Distributed Nash Equilibrium Seeking in Two-Coalition Zero-Sum Games Under Edge Agreements

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***Abstract*—This article investigates the Nash equilibrium seeking problem in two coalitions of agents constrained by edge agreements, where all agents within the same coalition are collectively cooperative to minimize the sum of their individual costs. The edge agreements are linear constraints imposed by adjacent agents, which integrate several coordination control objects (e.g., consensus control, cluster consensus control, and formation control) into a unified framework. We first propose a distributed algorithm based on the Rosen's gradient projection method and subgradient for Nash equilibrium seeking. Then, it is strictly proved that the proposed algorithm can solve the distributed Nash equilibrium seeking problem under edge agreements. Finally, a simulation example is provided to validate the effectiveness of the algorithm.**

***Index Terms*—Distributed algorithm, edge agreements, Nash equilibrium.**

Nomenclature

Variable Symbols	*Explanations*
$x_{i,i}$	State of agent $i \in \mathcal{V}_1$.
$y_{j,j}$	State of agent $j \in \mathcal{V}_2$.
$x_{i,l}$	i's estimate on l's state for $i, l \in \mathcal{V}_1$.
$y_{j,l}$	j's estimate on l's state for $j, l \in \mathcal{V}_2$.
$\hat{x}_i = \sum_{r\in\mathcal{V}_1}[\mathcal{A}_{11}]_i^r x_r$	Auxiliary variable of agent $i \in \mathcal{V}_1$.
$\hat{y}_j = \sum_{r\in\mathcal{V}_2}[\mathcal{A}_{22}]_j^r y_r$	Auxiliary variable of agent $j \in \mathcal{V}_2$.
$x = \mathrm{col}(x_{1,1}, \ldots, x_{m,m})$	Aggregation state of all agents in $\mathcal{V}_1$.
$y = \mathrm{col}(y_{m+1,m+1}, \ldots, y_{n,n})$	Aggregation state of all agents in $\mathcal{V}_2$.
$x_i = \mathrm{col}(x_{i,1}, \ldots, x_{i,m})$	Estimate of agent $i \in \mathcal{V}_1$ on x.
$y_j = \mathrm{col}(y_{j,m+1}, \ldots, y_{j,n})$	Estimate of agent $j \in \mathcal{V}_2$ on y.
$\widetilde{x}_j = \mathrm{col}(x_{j_1,j_1}, \ldots, x_{j_l,j_l})$	Aggregation states of all j's parent vertices in $\mathcal{V}_1$ for $j \in \mathcal{V}_2$.
$\widetilde{y}_i = \mathrm{col}(y_{i_1,i_1}, \ldots, y_{i_l,i_l})$	Aggregation states of all i's parent vertices in $\mathcal{V}_2$ for $i \in \mathcal{V}_1$.

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I. Introduction

In recent years, distributed computing problems have received widespread attention. Compared with traditional algorithms, distributed algorithms are more robust and cost-effective, in which all agents only need the information of themselves and their neighbors to achieve the global objective via local coordination. Distributed computing involves many fields, such as linear equations solving [1], [2], [3], optimization [4], [5], [6], [7] and Nash equilibrium seeking [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25].

In [9], [10], [11], [12], [13], and [14], multiplayer games were investigated, in which players were self-interested to minimize their own cost function. Hence, there is no agents' coalition whose members cooperatively minimize a common total cost function in these multiplayer games. However, in real life, competition is common among entities or coalitions, such as the Cournot duopoly model, in which two firms compete with each other. Therefore, it is necessary to study the competitive behaviors among coalitions. Recently, many efforts have been made to investigate this issue. In [15], two-coalition zero-sum games (ZSGs) were investigated, and the authors designed a distributed Nash equilibrium seeking algorithm based on saddle point dynamics for the games. Further, multicoalitions noncooperative games, which covered the two-coalition ZSGs, were investigated in [16] and a continuous-time distributed algorithm was designed by employing the average consensus protocol. In [17], the discrete-time gradient-free method was proposed to seek Nash equilibrium in multicoalition noncooperative games. In [18], multicoalition noncooperative games were considered under switching topologies. An approach based on extremum seeking was developed in [19], which reduced communication and computation costs compared with the counterpart in [16].

Unlike the unconstrained Nash equilibrium search problem considered in [15], [16], [17], [18], and [19], based on the gradient projection method, [20] investigated Nash equilibrium seeking for a two-coalition ZSG, where the agents' states in the same coalition were in the same closed convex set. Refs. [21] and [22] considered coupled linear and nonlinear constraints, respectively. But these constraints lacked specific forms for describing the relationship between agents within the same coalition. In [23] and [25], the consensus constraint was considered in multicoalition to characterize the consistent states of the agents in each coalition. Specifically, as mentioned in [26], [27], [28], [29] and [30], consensus means that agents in multiagent systems (MASs) reach an agreement regarding a common quantity and can be regarded as a global constraint, which requires that the states of all agents in MASs are equal. On this account, the consensus constraint implies that all agents in a coalition have to take consistent states, which are quite strict. The concept of edge agreements was proposed in [31], which imposed a group of corresponding local linear constraints on the edges of the network. Edge agreements are more flexible than consensus since consensus can be viewed as a special case of edge agreements.

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Although Lu and Mou [32] considered edge agreements as state constraints and proposed a continuous-time distributed algorithm based on the Lagrange multipliers method, they focused on the distributed optimization problem.

Motivated by the above observations, we consider distributed Nash equilibrium seeking for the two-coalition ZSGs subject to the edge agreements that are a series of linear constraints. Apparently, the edge agreement constraint has a wider applicability than the consensus constraint. The main contributions of the article are summarized as follows.

1) Distributed Nash equilibrium seeking problem in the two-coalition ZSGs is addressed. Different from related works [15], [16], [17], [18], [19], [20], [21], [22], [23], [25], the constraints of edge agreements considered in the two-coalition ZSGs integrate several coordination control objects (e.g., consensus control, cluster consensus control, and formation control) into a unified framework.
2) A distributed Nash equilibrium seeking algorithm is proposed for two-coalition ZSGs under the edge agreements. Our algorithm builds upon the Rosen's gradient projection method, projecting the gradient onto the kernel space of edge-agreement constraints' coefficient matrix, which is more computationally tractable than existing algorithms that rely on projections onto the constraint set. In addition, the proposed algorithm is capable of seeking Nash equilibrium in the presence of nonsmooth cost functions.

The rest of this article is organized as follows. Some preliminaries are presented in Section II. The problem formulation is presented in Section III. Our main results are given in Section IV. The numerical simulation is presented in Section V. Finally, Section VI concludes this article.

II. Preliminaries

A. Notations

Unless stated otherwise, all vectors in this article refer to column vectors. Let $\|\cdot\|$, $<\cdot,\cdot>$, and $\otimes$ denote the *Frobenius* norm, the *Euclidean* inner product, and the *Kronecker* product, respectively. For a set S, $|S|$ represents its cardinal number. For a matrix A, $[A]_i^j$ is used to denote its entry at the ith row and jth column, and $A^\top$ denotes its transpose. Let $\text{col}(x_1, x_2, \ldots, x_n)$ denote the column stack of vectors $x_1, x_2, \ldots, x_n$. $\text{blockdiag}(A_1, A_2, \ldots, A_n)$ denotes the block diagonal matrix whose ith diagonal block is a matrix A_i. Let I_d represent the d-dimensional identity matrix, $\mathbf{1}$ be the vector with appropriate dimension whose elements all equal one, and $\mathbf{0}_{m\times n}$ be the zero matrix with m rows and n columns. A matrix is said to be nonnegative if all its elements are nonnegative. A vector is called a stochastic vector if all its components are nonnegative and the sum of its components is equal to one. A matrix is called a doubly stochastic matrix if each of its row and column vectors is a stochastic vector. For a function $f(x)$, if it does not cause misunderstanding, it is simply written as f. For the sake of brevity, "with respect to" and "if and only if" are abbreviated as w.r.t. and iff, respectively.

B. Graph Theory

Let $\mathcal{G} = \{\mathcal{A}, \mathcal{V}, \mathcal{E}\}$ be a weighted digraph, where $\mathcal{V} = \{1, \ldots, n\}$ is the vertex set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a weighted adjacency matrix with $a_{ij} \geq 0$. A directed edge of $\mathcal{G}$ is represented as (j, i), where j is called a parent vertex of i and i is a child vertex of j, which means that vertex i can receive information from vertex j. Moreover, $a_{ij} > 0$ iff there is a directed edge $(j, i) \in \mathcal{E}$. An incidence matrix of digraph $\mathcal{G}$ is denoted by $H = [h_{kj}] \in \mathbb{R}^{\bar{m}\times n}$, where $\bar{m}$ represents the number of edges. The entry $h_{kj} = 1$ if j is the parent vertex in the kth edge; $h_{kj} = -1$ if j is the child vertex in the kth edge; elsewhere $h_{kj} = 0$. A path from vertex i_1 to vertex i_p is a finite-ordered sequence $(i_1, i_2), (i_2, i_3), \ldots, (i_{p-1}, i_p)$ with distinct vertices. A digraph is termed a strongly connected digraph if there exists a path from every vertex to every other vertex.

