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Max-consensus of multi-agent systems in random networks ☆

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ABSTRACT

This paper considers max-consensus of a discrete-time multi-agent system (MAS) in directed random networks. Interactions among agents in the MAS are probabilistic and independent with each other. By using max-plus algebra and random theory, a sufficient and necessary condition is given for achieving max-consensus of the MAS. Moreover, we demonstrate that the max-consensus in four probabilistic senses (almost surely, in probability, expectation and mean square) is equivalent when expected graph is strongly connected. This ensures that max-consensus can be achieved in multi-agent systems even if random failures occur in the communication network, which is of practical importance in the fields of wireless sensor networks and distributed computing. A simulation example is presented to illustrate the effectiveness of theoretical results.

1. Introduction

In recent decades, the coordination control of multi-agent systems (MASs) has gained attention in the control field, and is widely used in natural sciences, social sciences and other fields, including consensus [1–5], flocking control [6], containment control [7], formation control [8], iterative learning control [9], and etcetera. Consensus problem, as a fundamental issue in distributed control, which refers to agents updating their own states in local coordination and information communication, eventually achieving consensus. A common consensus problem is the average consensus (e.g., [10–14]). However, in some applications, for example, leader election, minimum-time rendezvous, a consensus algorithm called max-consensus is employed. The algorithm requires that all agents achieve the maximum state of the initial states, i.e., achieve max-consensus. Nejad et al. [15,16] studied the max-consensus under fixed and switching networks by max-plus algebra, and gave the sufficient and necessary conditions for solving strong and weak max-consensus, respectively. Giannini et al. [17] designed an asynchronous max-consensus protocol and showed a sufficient condition for discrete-time MASs to solve max-consensus in fixed networks. Zhang et al. [18] solved the max-consensus for MASs by using soft-max algorithm. It is worth noting that all the above studies are analyzed under fixed or switching networks. Besides, some researchers have also studied the dynamic max-consensus problem. Lippi et al. [19,20] designed an adaptive protocol and a distributed dynamic consensus protocol that solves the dynamic max-consensus of MASs under undirected networks. In [21,22], Deplano et al. designed dynamic max-consensus protocol, exact dynamic max-consensus (EDMC) protocol and self-tuning dynamic max-consensus (STDMC) protocol to solve the dynamic max-consensus problem of discrete-time MASs and applied the STDMC protocol to open MASs. Abdelrahim et al. [23] studied distributed maximal computation in open MASs. They addressed the challenge

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of agents leaving and arriving during the execution of the algorithm. The goal was to provide algorithms that can adapt to these changes while still computing the maximum value.

However, in practical applications, the communication network of systems may change over time due to agent failures, information channel failures, or packet losses. These phenomena are random events. As a result, it is crucial to investigate the consensus of MASs in random networks. In 2005, Hatano et al. [24] studied consensus for MASs in random networks. Later, Porfiri et al. [25] considered consensus of MASs in directed weighted random networks, and gave a sufficient and necessary criterion for systems to reach consensus almost surely. Tahbaz-Salehi et al. [26,27] considered consensus for discrete-time MASs with linear dynamics in random networks, and gave the sufficient and necessary criteria for systems to achieve consensus almost surely. Wu et al. [28] provided sufficient conditions for sampled-data MASs to achieve mean-square consensus under two different random networks in the case that the sampling time is small enough. Lin et al. [29] provided the sufficient and necessary conditions for switched MASs to achieve consensus almost surely and in mean-square, respectively. Sun et al. [30] provided criteria for second-order discrete-time MASs to achieve consensus almost surely. Wang et al. [31] provided a sufficient and necessary criterion for discrete-time high-order linear swarm systems to reach consensus almost surely.

Max-consensus is also investigated in the random case. Iutzeler et al. [32] proposed two consensus protocols random-broadcast-max and random-pairwise-max for MASs in random cases. In these two protocols, only one agent is randomly woken up and interacts with its neighbors at each time. Different from [32], this paper considers all agents interacting with their neighbors at every moment, thus these two protocols cannot be used for the MASs in this paper. Muniraju et al. [33] studied max-consensus of MASs with additive noise. Using sub-additive ergodic theory, the upper and lower growth rates were provided for the MASs achieving consensus under fixed and random networks, respectively. In [34], Golfar studied max-consensus in the case of communication networks satisfying Bernoulli packet loss, and gave sufficient conditions for MASs to converge with probability 1. However, in [33,34], detailed theoretical proofs are not given for the max-consensus of MASs in random networks.

Inspired by the above observation results, we intend to investigate max-consensus of discrete-time MASs in directed random networks by using a max-plus algebra. Then, we prove the consensus criterion for solving the max-consensus problems in the probabilistic sense. The following are the main contributions made in this paper:

(1) Based on matrix theory, stochastic analysis tools, max-plus algebra and random graph theory, the sufficient and necessary conditions are obtained for MASs to achieve max-consensus almost surely in random networks.
(2) The equivalent relations for the max-consensus of discrete-time MASs in different senses (almost surely, in probability, expectation, and mean square, respectively) are established when the expected graph is strongly connected.

Difficulties come down to the convergence analysis and modeling MASs in random networks. Besides, using max-plus algebra and probability theory to count all the cases in which any two agents have directed paths is also difficult.

The structure of this paper is given as follow. Some necessary notions and terminologies from max-plus algebra theory and directed random graph theory are provided in Section 2. The problem formulation is provided in Section 3. The criterion of max-consensus of MASs in a random network is obtained in Section 4. In Section 5, a simulation demonstrate our theoretical results. And Section 6 gives the conclusions.

