear equations
- A quicker, more sophisticated method for evaluating the $\{A_s^{(j)}\}$ is illustrated in the example on page 106
- The deduced conclusion is if $R_{m,n}$ can be written in the form (3), then a general expression for the coefficients is
$$A_s^{(j)} = \lim_{z\to\xi_j} \frac{1}{s!}\frac{d^s}{dz^s}\left[(z-\xi_j)^{d_j} R_{m,n}(z)\right]$$

The Complex Function e^z

- The complex exponential function e^z plays a prominent role in analytic function theory, not only because of its own important properties but because it is used to define the complex trigonometric and hyperbolic functions
- If $z = x + iy$, $e^z = e^x(\cos y + i \sin y)$ according to the Euler's Equation
- e^z is an entire function and its arbitrary degree of derivative is itself, i.e.,

$$\frac{d}{dz}e^z = e^z \implies \frac{d^n}{dz^n}e^z = e^z$$

Polar Form of the Complex Function e^z

- The polar components of e^z is

$$|e^z| = e^x, \quad \arg e^z = y + 2k\pi \quad (k = 0, \pm 1, \pm 2, \ldots)$$

- From the above expression, we can see that e^z is never zero. However, e^z does assume every other complex value
- The exponential function is one-to-one on the real axis, but it is not one-to-one on the complex plane. In fact, we have
  1. The equation $e^z = 1$ holds if, and only if, $z = 2k\pi i$, where k is an integer
  2. The equation $e^{z_1} = e^{z_2}$ holds if, and only if, $z_1 = z_2 + 2k\pi i$, where k is an integer

Polar Form of the Complex Function e^z (Cont'd)

- The Theorem tell us e^z is periodic with complex period $2\pi i$
- If we divide up the z-plane into the infinite horizontal strips:

  $S_n := \{x + iy | -\infty < x < \infty, (2n-1)\pi < y \leq (2n+1)\pi\}$

  where $n = 0, \pm 1, \pm 2, \ldots$
- e^z behaves in the same manner on each strip. Furthermore, e^z is one-to-one on each strip
- Any one of these strips is called a fundamental region for e^z

Trigonometric Functions

- For real variables, we have the identities

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \quad \cos y = \frac{e^{iy} + e^{-iy}}{2}$$

- We extend the identities to the complex case: Given any complex number z, we define

$$\sin z := \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z := \frac{e^{iz} + e^{-iz}}{2}$$

- Since e^{iz} and e^{-iz} are entire functions, so are $\sin z$ and $\cos z$. Some further identities remain valid in the complex case (See page 113)

The Distinction Between the Real and Complex Cases

- The real cosine function is bounded by 1, i.e.,
$$|\cos x| \le 1, \quad \text{for all real } x$$
- But in the complex case, the cosine function
$$|\cos(iy)| = \left|\frac{e^{-y}+e^{y}}{2}\right| = \cosh y$$
which is unbounded and, in fact, is never less than 1
- But note that, this does not mean $|\sin z|$ or $|\cos z|$ is always greater than 1!

Hyperbolic Functions

- For any complex number z, we define
$$\sinh z := \frac{e^z - e^{-z}}{2}, \quad \cosh z := \frac{e^z + e^{-z}}{2}$$
- One nice feature of the complex variable perspective is that it reveals the intimate connection between hyperbolic functions and their trigonometric analogues (See page 114-115 for details)

Definition of Logarithmic Functions

- $\log z$ is defined as the inverse of the exponential function; i.e.,
$$w = \log z \quad \text{if} \quad z = e^w$$
- Since e^w is never zero, we presume that $z \neq 0$. Let us write z in the polar form as $z = re^{i\theta}$ and w in the standard form as $w = u + iv$. Then the equation $z = e^w$ becomes
$$re^{i\theta} = e^{u+iv} = e^u e^{iv}$$
- Taking magnitudes of both sides we deduce that $r = e^u$, or that u is the ordinary logarithm of r: $u = \text{Log } r = \text{Log } |z|$
- The equality of the remaining factors, $e^{i\theta} = e^{iv}$, identifies v as the (multiple-valued) polar angle θ=$\arg z$: $v = \arg z = \theta$