C. Convex Analysis

This subsection introduces some concepts about convex (concave) functions. $f : \mathbb{R}^m \to \mathbb{R}$ is said to be convex if $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2) \quad \forall x_1, x_2, \lambda \in [0, 1]$. $g : \mathbb{R}^m \to \mathbb{R}$ is said to be concave if $g(\lambda x_1 + (1-\lambda)x_2) \geq \lambda g(x_1) + (1-\lambda)g(x_2) \quad \forall x_1, x_2, \lambda \in [0, 1]$. $\phi(x, y) : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ is said to be convex-concave if ϕ is convex w.r.t x and concave w.r.t y. In addition, (x_0, y_0) is called a saddle point of $\phi(x, y)$ if

$$\phi(x_0, y) \leq \phi(x_0, y_0) \leq \phi(x, y_0) \quad \forall x, y. \tag{1}$$

For a convex function f, $\partial f(\hat{x}) \in \mathbb{R}^m$ is a subgradient of f at $\hat{x}$ if $f(x) \geq f(\hat{x}) + (x - \hat{x})^\top \partial f(\hat{x}) \quad \forall x$. Similarly for a concave function g, $\partial g(\hat{x}) \in \mathbb{R}^m$ is a subgradient of g at $\hat{x}$ if $g(x) \leq g(\hat{x}) + (x - \hat{x})^\top \partial g(\hat{x}) \quad \forall x$.

D. Two-Player ZSGs

Two-player ZSGs are described as a triple $\{P, X, U\}$, where $P = \{p_1, p_2\}$ is a player set; $X = X_1 \times X_2$ and $X_i \subseteq \mathbb{R}^{m_i}$ is an action set only available to player p_i; $U = (u_1, u_2)$, $u_i : X \to \mathbb{R}$ is the cost function of player p_i and $u_1(x_1, x_2) + u_2(x_1, x_2) = 0 \quad \forall x_1 \in X_1 \quad \forall x_2 \in X_2$.

In two-player ZSGs, a joint action (x_1', x_2') is called Nash equilibrium if no player can reduce its cost by changing its action under the current actions of another player, i.e., $u_1(x_1', x_2') \leq u_1(x_1, x_2')$ and $u_2(x_1', x_2') \leq u_2(x_1', x_2) \quad \forall x_1 \in X_1 \quad \forall x_2 \in X_2$.

Remark 1: Note that $u_1(x_1', x_2') + u_2(x_1', x_2') = 0$ and $u_1(x_1', x_2) + u_2(x_1', x_2) = 0$. Thus, $-u_2(x_1', x_2') \geq -u_2(x_1', x_2)$ iff $u_1(x_1', x_2') \geq u_1(x_1', x_2)$. By (1), a Nash equilibrium of a two-player ZSG is a saddle point of u_i.

The following lemma about the two-player ZSGs is useful.

Lemma 1 (Theorem 2.5 in [15]): Let $X_1 \subseteq \mathbb{R}^{m_1}$ and $X_2 \subseteq \mathbb{R}^{m_2}$ be nonempty, compact, and convex. Let $u_1(x_1, x_2)$ be continuous. If the set $\{x \in X_1 | u_1(x, x_2) \leq c \quad \forall c \in \mathbb{R}\}$ and the set $\{y \in X_2 | u_1(x_1, y) \geq c \forall c \in \mathbb{R}\}$ are convex, then

$$\max_{x_2 \in X_2} \min_{x_1 \in X_1} u_1(x_1, x_2) = \min_{x_1 \in X_1} \max_{x_2 \in X_2} u_1(x_1, x_2).$$

Remark 2: Since the equivalence between the convexity of the lower level set and the quasi-convexity of the function, Lemma 1 is an equivalent statement of the well-known Sion's minmax theorem ([33, Th. 3.4]). Moreover, if $u_1(x_1, x_2)$ is convex–concave, then $u_1(x_1, x_2)$ is continuous, and $\{x \in X_1 | u_1(x, x_2) \leq c \quad \forall c \in \mathbb{R}\}$ and $\{y \in X_2 | u_1(x_1, y) \geq c \quad \forall c \in \mathbb{R}\}$ are convex. Therefore, for compact sets $X_1 \subseteq \mathbb{R}^{m_1}$ and $X_2 \subseteq \mathbb{R}^{m_2}$, combining with Lemma 1, there exists a pair (x_1^*, x_2^*) such that $\max_{x_2 \in X_2} \min_{x_1 \in X_1} u_1(x_1, x_2) = \min_{x_1 \in X_1} \max_{x_2 \in X_2} u_1(x_1, x_2) = u_1(x_1^*, x_2^*)$. Then, according to [20, Lemma 2.1], (x_1^*, x_2^*) is a saddle point of $u_1(x_1, x_2)$. Hence the two-player ZSG has a Nash equilibrium.

III. Problem Setup

In this section, the problem formulation will be presented. First, we will describe the network models.

Denote a group of agents as $\mathcal{V} = \{1, \ldots, n\}$ composed of two coalitions $\mathcal{V}_1 = \{1, \ldots, m\}$ and $\mathcal{V}_2 = \{m+1, \ldots, n\}$. A weighted digraph $\mathcal{G}(\mathcal{A}, \mathcal{V}, \mathcal{E})$ that contains self-loops is used to model relationships

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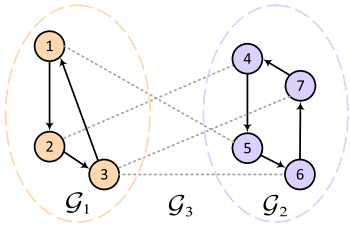

Fig. 1. Communication network in a group of agents composed of two coalitions.

between two coalitions, where $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Obviously, $\mathcal{A}$ can be partitioned as

$$\begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}$$

where $\mathcal{A}_{11} \in \mathbb{R}^{m\times m}$, $\mathcal{A}_{12} \in \mathbb{R}^{m\times(n-m)}$, $\mathcal{A}_{21} \in \mathbb{R}^{(n-m)\times m}$, and $\mathcal{A}_{22} \in \mathbb{R}^{(n-m)\times(n-m)}$. Then, $\mathcal{G}$ can be divided into $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ (see, e.g., Fig. 1). $\mathcal{G}_1 = (\mathcal{A}_1, \mathcal{V}_1, \mathcal{E}_1)$ describes the communication among $\mathcal{V}_1$, where $\mathcal{A}_1 = \mathcal{A}_{11}$ and $\mathcal{E}_1 \subseteq \mathcal{V}_1 \times \mathcal{V}_1$. $\mathcal{G}_2 = (\mathcal{A}_2, \mathcal{V}_2, \mathcal{E}_2)$ describes the communication among $\mathcal{V}_2$, where $\mathcal{A}_2 = \mathcal{A}_{22}$ and $\mathcal{E}_2 \subseteq \mathcal{V}_2 \times \mathcal{V}_2$. $\mathcal{G}_3 = (\mathcal{A}_3, \mathcal{V}, \mathcal{E}_3)$ depicts the communication between $\mathcal{V}_1$ and $\mathcal{V}_2$, where $\mathcal{A}_3 = \mathcal{A} - \text{blockdiag}\{\mathcal{A}_{11}, \mathcal{A}_{22}\}$ and $\mathcal{E}_3 \subseteq \bigcup_{r=1}^{2} \mathcal{V}_{3-r} \times \mathcal{V}_r$. Furthermore, each vertex in $\mathcal{G}_r$ has at least one parent vertex in $\mathcal{G}_{3-r}$ $(r = 1, 2)$, i.e, the graph $\mathcal{G}_3$ has no isolate vertices.

The following assumptions on network model are useful.

Assumption 1: Nonnegative matrices $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are doubly stochastic matrices, i.e, $\mathcal{A}_{11}\mathbf{1} = \mathbf{1}, \mathbf{1}^\top\mathcal{A}_{11} = \mathbf{1}^\top, \mathcal{A}_{22}\mathbf{1} = \mathbf{1}, \mathbf{1}^\top\mathcal{A}_{22} = \mathbf{1}^\top$.

Assumption 2: $\mathcal{G}_1$ and $\mathcal{G}_2$ are strongly connected.

The states of agent $i \in \mathcal{V}_1$ and agent $j \in \mathcal{V}_2$ are represented by $x_{i,i}(k) \in \mathbb{R}^d$ and $y_{j,j}(k) \in \mathbb{R}^d$ at time k, respectively, where $k = 0, 1, 2, \cdots$. Then, the coalition states of two coalitions are $x = \text{col}(x_{1,1}, \ldots, x_{m,m})$ and $y = \text{col}(y_{m+1,m+1}, \ldots, y_{n,n})$, respectively. In addition, agent i in $\mathcal{V}_1$ is equipped with $x_i = \text{col}(x_{i,1}, \ldots, x_{i,m})$ to estimate x, where $x_{i,r}$ is an estimate of agent i on the state of agent $r \in \mathcal{V}_1$ $(i \neq r)$. Agent j in $\mathcal{V}_2$ is equipped with $y_j = \text{col}(y_{j,m+1}, \ldots, y_{j,n})$ to estimate y, where $y_{j,r}$ is an estimate of agent j on the state of agent $r \in \mathcal{V}_2$ $(j \neq r)$. Subsequently, the agents will send their own estimates to their child vertices.

Second, we consider that each agent in the group possesses a associated cost function related to the coalition state to which the agent belongs and states of parent vertices in the other coalition. Agent $i \in \mathcal{V}_1$ has a convex–concave cost function $f_i(x, \widetilde{y}_i)$, where $\widetilde{y}_i = \text{col}(y_{i_1,i_1}, \ldots, y_{i_l,i_l})$ and $i_1, \ldots, i_l$ are all i's parent vertices in $\mathcal{V}_2$. Agent $j \in \mathcal{V}_2$ has a convex–concave cost function $-g_j(\widetilde{x}_j, y)$, where $\widetilde{x}_j = \text{col}(x_{j_1,j_1}, \ldots, x_{j_l,j_l})$ and $j_1, \ldots, j_l$ are all j's parent vertices in $\mathcal{V}_1$. Then, the sum cost function of each coalition can be expressed as $\sum_{i\in\mathcal{V}_1} f_i(x, \widetilde{y}_i)$ and $-\sum_{j\in\mathcal{V}_2} g_j(\widetilde{x}_j, y)$ respectively. Moreover, the sum of the cost functions of two coalitions are opposite to each other, (i.e., $\sum_{i\in\mathcal{V}_1} f_i(x, \widetilde{y}_i) - \sum_{j\in\mathcal{V}_2} g_j(\widetilde{x}_j, y) = 0$), and agents within the same coalition are cooperative to minimize the sum of their individual costs. This setting and the communication network constitute the so-called two-coalition ZSG, where two coalitions can be viewed as two virtual players engaging in a two-player ZSG. Consequently, the Nash equilibrium of the two-coalition ZSG can be naturally described as $(x^*, y^*) \in \arg\sup_y \inf_x U(x, y)$, where $U(x, y) = \sum_{i\in\mathcal{V}_1} f_i(x, \widetilde{y}_i) = \sum_{j\in\mathcal{V}_2} g_j(\widetilde{x}_j, y)$.

In this article, our goal is to design a distributed algorithm to seek the above Nash equilibrium under the following edge agreements (c.f., [31]):

$$A_{i,j}(x_{i,i} - x_{j,j}) = b_{i,j} \quad \forall (j,i) \in \mathcal{E}_1 \tag{2a}$$

$$A_{i,j}(y_{i,i} - y_{j,j}) = b_{i,j} \quad \forall (j,i) \in \mathcal{E}_2 \tag{2b}$$

where for $(j,i) \in \mathcal{E}_1$, there is $A_{i,j} \in \mathbb{R}^{m_1\times d}$ and $b_{i,j} \in \mathbb{R}^{m_1}$; for $(j,i) \in \mathcal{E}_2$, there is $A_{i,j} \in \mathbb{R}^{m_2\times d}$ and $b_{i,j} \in \mathbb{R}^{m_2}$. Notably, for any undirected edge $(i,j) \in \mathcal{E}_r$ $(r = 1, 2)$, the conditions $A_{i,j} = A_{j,i}$ and $b_{i,j} = -b_{j,i}$ must hold to ensure the feasibility of the edge agreements. In this article, the edge agreements are supposed to be solvable.

Remark 3: Note that the edge agreements associated with two vertices are constructed only if there is an edge between them. These edge agreements mean that both vertices connected with an edge admit common linear constraints of their states. Hence, such constraints are called edge agreements. Moreover, the edge agreements integrate several coordination control objects into a unified framework. Here, we take coalition $\mathcal{V}_2$ in Fig. 1 as an example, where states of all agents are subject to (2b). If $A_{i,j} = I_d, b_{i,j} = 0$ for any $(j,i) \in \mathcal{E}_2$, then agents will achieve consensus, i.e., $y_{4,4} = \cdots = y_{7,7}$. If $A_{i,j} = I_d$ for any $(j,i) \in \mathcal{E}_2$ and $b_{4,7} = b_{6,5} = \mathbf{1}, b_{5,4} = b_{7,6} = \mathbf{0}_{d\times 1}$, then the cluster consensus will be achieved. Specifically, agents separate into two clusters, achieving consensus within each cluster, but not achieving consensus between clusters, i.e., $y_{4,4} = y_{5,5} \neq y_{6,6} = y_{7,7}$. If $A_{i,j} = I_d$ $(d = 2)$ for any $(j,i) \in \mathcal{E}_2$ and $b_{4,7} = (-4, 0)^\top, b_{5,4} = (0, -3)^\top, b_{6,5} = (4, 0)^\top, b_{7,6} = (0, 3)^\top$, then agents will form a rectangle with a length of 4 and a width of 3 in 2-D.

In summary, we aim to solve the following problem to obtain the Nash equilibrium of the two-coalition ZSGs under the edge agreements:

$$\begin{aligned} &\sup_y \inf_x U(x,y) \\ \text{s.t.}\quad &A_{i,j}(x_{i,i} - x_{j,j}) = b_{i,j} \quad \forall (j,i) \in \mathcal{E}_1 \\ &A_{i,j}(y_{i,i} - y_{j,j}) = b_{i,j} \quad \forall (j,i) \in \mathcal{E}_2. \end{aligned} \tag{3}$$

IV. Main Result

In this section, we will propose a discrete-time distributed algorithm to solve problem (3) and strictly prove its feasibility. In order to facilitate the algorithm design, we make the following transformation to the edge agreements.

Let $\bar{m}_1 = |\mathcal{E}_1|$ and $\bar{m}_2 = |\mathcal{E}_2|$, where $|\mathcal{E}_1|$ and $|\mathcal{E}_2|$ are the numbers of elements in $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Let $H_1 \in \mathbb{R}^{\bar{m}_1\times m}$ be the incidence matrix of $\mathcal{G}_1$ and $H_2 \in \mathbb{R}^{\bar{m}_2\times(n-m)}$ be the incidence matrix of $\mathcal{G}_2$. Then, (2a) and (2b) become

$$-\bar{A}_1\bar{H}_1 x = \bar{b}_1 \text{ and } -\bar{A}_2\bar{H}_2 y = \bar{b}_2$$

where $\bar{H}_1 = H_1 \otimes I_d$, $\bar{b}_1 = \text{col}(b_{i_1,j_1}, \cdots, b_{i_{\bar{m}_1},j_{\bar{m}_1}})$, $\bar{H}_2 = H_2 \otimes I_d$, $\bar{b}_2 = \text{col}(b_{i'_1,j'_1}, \ldots, b_{i'_{\bar{m}_2},j'_{\bar{m}_2}})$, $\bar{A}_1 = \text{blockdiag}\{A_{i_1,j_1}, \ldots, A_{i_{\bar{m}_1},j_{\bar{m}_1}}\}$, and $\bar{A}_2 = \text{blockdiag}\{A_{i'_1,j'_1}, \ldots, A_{i'_{\bar{m}_2},j'_{\bar{m}_2}}\}$. Moreover, i_r and j_r denote the child and parent vertices of the rth edge of $\mathcal{G}_1$, respectively. Similarly, i'_r and j'_r denote the child and parent vertices of the rth edge of $\mathcal{G}_2$, respectively. For simplicity, denote $-\bar{A}_1\bar{H}_1$ as M_1. Then we can convert (2a) into

$$M_1 x = \bar{b}_1. \tag{4}$$

Analogously, we can convert (2b) into

$$M_2 y = \bar{b}_2 \tag{5}$$

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where $M_2 = -\bar{A}_2\bar{H}_2$. Denote $\Omega_1 = \{x|M_1x = \bar{b}_1, x \in \mathbb{R}^{md}\}$ and $\Omega_2 = \{y|M_2y = \bar{b}_2, y \in \mathbb{R}^{(n-m)d}\}$. Then, our goal is turned to seek (x^*, y^*) of the function $U(x, y)$ in $\Omega_1 \times \Omega_2$.