Definition of Logarithmic Functions (Cont'd)

- Thus $w = \log z$ is also a multiple-valued function. The explicit definition is as follows
- Definition 3: If $z \neq 0$, then we define $\log z$ to be the set of infinitely many values
$$\begin{aligned}\log z: \; &= \text{Log}|z| + i \arg z \\ &= \text{Log}|z| + i\text{Arg}z + i2k\pi \quad (k = 0, \pm 1, \pm 2, \ldots)\end{aligned}$$
- The multiple-valuedness of $\log z$ simply reflects the fact that the imaginary part of logarithm is the polar angle θ (multiple-valued); the real part is single-valued

Properties of Logarithmic Functions

- Many familiar properties of the real logarithmic function can be extended to the complex case, but it should be noted that $\log z$ is multiple-valued. Hence, the precise statements of these extensions are more complicated
- If $z \neq 0$, we have $z = e^{\log z}$, but

$$\log e^z = z + 2k\pi i \quad (k = 0, \pm 1, \pm 2, \ldots)$$

The Principle Value of Logarithm $\mathrm{Log} z$

- The notation of branch cut is used to resolve the ambiguity in the designation of the polar angle $\theta = \arg z$
- We take $\mathrm{Arg} z$ to be the principal value of $\arg z$, in the interval $(\tau, \tau + 2\pi]$ which shifts the 2π-discontinuities to the ray $\theta = \tau$
- Similarly, we generate single-valued branches of $\log z$. The principle value of the logarithm $\mathrm{Log} z$ is the value inherited from the principal value of the argument:

$$\mathrm{Log} z := \mathrm{Log}|z| + i\mathrm{Arg} z$$

(Note that we use the same convention 'capital L' for the principal value as for the real value, since $\mathrm{Arg} z = 0$ if z is positive real)

Analyticity and Derivative of $\mathrm{Log} z$

- $\mathrm{Log} z$ also inherits, from $\mathrm{Arg} z$, the discontinuities along the branch cut
- However, at all points off the nonpositive real axis, $\mathrm{Log} z$ is continuous when it is defined on the interval $(-\pi, \pi]$ and we have the following theorem
- Theorem 4: The function $\mathrm{Log} z$ is analytic in the domain D^* consisting of all points of the complex plane except those lying on the nonpositive real axis. Furthermore

$$\frac{d}{dz}\mathrm{Log} z = \frac{1}{z}, \quad \text{for } z \text{ in } D^*$$

Other Branches of $\log z$

- Other branches $\log z$ can be employed if the location of the discontinuities on the negative axis is inconvenient. Clearly, the specification

$$\mathcal{L}_\tau(z) := \mathrm{Log} z + i \arg_\tau z$$

results in a single-valued function whose imaginary part lies in the interval $(\tau, \tau + 2\pi]$
- Also, Theorem 4 shows that this function is analytic in the complex plane excluding the ray $\theta = \tau$ and the origin
- When complex arithmetic is incorporated into computer packages, all functions must of necessity be programmed as single-valued

Complex Powers Functions

- One important theoretical use of the logarithmic function is to define complex powers of z. The definition is motivated by the identity

$$z^n = \left(e^{\log z}\right)^n = e^{n \log z}$$

which holds for any integer n

- Definition 5: If α is a complex constant and $z \neq 0$, then we define z^α by

$$z^\alpha := e^{\alpha \log z}$$

This means that each value of $\log z$ leads to a particular value of z^α

Complex Powers Functions (Cont'd)

- Since $\log z = \text{Log}z + i\text{Arg}z + 2k\pi i$, we can get the following expression

$$z^\alpha = e^{\alpha(\text{Log}|z| + i\text{Arg}z + 2k\pi i)} = e^{\alpha(\text{Log}|z| + i\text{Arg}z)} e^{\alpha 2k\pi i} \quad (4)$$

where $k = 0, \pm 1, \pm 2, \cdots$ (See Example 1 on page 132)