Let $\mathcal{E}_{r,i} = \{(j,i)|\ (j,i) \in \mathcal{E}_r\}$, where i is the child node of the edges of $\mathcal{E}_r$ for $r = 1, 2$. Let $k_{i,1}, k_{i,2}, \ldots, k_{i,|\mathcal{E}_{1,i}|}$ and $k'_{i,1}, k'_{i,2}, \ldots, k'_{i,|\mathcal{E}_{2,i}|}$ represent the numerical identifiers to the edges in $\mathcal{E}_{1,i}$ and $\mathcal{E}_{2,i}$, respectively. In addition, let $K_i = \{k_{i,1}, k_{i,2}, \ldots, k_{i,|\mathcal{E}_{1,i}|}\}$ and $K'_i = \{k'_{i,1}, k'_{i,2}, \ldots, k'_{i,|\mathcal{E}_{2,i}|}\}$. Obviously, $\bigcup_{i\in\mathcal{V}_1} K_i = \{1, 2, \ldots, \bar{m}_1\}$ and $\bigcup_{i\in\mathcal{V}_2} K'_i = \{1, 2, \ldots, \bar{m}_2\}$. Then, we get

$$A_{i,j}(x_{i,i} - x_{j,j}) = b_{i,j}, \quad i \in \mathcal{V}_1 \quad \forall (j,i) \in \mathcal{E}_{1,i} \tag{6a}$$

$$A_{i,j}(y_{i,i} - y_{j,j}) = b_{i,j}, \quad i \in \mathcal{V}_2 \quad \forall (j,i) \in \mathcal{E}_{2,i}. \tag{6b}$$

The compact forms of (6a) and (6b) are as follows:

$$M_{1,i}x = \bar{b}_{1,i} \quad \forall i \in \mathcal{V}_1 \tag{7a}$$

$$M_{2,i}y = \bar{b}_{2,i} \quad \forall i \in \mathcal{V}_2 \tag{7b}$$

where $M_{1,i} = -\bar{A}_{1,i}\bar{H}_{1,i}$, $M_{2,i} = -\bar{A}_{2,i}\bar{H}_{2,i}$, $\bar{A}_{1,i} = \text{blockdiag}\left\{A_{i_{k_{i,1}},j_{k_{i,1}}}, \ldots, A_{i_{k_{i,|\mathcal{E}_{1,i}|}},j_{k_{i,|\mathcal{E}_{1,i}|}}}\right\}$, and $\bar{A}_{2,i} = \text{blockdiag}\left\{A_{i'_{k'_{i,1}},j'_{k'_{i,1}}}, \ldots, A_{i'_{k'_{i,|\mathcal{E}_{2,i}|}},j'_{k'_{i,|\mathcal{E}_{2,i}|}}}\right\}$. Furthermore, $\bar{H}_{1,i} = H_{1,i} \otimes I_d$ and $\bar{H}_{2,i} = H_{2,i} \otimes I_d$, where $H_{1,i}$ is composed of the $k_{i,1}$th, $k_{i,2}$th, $\cdots$, $k_{i,|\mathcal{E}_{1,i}|}$th row vectors of H_1, and $H_{2,i}$ is composed of the $k'_{i,1}$th, $k'_{i,2}$th, $\cdots$, $k'_{i,|\mathcal{E}_{2,i}|}$th row vectors of H_2. In the same way, $\bar{b}_{1,i} = \text{col}(b_{i_{k_{i,1}},j_{k_{i,1}}}, \ldots, b_{i_{k_{i,|\mathcal{E}_{1,i}|}},j_{k_{i,|\mathcal{E}_{1,i}|}}})$ and $\bar{b}_{2,i} = \text{col}(b_{i'_{k'_{i,1}},j'_{k'_{i,1}}}, \ldots, b_{i'_{k'_{i,|\mathcal{E}_{2,i}|}},j'_{k'_{i,|\mathcal{E}_{2,i}|}}})$. Let $P_{r,i} = I_{|\mathcal{V}_r|d} - M_{r,i}^\top(M_{r,i}M_{r,i}^\top)^{-1}M_{r,i}$, which is a projection onto kernel space of $M_{r,i}$, where $i \in \mathcal{V}_r$ and $r = 1, 2$.

In this article, we consider the following algorithm based on Rosen's gradient projection method:

$$\begin{aligned} x_i(k+1) = x_i(k) &+ P_{1,i}\big(\widehat{x}_i(k) - x_i(k)\big) \\ &- \alpha(k)\left(\partial_x f_i\left(\widehat{x}_i(k), \widetilde{y}_i(k)\right)\right),\ i \in \mathcal{V}_1 \end{aligned} \tag{8a}$$

$$\begin{aligned} y_i(k+1) = y_i(k) &+ P_{2,i}\big(\widehat{y}_i(k) - y_i(k)\big) \\ &+ \alpha(k)\left(\partial_y g_i\left(\widetilde{x}_i(k), \widehat{y}_i(k)\right)\right),\ i \in \mathcal{V}_2 \end{aligned} \tag{8b}$$

where $\widehat{x}_i(k) = \sum_{j\in\mathcal{V}_1}[\mathcal{A}_{11}]_i^j x_j(k) \in \mathbb{R}^{md}$, $\widehat{y}_i(k) = \sum_{j\in\mathcal{V}_2}[\mathcal{A}_{22}]_i^j y_j(k) \in \mathbb{R}^{(n-m)d}$, $\alpha(k)$ is the step size at time k and the gradients are projected onto the kernel space of $M_{r,i}$. Furthermore, the step size satisfies the following assumption.

Assumption 3: $\alpha(k) > 0$, $\sum_{k=0}^{\infty}\alpha(k) = \infty$ and $\sum_{k=0}^{\infty}\alpha^2(k) < \infty$.

Remark 4: It is worth highlighting that in the proposed algorithm, the projection onto the kernel space of $M_{1,i}$ ($M_{2,i}$) is easier obtained than the projection onto the solution space of (7a) and (7b). Thus, our algorithm is more readily applicable than algorithms that based on the projection onto the solution space.

In addition, in this article, we always assume that the subgradients are bounded, which can be expressed as the following universal assumption (see [5] and [20]).

Assumption 4: There exists a positive constant, denoted by L, such that $\|\partial_x f_i(x, \widetilde{y}_i)\| < L$ and $\|\partial_y g_j(\widetilde{x}_j, y)\| < L$ $\forall i \in \mathcal{V}_1$ $\forall j \in \mathcal{V}_2, \forall x \in \Omega_{1,i}, y \in \Omega_{2,j}$, where $\Omega_{1,i} = \{x|M_{1,i}x = \bar{b}_{1,i}, x \in \mathbb{R}^{md}\}$ and $\Omega_{2,j} = \{y|M_{2,j}y = \bar{b}_{2,j}, y \in \mathbb{R}^{(n-m)d}\}$.

Remark 5: If the cost functions are nonsmooth, then its gradient at some points may not exist. The majority of existing algorithms that rely on gradient information are difficult to deal with this challenge. In this article, this problem is addressed by utilizing the subgradient to replace the gradient at points where the gradient does not exist.

Obviously, the proposed algorithm is distributed because it only requires local information. Then, using the Sandwich Theorem, we can get the main result of this article.

Theorem 1: Let Assumptions 1–4 hold, and $f_i(x, \widetilde{y}_i)$ and $g_j(\widetilde{x}_j, y)$ are convex–concave functions. Let $x_i(0)$ and $y_j(0)$ satisfy (7a) and (7b), respectively. Then, the algorithm (8) solves the problem (3).

Proof: The proof is presented in Appendix. ■

V. Simulation

In this section, we present an illustrative example to demonstrate the result. A network comprising seven agents (see Fig. 1) is considered, where $n = 7$, $m = 3$, $d = 2$. The incidence matrices of the network in Fig. 1 are given by

$$H_1 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, H_2 = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

The corresponding weight matrix is as follows:

$$\mathcal{A} = \begin{bmatrix} 0.5 & 0 & 0.5 & 1 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}.$$

The cost function of each agent is as follows:

$$\begin{cases} f_1(x, \widetilde{y}_1) = \|x_{1,1} - (0,2)^\top\| + \|x_{2,2} - (0,0)^\top\| \\ \qquad\qquad +\|x_{3,3} - (2,0)^\top\| - 4\|y_{5,5}\|, \\ f_2(x, \widetilde{y}_2) = \|x_{1,1} - (0,2)^\top\| + \|x_{2,2} - (0,0)^\top\| \\ \qquad\qquad +\|x_{3,3} - (2,0)^\top\| - 4\|y_{4,4}\|, \\ f_3(x, \widetilde{y}_3) = \|x_{1,1} - (0,2)^\top\| + \|x_{2,2} - (0,0)^\top\| \\ \qquad\qquad +\|x_{3,3} - (2,0)^\top\| - 4\|y_{6,6}\| - 4\|y_{7,7}\| \\ g_4(\widetilde{x}_4, y) = \|y_{4,4}\| + \|y_{5,5}\| + \|y_{6,6}\| + \|y_{7,7}\| - \|x_{2,2}\| \\ g_5(\widetilde{x}_5, y) = \|y_{4,4}\| + \|y_{5,5}\| + \|y_{6,6}\| + \|y_{7,7}\| - 3\|x_{1,1}\| \\ g_6(\widetilde{x}_6, y) = \|y_{4,4}\| + \|y_{5,5}\| + \|y_{6,6}\| + \|y_{7,7}\| - \|x_{3,3}\| \\ g_7(\widetilde{x}_7, y) = \|y_{4,4}\| + \|y_{5,5}\| + \|y_{6,6}\| + \|y_{7,7}\| - 2\|x_{3,3}\|. \end{cases}$$

It is easy to see that $\sum_{i=1}^{3} f_i(x, \widetilde{y}_i) = \sum_{j=4}^{7} g_j(\widetilde{x}_j, y)$, and all cost functions are convex–concave and satisfy Assumption 4. Furthermore, the agents' states in the two coalitions present the positions in $\mathbb{R}^2$.

Configure $A_{i,j}$ and $b_{i,j}$ as follows:

$$A_{i,j} = I_2 \quad \forall (j,i) \in \mathcal{E}_1 \cup \mathcal{E}_2,$$

$$b_{1,3} = (-2,2)^\top, b_{2,1} = (0,-2)^\top, b_{3,2} = (2,0)^\top$$

$$b_{4,7} = (-2,0)^\top, b_{5,4} = (0,-2)^\top, b_{6,5} = (2,0)^\top, b_{7,6} = (0,2)^\top.$$

Then, if the states of all agents satisfy the edge agreements, coalition $\mathcal{V}_1$ and coalition $\mathcal{V}_2$ will form an isosceles right triangle with a right angle length of 2 and a square with a side length of 2, respectively.

Take following initial conditions:

$$x_1(0) = (1,3,0,0,3,1)^\top, x_2(0) = (-1,5,-1,3,0,0)^\top$$

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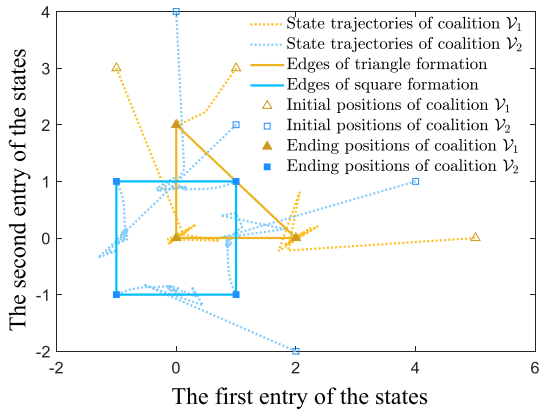

Fig. 2. Formations of two coalitions.

$$x_3(0) = (0, 0, 3, 0, 5, 0)^\top$$

$$y_4(0) = (1, 2, 0, 0, 0, 0, 3, 2)^\top, y_5(0) = (2, 0, 2, -2, 0, 0, 0, 0)^\top$$

$$y_6(0) = (0, 0, 2, 1, 4, 1, 2, 1)^\top, y_7(0) = (0, 0, 0, 0, 0, 2, 0, 4)^\top.$$

Let $\alpha(k) = 1/(k+1)$. It is easy to see that the norm $\|\cdot\|$ on $\mathbb{R}^2$ is nondifferentiable at zero vector. Clearly, the vectors within the unit circle are subgradients of the $\|\cdot\|$. In the simulation, the subgradient of $\|\cdot\|$ is selected as $(1, 0)^\top$.

The simulation result are shown in Fig. 2, where the blue and golden dashed lines represent the state trajectories of agents in coalition $\mathcal{V}_1$ and coalition $\mathcal{V}_2$, respectively. The blue and golden solid lines represent the edges of formation shapes, respectively. Obviously, the agents of coalition $\mathcal{V}_1$ form a triangle formation, while the agents of coalition $\mathcal{V}_2$ form a square formation, which is consistent with the edge agreements. Further, it can be observed from Fig. 2 that the states of all agents reach a Nash equilibrium (x^*, y^*), where $x^* = (0, 2, 0, 0, 2, 0)^\top$ and $y^* = (-1, 1, -1, -1, 1, -1, 1, 1)^\top$, which can be verified by the KKT conditions.

VI. Conclusion

In this article, the two-coalition ZSGs constrained by edge agreements have been investigated, where agents within the same coalition are collectively cooperative to minimize the total of their individual costs and the total costs between two coalitions are conflictive. Based on Rosen's gradient projection method and subgradient, we have designed a discrete-time distributed Nash equilibrium seeking algorithm for the games with a weight directed network and the nonsmooth cost functions. We rigorously prove that the proposed algorithm solves the distributed Nash equilibrium seeking problem under edge agreements and the numerical test verify its effectiveness.

Appendix

In Appendix, we will prove Theorem 1. We begin by presenting several lemmas, before embarking on the proof.

Lemma 2: Let Assumptions 1 and 2 hold and $\eta = \min_{i,j\in\mathcal{V}} a_{ij}$. Then, for all $i_1, i_2 \in \{1, 2, \ldots, m\}$ and $j_1, j_2 \in \{m+1, m+2, \ldots, n\}$, we have $\|[\mathcal{A}_1^k]_{i_1}^{i_2} - \frac{1}{m}\| \le C_1\beta_1^k$, and $\|[\mathcal{A}_2^k]_{j_1}^{j_2} - \frac{1}{n-m}\| \le C_2\beta_2^k$, where $C_1 = 2\frac{(1+\eta^{-(m-1)})}{(1-\eta^{m-1})}$, $\beta_1 = (1-\eta^{m-1})^{\frac{1}{m-1}}$, $C_2 = 2\frac{(1+\eta^{-(n-m-1)})}{(1-\eta^{n-m-1})}$ and $\beta_2 = (1-\eta^{n-m-1})^{\frac{1}{n-m-1}}$. Moreover, the entries of $\mathcal{A}_1^k$ and $\mathcal{A}_2^k$ converge to $\frac{1}{m}$ and $\frac{1}{n-m}$ as $k \to \infty$, respectively.

Lemma 2 is the fixed graph version of [5, Proposition 1]. The specific proof can be seen in [5].

Lemma 3 (Lemma 1 in [6]): Suppose $X \subseteq \mathbb{R}^n$ is a nonempty closed convex set. Then, $\|P_X x - z\|^2 \le \|x - z\|^2 - \|P_X x - x\|^2 \quad \forall x \in \mathbb{R}^n \quad \forall z \in X$, where P_X is the projection onto X.

Lemma 4 (Lemma 7 in [6]): Suppose $0 < \beta < 1$, $\gamma(k)$ is a positive scalar sequence and $\lim_{k\to\infty}\gamma(k) = 0$. Then $\lim_{k\to\infty}\sum_{l=0}^{k}\beta^{k-l}\gamma(l) = 0$. Moreover, if $\sum_{k=0}^{\infty}\gamma(k) < \infty$, then $\sum_{k=0}^{\infty}\sum_{l=0}^{k}\beta^{k-l}\gamma(l) < \infty$.

Lemma 5 (Lemma 5.1 in [20]): Assume $a(k)$, $b(k)$ and $c(k)$ are nonnegative sequences. Let $\sum_{k=0}^{\infty} b(k) < \infty$. If $a(k+1) \le a(k) + b(k) - c(k)$ for $k > 0$, $\lim_{k\to\infty} a(k)$ is a finite number.

Definition 1 (See [4]): Let $C \in \mathbb{R}^{mn\times mn}$ be a block matrix divided into m blocks of $n \times n$ matrices, where ijth block is denoted as C_{ij}. A norm $|||\cdot|||$ is defined as $|||C||| = \| < C > \|_\infty$, where $< C >$ is the matrix whose entry at ith row and jth column is $\|C_{ij}\|_2$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are the induced 2-norm and ∞-norm, respectively.

Lemma 6: Let $h_x(k) = \max_{i_1,i_2\in\mathcal{V}_1}\|x_{i_1}(k) - x_{i_2}(k)\|$ and $h_y(k) = \max_{j_1,j_2\in\mathcal{V}_2}\|y_{j_1}(k) - y_{j_2}(k)\|$. Suppose $x_i(0)$ and $y_j(0)$ satisfy (7a) and (7b), respectively. Then, $\lim_{k\to\infty} h_x(k) = 0$ and $\lim_{k\to\infty} h_y(k) = 0$. Furthermore, $\sum_{k=0}^{\infty}\alpha(k)h_x(k) < \infty$ and $\sum_{k=0}^{\infty}\alpha(k)h_y(k) < \infty$.