- The values of z^α obtained by taking $k = k_1$ and $k = k_2$ ($\neq k_1$) in Eq.(4) will therefore be the same when

$$e^{\alpha 2k_1\pi i} = e^{\alpha 2k_2\pi i}$$

By Theorem 3 of Sec. 3.2 this occurs only if

$$\alpha 2k_1\pi i = \alpha 2k_2\pi i + 2m\pi i$$

where m is an integer. By solving this equation, we get $\alpha = m/(k_1 - k_2)$

Complex Powers Functions (Cont'd)

- This means only when α is a real rational number, (4) yields some identical values of z^α
- If α is not a real rational number, we obtain infinitely many different values for z^α, one for each choice of the integer k in Eq.(4)
- One the other hand, if $\alpha = m/n$, where m and $n > 0$ are integers having no common factor, then one can verify that there are exactly n distinct values of $z^{m/n}$, namely

$$z^{m/n} = \exp\left(\frac{m}{n}\text{Log}|z|\right) \exp\left(i\frac{m}{n}(\text{Arg}z + 2k\pi)\right) \quad (5)$$

where $k = 0, 1, \ldots, n-1$

Complex Powers Functions (Cont'd)

- Eq.(5) is entirely consistent with the theory of roots discussed in Sec. 1.5
- In summary,
  - z^α is single-valued when α is a real integer
  - z^α takes finitely many values when α is a real rational number
  - z^α takes infinitely many values in all other cases
- From Definition 4 and 5, we know that each branch of $\log z$ yields a branch of z^α. For example, using the principal branch of $\log z$ we obtain the principal branch of z^α, namely, $e^{\alpha \text{Log} z}$

Complex Powers Functions (Cont'd)

- Since e^z in entire and $\text{Log}z$ is analytic in the slit domain D^*, the chain rule implies that the principal branch of z^α is also analytic in D^*
- For z in D^*, we have
$$\frac{d}{dz}\left(e^{\alpha \text{Log}z}\right) = e^{\alpha \text{Log}z}\frac{d}{dz}(\alpha \text{Log}z) = e^{\alpha \text{Log}z}\frac{\alpha}{z}$$
- Other branches of z^α can be constructed by using other branches of $\log z$, and since each branch of the latter has derivative $1/z$, the formula
$$\frac{d}{dz}(z^\alpha) = \alpha z^\alpha \frac{1}{z}$$
is valid for each corresponding branch of z^α

Inverse Trigonometric Functions

- We have exponentials expressed in terms of trig functions, trig functions expressed as exponentials, and logs interpreted as inverse of exponentials
- Similarly, we can get the inverse trigonometric functions for complex numbers
- We start with the inverse sin function $w = \sin^{-1} z$. From the identity: $z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$, we can deduce that
$$e^{2iw} - 2ize^{iw} - 1 = 0$$
- By solving the above quadratic formula, we arrive at
$$e^{iw} = iz + (1 - z^2)^{1/2}$$
- Next, by taking logarithms, we get:
$$\sin^{-1} z = -i \log[iz + (1 - z^2)^{1/2}]$$

Inverse Trigonometric Functions (Cont'd)

- We can obtain a branch of the multiple-valued function $\sin^{-1} z$ by first choosing a branch of the square root and then selecting a suitable branch of the logarithm
- Using the chain rule and the formula of $\sin^{-1} z$, one can show that any such branch of $\sin^{-1} z$ satisfies
$$\frac{d}{dz}(\sin^{-1} z) = \frac{1}{(1 - z^2)^{1/2}} \quad (z \neq \pm 1)$$
where the choice of the square root on the right must be the same as that used in the branch of $\sin^{-1} z$
- The same methods can be applied to inverse cosine, tangent, and hyperbolic functions