Proof of Lemma 6: We first demonstrate $\lim_{k\to\infty} h_x(k) = 0$ and $\sum_{k=0}^{\infty}\alpha(k)h_x(k) < \infty$. Suppose that $M_1 x = \bar{b}_1$ has many solutions. By mathematical induction, the properties of the projection and $M_{1,i}x_i(0) = \bar{b}_{1,i}$, then we have $M_{1,i}x_i(k) = \bar{b}_{1,i}$ for all k and $i \in \mathcal{V}_1$. Let $\varepsilon_{1,i} = x_i - x_0$ for any $x_0 \in \Omega_1$. Combining $P_{1,i}\varepsilon_{1,i} = \varepsilon_{1,i}$, (8a) and Assumption 1, we obtain

$$\begin{aligned}
&\varepsilon_{1,i}(k+1)\\
&= x_i(k) - x_0 + P_{1,i}\left(\hat{x}_i(k) - x_0 + x_0 - x_i(k)\right.\\
&\left.-\alpha(k)\partial_x f_i(\hat{x}_i(k), \widetilde{y}_i(k))\right)\\
&= P_{1,i}\sum_{j\in\mathcal{V}_1}[\mathcal{A}_{11}]_i^j P_{1,j}\varepsilon_{1,j}(k) - P_{1,i}\alpha(k)\partial_x f_i(k) \qquad (9)
\end{aligned}$$

where $\partial_x f_i(k) = f_i(\hat{x}_i(k), \widetilde{y}_i(k))$.

Let $Q^\top \in \mathbb{R}^{md\times s}$ be a matrix whose columns are an orthogonal basis of the orthogonal complement space of $\bigcap_{i=1}^m \mathcal{P}_{1,i}$, where s is the dimension of the orthogonal complement space and $\mathcal{P}_{1,i} = \{y | y = P_{1,i}x, \forall x \in \mathbb{R}^{md}\}$. Denote $\hat{P}_{1,i} = QP_{1,i}Q^\top$ and $\hat{\mathcal{P}}_{1,i} = \{y | y = \hat{P}_{1,i}x, \forall x \in \mathbb{R}^s\}$. Then, by lemma 1 in [2], we have that $\hat{P}_{1,i}$ is an orthogonal projection, $QP_{1,i} = \hat{P}_{1,i}Q$, $P_{1,i}Q^\top = Q^\top\hat{P}_{1,i}$ and $\bigcap_{i=1}^m \hat{\mathcal{P}}_{1,i} = \{0\}$. Multiplying both sides of (9) by Q from the left, it follows that:

$$\hat{\varepsilon}_{1,i}(k+1) = \hat{P}_{1,i}\sum_{j\in\mathcal{V}_1}[\mathcal{A}_{11}]_i^j\hat{P}_{1,j}\hat{\varepsilon}_{1,j}(k) - \alpha(k)\hat{P}_{1,i}\partial_x\hat{f}_i(k) \quad (10)$$

where $\hat{\varepsilon}_{1,i} = Q\varepsilon_{1,i}$ and $\partial_x\hat{f}_i(k) = Q\partial_x f_i(k)$. Notably, there is a positive constant $\hat{L}$ such that $\|\partial_x\hat{f}_i(k)\| < \hat{L}$.

Denote $\hat{\varepsilon}_1 = \mathrm{col}(\hat{\varepsilon}_{1,1}, \ldots, \hat{\varepsilon}_{1,m}) \in \mathbb{R}^{ms}$, $\hat{P}_1 = \mathrm{blockdiag}(\hat{P}_{1,1}, \ldots, \hat{P}_{1,m}) \in \mathbb{R}^{ms\times ms}$, and $\partial_x\hat{F} = \mathrm{col}(\partial_x\hat{f}_1, \cdots\partial_x\hat{f}_m) \in \mathbb{R}^{ms}$. Then, according to (10), $\hat{\varepsilon}_1(k+1) = \hat{\Lambda}\hat{\varepsilon}_1(k) - \alpha(k)\hat{P}_1\partial_x\hat{F}(k)$, where $\hat{\Lambda} = \hat{P}_1(\mathcal{A}_{11}\otimes I_s)\hat{P}_1$. According to [2, Th. 3], we have $|||\hat{\Lambda}|||^k \le \hat{\lambda}_1^{k-\hat{\omega}}$, where $0 < \hat{\lambda}_1 < 1$ and $\hat{\omega}$ is a constant. Moreover, by the norm-equivalence of the norm, there is a constant c such that $\|\cdot\| \le c|||\cdot|||$. Further, it follows Assumption 4 and compatibility of

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Frobenius norm that:

$$\|\hat{\varepsilon}_1(k+1)\| \le c\hat{\lambda}^{k+1-\hat{\omega}}\|\hat{\varepsilon}_1(0)\| + c\sqrt{m}\hat{L}\|\hat{P}_1\|\sum_{r=0}^{k}\hat{\lambda}^{k-r-\hat{\omega}}\alpha(r) \tag{11}$$

$$\begin{aligned}&\sum_{k=0}^{\infty}\alpha(k+1)\|\hat{\varepsilon}_1(k+1)\|\\ &\le c\sum_{k=0}^{\infty}(\alpha^2(k+1)+\hat{\lambda}_1^{2(k+1-\hat{\omega})})\|\hat{\varepsilon}_1(0)\|\\ &\quad+\frac{c\sqrt{m}\hat{L}\|\hat{P}_1\|\hat{\lambda}_1^{-\hat{\omega}}}{1-\hat{\lambda}_1}\sum_{k=0}^{\infty}\alpha^2(k+1)\\ &\quad+c\sqrt{m}\hat{L}\|\hat{P}_1\|\sum_{k=0}^{\infty}\sum_{r=0}^{k}\alpha^2(r)\hat{\lambda}_1^{k-r-\hat{\omega}}.\end{aligned} \tag{12}$$

Since $\lim_{k\to\infty}\alpha(k)=0$ and Lemma 4, we have $\lim_{k\to\infty}\|\hat{\varepsilon}_1(k)\|=0$ and $\sum_{k=0}^{\infty}\alpha(k)\|\hat{\varepsilon}_1(k)\|<\infty$.

It follows from $P_{1,i}Q^\top = Q^\top\hat{P}_{1,i}$ that $Q^\top\hat{\varepsilon}_{1,i}(k+1) = P_{1,i}\sum_{j\in\mathcal{V}_1}[\mathcal{A}_{11}]_i^j P_{1,j}Q^\top\hat{\varepsilon}_{1,j} - \alpha(k)P_{1,i}Q^\top\partial_x\hat{f}_i(\hat{x}_i(k),\widetilde{y}_i(k))$. Let $z_i = \varepsilon_i - Q^\top\hat{\varepsilon}_i$. It is easy to see $Qz_{1,i}=0$, which implies that $z_{1,i}\in\bigcap_{i=1}^m\mathcal{P}_{1,i}$. Then, it follows that $z_{1,i}(k+1)=\sum_{j\in\mathcal{V}_1}[\mathcal{A}_{11}]_i^j z_{1,j}(k) - \alpha(k)(I_{md}-Q^\top Q)\partial_x f_i(k)$.

Denote $z_1=\mathrm{col}(z_{1,1},\dots,z_{1,m})$ and $\Gamma=\mathrm{blockdiag}(I_{md}-Q^\top Q,\dots,I_{md}-Q^\top Q)\in\mathbb{R}^{m^2d\times m^2d}$. It can be readily derived that $z_1(k+1)=(\mathcal{A}_{11}\otimes I_{md})^{k+1}z_1(0)-\sum_{r=0}^{k}\alpha(r)(\mathcal{A}_{11}\otimes I_{md})^{k-r}\Gamma\partial_x F(k)$.

Then, according to the Lemma 2, it follows that $\forall i_1,i_2\in\mathcal{V}_1$:

$$\begin{aligned}&\|z_{1,i_1}(k+1)-z_{1,i_2}(k+1)\|\\ &\le\sum_{j\in\mathcal{V}_1}\|[\mathcal{A}_{11}^{k+1}]_{i_1}^j-\frac{1}{m}+\frac{1}{m}-[\mathcal{A}_{11}^{k+1}]_{i_2}^j\|\|z_{1,j}(0)\|\\ &\quad+L\|\Gamma\|\sum_{r=0}^{k}\alpha(r)\|[\mathcal{A}_{11}^{k-r}]_{i_1}^j-\frac{1}{m}+\frac{1}{m}-[\mathcal{A}_{11}^{k-r}]_{i_2}^j\|\\ &\le 2C_1\beta_1^{k+1}\sum_{j\in\mathcal{V}_1}\|z_{1,j}(0)\|+2LC_1\|\Gamma\|\sum_{r=0}^{k}\alpha(r)\beta_1^{k-r}.\end{aligned}$$

By Lemma 4, we have $\lim_{k\to\infty}\|z_{1,i_1}(k)-z_{1,i_2}(k)\|=0$ and $\sum_{k=0}^{\infty}\alpha(k+1)\|z_{1,i_1}(k+1)-z_{1,i_2}(k+1)\|\le C_1\sum_{k=0}^{\infty}(\alpha^2(k+1)+\beta_1^{2(k+1)})\sum_{j\in\mathcal{V}_1}\|z_{1,j}(0)\|+LC_1\|\Gamma\|\sum_{k=0}^{\infty}\sum_{r=0}^{k}\beta_1^{k-r}(\alpha(k+1)^2+\alpha^2(r))$. Similar to the derivation of (12), $\sum_{k=0}^{\infty}\alpha(k+1)\|z_{1,i_1}(k+1)-z_{1,i_2}(k+1)\|<\infty$. Then, combining (11) and (12), it can be derived that $\forall i_1,i_2\in\mathcal{V}_1$, $\|x_{i_1}(k)-x_{i_2}(k)\|\le\|z_{1,i_1}(k)-z_{1,i_2}(k)\|+\|Q\|\|\hat{\varepsilon}_{1,i_1}(k)-\hat{\varepsilon}_{1,i_2}(k)\|$, and $\sum_{k=0}^{\infty}\alpha(k)\|x_{i_1}(k)-x_{i_2}(k)\|<\infty$. Therefore, $\lim_{k\to\infty}h_x(k)=0$ and $\sum_{k=0}^{\infty}\alpha(k)h_x(k)<\infty$. When $M_1x=\bar{b}_1$ has a unique solution, the above results can also be obtained in the same way. Analogously, we have $\lim_{k\to\infty}h_y(k)=0$ and $\sum_{k=0}^{\infty}\alpha(k)h_y(k)<\infty$. In conclusion, Lemma 6 holds. ■

Proof of Theorem 1

The proof will be divided into three steps.

Step 1: We show that $x_i(k)$ and $y_i(k)$ are bounded. By using Lemma 3 and (8a), it follows that $\forall x_0\in\Omega_1$, $\forall y_0\in\Omega_2$:

$$\begin{aligned}&\|x_i(k+1)-x_0\|^2\\ &=\left\|P_{1,i}\left(\widehat{x}_i(k)-x_i(k)-\alpha(k)\partial_x f_i\left(\widehat{x}_i(k),\widetilde{y}_i(k)\right)\right)+x_i(k)-x_0\right\|^2\\ &\le\|\widehat{x}_i(k)-x_0-\alpha(k)\partial_x f_i\left(\widehat{x}_i(k),\widetilde{y}_i(k)\right)\|^2\\ &\le 2\alpha(k)<\widehat{x}_i(k)-x_0,-\partial_x f_i\left(\widehat{x}_i(k),\widetilde{y}_i(k)\right)>+\|\widehat{x}_i(k)-x_0\|^2\\ &\quad+\alpha^2(k)L^2.\end{aligned} \tag{13}$$

Obviously, $\|\cdot\|^2$ is a convex function. Denote $\bar{x}=\frac{1}{m}\sum_{i\in\mathcal{V}_1}x_i=\mathrm{col}(\bar{x}_{1,1},\dots,\bar{x}_{m,m})$ and $\bar{y}=\frac{1}{n-m}\sum_{j\in\mathcal{V}_2}y_j=\mathrm{col}(\bar{y}_{m+1,m+1},\dots,\bar{y}_{n,n})$, where $\bar{x}_{i,i}\in\mathbb{R}^d$ $(i\in\mathcal{V}_1)$ and $\bar{y}_{j,j}\in\mathbb{R}^d$ $(j\in\mathcal{V}_2)$. Denote $\bar{x}_j=\mathrm{col}(\bar{x}_{j_1,j_1},\dots,\bar{x}_{j_l,j_l})$ for $j\in\mathcal{V}_2$ and $\bar{y}_i=\mathrm{col}(\bar{y}_{i_1,i_1},\dots,\bar{y}_{i_l,i_l})$ for $i\in\mathcal{V}_1$. Furthermore, it is obtained that for the convexity of $f_i(x,\cdot)$, we have that $\forall w_1,w_2$, $f_i(w_1,\cdot)\ge f_i(w_2,\cdot)+\partial_x f_i(w_2,\cdot)^\top(w_1-w_2)$. Without loss of generality, assume $f_i(w_1,\cdot)\le f_i(w_2,\cdot)$. Then, we have $0>f_i(w_1,\cdot)-f_i(w_2,\cdot)\ge\partial_x f_i(w_2,\cdot)^\top(w_1-w_2)$. Therefore, by Assumption 4, $\|f_i(w_1,\cdot)-f_i(w_2,\cdot)\|\le L\|w_1-w_2\|$. Analogously, by the concavity of $f_i(\cdot,\widetilde{y}_i)$, we have that $\forall z_1,z_2$, $\|f_i(\cdot,z_1)-f_i(\cdot,z_2)\|\le L\|z_1-z_2\|$.

Therefore, according to properties of convex–concave function, inequality (13) and Assumption 4, we have

$$\begin{aligned}&\|f_i\left(\bar{x}(k),\bar{y}_i(k)\right)-f_i\left(\widehat{x}_i(k),\widetilde{y}_i(k)\right)\|\\ &\le\|f_i\left(\bar{x}(k),\bar{y}_i(k)\right)-f_i\left(\widehat{x}_i(k),\bar{y}_i(k)\right)\\ &\quad+f_i\left(\widehat{x}_i(k),\bar{y}_i(k)\right)-f_i\left(\widehat{x}_i(k),\widetilde{y}_i(k)\right)\|\\ &\le L\left(\|\widehat{x}(k)-\bar{x}(k)\|+\|\bar{y}_i(k)-\widehat{y}(k)\|\right)\\ &\le L\left(h_x(k)+\|\bar{y}_i(k)-\widehat{y}(k)\|\right)\end{aligned}$$

It follows that:

$$\begin{aligned}&\|x_i(k+1)-x_0\|^2\\ &\le\sum_{j\in\mathcal{V}_1}a_{ij}\|x_j(k)-x_0\|^2+\alpha^2(k)L^2\\ &\quad+2\alpha(k)\left(f_i\left(x_0,\widetilde{y}_i(k)\right)-f_i\left(\widehat{x}_i(k),\widetilde{y}_i(k)\right)\right)\\ &\le\sum_{j\in\mathcal{V}_1}a_{ij}\|x_j(k)-x_0\|^2+\alpha^2(k)L^2+2\alpha(k)\left(f_i\left(x_0,\widetilde{y}_i(k)\right)\right.\\ &\quad-f_i\left(x_0,\bar{y}_i(k)\right)+f_i\left(x_0,\bar{y}_i(k)\right)-f_i\left(\bar{x}(k),\bar{y}_i(k)\right)\\ &\quad\left.+f_i\left(\bar{x}(k),\bar{y}_i(k)\right)-f_i\left(\widehat{x}_i(k),\widetilde{y}_i(k)\right)\right)\\ &\le\sum_{j\in\mathcal{V}_1}a_{ij}\|x_j(k)-x_0\|^2+\alpha^2(k)L^2+2\alpha(k)Le_{1i}(k)\\ &\quad+2\alpha(k)\left(f_i\left(x_0,\bar{y}_i(k)\right)-f_i\left(\bar{x}(k)\right),\bar{y}_i(k)\right)\right)\end{aligned} \tag{14}$$

where $e_{1i}(k)=h_x(k)+2\|\widetilde{y}_i(k)-\bar{y}_i(k)\|$. Further, it follows that:

$$\begin{aligned}&\sum_{i\in\mathcal{V}_1}\|x_i(k+1)-x_0\|^2\\ &\le\sum_{i\in\mathcal{V}_1}\|x_i(k)-x_0\|^2+m\alpha^2(k)L^2+2\alpha(k)L\sum_{i\in\mathcal{V}_1}e_{1i}(k)\\ &\quad+2\alpha(k)\left(U(x_0,\bar{y}(k))-U(\bar{x}(k),\bar{y}(k))\right).\end{aligned} \tag{15}$$

Analogously

$$\begin{aligned}\sum_{i\in\mathcal{V}_2}\|y_i(k+1)-y_0\|^2\le&\sum_{i\in\mathcal{V}_2}\|y_i(k)-y_0\|^2+(n-m)\alpha^2(k)L^2\\ &+2\alpha(k)\left[U(\bar{x}(k),\bar{y}(k))-U\left(\bar{x}(k),y_0\right)\right]\\ &+2\alpha(k)L\sum_{i\in\mathcal{V}_2}e_{2i}(k)\end{aligned} \tag{16}$$

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where $e_{2i}(k) = h_y(k) + 2\|\widetilde{x}_i(k) - \overline{x}_i(k)\|$.

Let $(x_0, y_0) = (x^*, y^*)$, where (x^*, y^*) is a saddle point of function $U(x, y)$. Denote $\Xi(k) = \sum_{i\in\mathcal{V}_1}\|x_i(k) - x^*\|^2 + \sum_{i\in\mathcal{V}_2}\|y_i(k) - y^*\|^2$. Adding (15) and (16) together, it follows that:

$$\Xi(k+1) \le \Xi(k) + n\alpha^2(k)L^2 - 2\alpha(k)\Delta + 2\alpha(k)L\sum_{l=1}^{2}\sum_{i\in\mathcal{V}_l} e_{li}(k) \tag{17}$$

where $\Delta = U(\overline{x}(k), y^*) - U(x^*, \overline{y}(k)) \ge 0$. According to Lemma 5, Lemma 6, Assumption 3, and (17), it can be verified that $\lim_{k\to\infty}\Xi(k)$ exists, which implies that $\sum_{i\in\mathcal{V}_1}\|x_i(k) - x_0\|^2$ and $\sum_{i\in\mathcal{V}_2}\|y_i(k) - y_0\|^2$ are bounded. Hence, $x_i(k)$ and $y_i(k)$ are bounded for all i, j.

Step 2: We will prove $x_i(k)$ converges to a constant and satisfies (4). Let $\gamma(k) = \sum_{i\in\mathcal{V}_1}\|x_i(k) - x_0\|^2$ and $\zeta(k) = m\alpha^2(k)L^2 + 2\alpha(k)L\sum_{i\in\mathcal{V}_1}\|x_0 - \widehat{x}_i(k)\|$. By (14), we have $\|x_i(k+1) - x_0\|^2 \le \sum_{j\in\mathcal{V}_1} a_{ij}\|x_j(k) - x_0\|^2 + 2\alpha(k)L\|x_0 - \widehat{x}_i(k)\| + \alpha^2(k)L^2$. Then, it is easy to get that

$$\gamma(k+1) \le \gamma(k) + \zeta(k). \tag{18}$$

According to the Lemma 6, it is true that $\gamma(k)$ and $\|x_i(k) - x_0\|^2$ exhibit the same converge or diverge properties. By the properties of the projection, the mathematical induction and $M_{1i_1}x_{i_1}(0) = \bar{b}_{1i_1}$, it follows that $M_{1i_1}x_{i_1}(k) = \bar{b}_{1i_1} \quad \forall i_1 \in \mathcal{V}_1 \quad \forall k$. Since $\lim_{k\to\infty} h_x(k) = 0$, it follows that $\lim_{k\to\infty} M_1 x_i(k) = \bar{b}_1, \quad \forall i \in \mathcal{V}_1$. This implies that all limit points of $x_i(k)$ must belong to Ω_1 (satisfies (4)). Furthermore, since x_0 can be arbitrarily chosen as any limit point of $x_i(k)$, it follows that $\|x_i(k) - x_0\|^2$ and $x_i(k)$ converge or diverge simultaneously. Then, the convergence of $x_i(k)$ is both necessary and sufficient for the convergence of $\gamma(k)$.

Next, we transform the problem into the convergence analysis of $\gamma(k)$. Since $x_i(k)$ is bounded for all i, $\gamma(k)$ has superior and inferior limits. Denote the superior and inferior limits of $\gamma(k)$ as a_1 and a_2, respectively. If $a_1 = a_2$, then the limit of $\gamma(k)$ exists. Assume that $a_1 \ne a_2$. Then, by Assumption 3 and boundedness of $x_i(k)$, we have $\lim_{k\to\infty}\zeta(k) = 0$. Then $\forall\varepsilon > 0$, there is an integer $N_1 > 0$ such that

$$-\varepsilon < \zeta(k) < \varepsilon \quad \forall k > N_1. \tag{19}$$

Then, by (18) and (19), it follows that:

$$\gamma(k+1) \le \gamma(k) + \varepsilon \quad \forall k > N_1. \tag{20}$$

Let $\gamma_{k_{n_1}}$ and $\gamma_{k_{n_2}}$ be subsequences of $\gamma(k)$ such that $\lim_{k_{n_1}\to\infty}\gamma_{k_{n_1}} = a_1$ and $\lim_{k_{n_2}\to\infty}\gamma_{k_{n_2}} = a_2$. Let c be a positive integer greater than 1. Based on the arbitrariness of ε, we take $\varepsilon = \frac{a_1 - a_2}{c+3}$. Obviously, $(a_1 - \varepsilon) - (a_2 + \varepsilon) = \frac{(c+2)a_1 + a_2}{c+3} - \frac{a_1 + (c+2)a_2}{c+3} = \frac{(c+1)(a_1 - a_2)}{c+3} > 0$. According to the definition of the limit, there are integers $N_2, N_3 > 0$, such that $-\varepsilon < \gamma_{k_{n_1}} - a_1 < \varepsilon$, and $-\varepsilon < \gamma_{k_{n_2}} - a_2 < \varepsilon$, when $k_{n_1} > N_2$ and $k_{n_2} > N_3$. Then, $\forall k_{n_1} > N_2, k_{n_2} > N_3$

$$\gamma_{k_{n_1}} > a_1 - \varepsilon > a_2 + \varepsilon > \gamma_{k_{n_2}}. \tag{21}$$

Moreover, let $N = \max\{N_1, N_2, N_3\}$ and $0 < k_{n_1} - k_{n_2} < c$. According to (20), when $k_{n_1} > N$ and $k_{n_2} > N$ we have

$$\gamma_{k_{n_1}} \le \gamma_{k_{n_2}} + (k_{n_1} - k_{n_2} + 1)\varepsilon < \gamma_{k_{n_2}} + (c+1)\varepsilon. \tag{22}$$

By (21) and (22), we have $(c+1)\varepsilon = a_1 - a_2 - 2\varepsilon < \gamma_{k_{n_1}} - \gamma_{k_{n_2}} < (c+1)\varepsilon$. It is easy to see that the inequality above creates a contradiction. Hence, the previous assumption (i.e., $a_1 \ne a_2$) is invalid, which means that $x_i(k)$ is convergent. Analogously, $y_i(k)$ is also convergent and satisfies (5).

Step 3: We show that the limits of $x_i(k)$ and $y_i(k)$ is a saddle point of $U(x, y)$. Let $\lim_{k\to\infty} x_i(k) = \theta$ and $\lim_{k\to\infty} y_i(k) = \iota$. Suppose (θ, ι) is not a saddle point. There is $(x_0', y_0') \in \Omega_1 \times \Omega_2$ such that at least one of the following inequalities holds:

$$U(\theta, \iota) > U(x_0', \iota) \tag{23}$$

$$U(\theta, y_0') > U(\theta, \iota). \tag{24}$$

Without loss of generality, assume that (23) holds. Then, there is $\sigma > 0$ such that $U(\theta, \iota) = U(x_0', \iota) + \sigma$.

Since the continuity of $U(x, y)$, $\lim_{k\to\infty} h_x(k) = 0$ and $\lim_{k\to\infty} h_y(k) = 0$, there are ε_0 ($\varepsilon_0 < \sigma$) and a constant k_0 such that, $\forall k > k_0$

$$U(x_0', \overline{y}(k)) < \frac{\varepsilon_0}{2} + U(x_0', \iota) \tag{25}$$

$$U(\overline{x}(k), \overline{y}(k)) > -\frac{\varepsilon_0}{2} + U(\theta, \iota). \tag{26}$$

Let $(x_0, y_0) = (x_0', y_0')$. Substituting (25) and (26) into (15), then $\sum_{i\in\mathcal{V}_1}\|x_i(k+1) - x_0'\|^2 \le \sum_{i\in\mathcal{V}_1}\|x_i(k_0+1) - x_0'\|^2 + mL^2\sum_{\tau=k_0+1}^{k}\alpha^2(\tau) + 2\sum_{\tau=k_0+1}^{k}\alpha(\tau)(\varepsilon_0 - \sigma) + 2L\sum_{\tau=k_0+1}^{k}\alpha(\tau)\sum_{i\in\mathcal{V}_1} e_{1i}(\tau)$.

Since $\sum_{k=1}^{\infty}\alpha(k) = \infty$, $\varepsilon_0 - \sigma < 0$, $\sum_{k=0}^{\infty}\alpha(k)h_x(k) < \infty$ and $x_i(k)$ are bounded for all k, the above inequality yields a contradiction when k is taken to be sufficiently large. This contradiction indicates that (23) cannot hold. Similarly, combining with (16), (24) also cannot hold. This means that (θ, ι) is a saddle point of $U(x, y)$.

In summary, it can be seen that Theorem 1 holds. ■

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