Notations: $\mathbb{N}$ and $\mathbb{R}$ are the sets of non-negative integers and real numbers, respectively. The real vector space with n dimensions is represented by the symbol $\mathbb{R}^n$. $\mathbb{R}^{m\times n}_{max}$ is $m \times n$ matrix space with elements including real numbers and infinity. $\|\cdot\|_\infty$ denotes the infinite norm of a vector or matrix. The term x^T is the transpose of a given vector x. $|\cdot|$ denotes the absolute value of a real number. The probability of event "$x > a$" is represented by $P(x > a)$ and the mathematical expectation that comes with the random variable x is expressed by $E(x)$. $\bigcup_{k=1}^{n} A_k$ is the union of $A_1, \dots, A_n$. $\bigotimes_{l=0}^{k} A(l)$ indicates that $A(k) \otimes A(k-1) \otimes \cdots \otimes A(0)$. $\bigoplus_{r=1}^{n}(a_{ir} \otimes b_{rj})$ is expressed as $(a_{i1} \otimes b_{1j}) \oplus (a_{i2} \otimes b_{2j}) \oplus \cdots \oplus (a_{in} \otimes b_{nj})$. $X_n \to X$ indicates that $\lim_{n\to\infty} X_n = X$. I is an $n \times n$ dimensional matrix in which each element is an element e. $\times$ represents the multiplication in conventional algebra. $\binom{k+1}{r}$ represents a combination, i.e., $\binom{k+1}{r} = \frac{(k+1)\times(k)\times\cdots\times(k+2-r)}{(r)\times(r-1)\times\cdots\times 1}$.

2. Preliminaries

2.1. Random graph

A vertex set $V = \{1, 2, \dots, n\}$ and an edge set $\mathcal{E} \subseteq V \times V$ make up a directed graph $G = (V, \mathcal{E})$. In graph G, let (j, i) be a directed edge. $(j, i) \in \mathcal{E}$ demonstrates that agent j can obtain the state of agent i. If the existence of (j, i) is randomly determined by probability $p_{ij} \in [0, 1]$, the graph G is referred to as a random directed graph. In the random directed graph, interactions among agents can be characterized as a sequence of directed graphs $G(k) = (V, \mathcal{E}(k)), k \in \mathbb{N}$, denoted as $\{G(k)\}$. $N_{i,k} = \{(j, i) \mid (j, i) \in \mathcal{E}(k), j \in V\}$ represents the set of neighbors of vertex i at the kth time. We assume that edges exist with probabilities independent of other edges, then the random graphs are also independent of each other. Meanwhile, we assume that each agent in the random graph has a fixed self-loop, i.e., $p_{ii} = 1, i \in V$. We represent $\overline{G} = (V, \overline{\mathcal{E}})$ as the expected graph of random graph sequence $\{G(k)\}$. If for any $i, j \in V$, (j, i) exists with probability $p_{ij} > 0$, then $(j, i) \in \overline{\mathcal{E}}$, otherwise, $(j, i) \notin \overline{\mathcal{E}}$. A directed path between two different vertices i and j of the expected graph $\overline{G}$ is a finite-order sequence with different edges $(i, i_1), (i_1, i_2), \dots, (i_s, j)$. The expected graph $\overline{G}$ is claimed to be strongly connected if for any $i, j \in V, i \neq j$, there is a directed path from vertex i to vertex j.

2.2. Max-plus algebra

Next, the fundamental knowledge of max-plus algebra are introduced in detail in this subsection (refer to [35,36]). The max-plus algebra theory is a mathematical framework that extends the conventional algebraic operations of addition and multiplication to the operations of maximization and addition, respectively. It has been applied in various fields, including optimization, scheduling, and control systems. In consensus problems for MASs, max-plus algebra is used for the modeling and analysis of discrete-event or discrete-time systems. Besides, the weighted graph can be expressed as a compact expression.

Let $e = 0, \varepsilon = -\infty$ and $\mathbb{R}_{max} = \mathbb{R} \cup \{\varepsilon\}$. The definitions of the two operations $\oplus$ and $\otimes$ on $\mathbb{R}_{max}$ are the following. For elements $a, b \in \mathbb{R}_{max}$,

$$a \oplus b = max(a, b),$$
$$a \otimes b = a + b.$$

Then $(\mathbb{R}_{max}, \oplus, \otimes, \varepsilon, e)$ is known as the max-plus algebra. The zero and one element of max-plus algebra are described by the elements ε and e, respectively. And two operations $\oplus$ and $\otimes$ satisfy the following operation properties. $\forall a, b, c \in \mathbb{R}_{max}$,

(1) Commutativity: $a \oplus b = b \oplus a$ and $a \otimes b = b \otimes a$.
(2) Associativity: $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ and $a \otimes (b \otimes c) = (a \otimes b) \otimes c$.
(3) Distributivity: $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$.
(4) Existence of a zero element: $a \oplus \varepsilon = \varepsilon \oplus a = a$.
(5) Existence of a unit element: $a \otimes e = e \otimes a = a$.
(6) The zero is absorbing for $\otimes$: $a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$.

In addition, it is possible to apply the two operations of the max-plus algebra to matrices. For matrices $A = [a_{ij}]_{m\times n}, B = [b_{ij}]_{m\times n} \in \mathbb{R}^{m\times n}_{max}$,

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}, i = 1, \dots, m, j = 1, \dots, n.$$

For $A = [a_{ij}]_{m\times n} \in \mathbb{R}^{m\times n}_{max}, B = [b_{ij}]_{n\times q} \in \mathbb{R}^{n\times q}_{max}$,

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} (a_{ik} \otimes b_{kj}) = \max_{k=1,\dots,n} (a_{ik} + b_{kj}), i = 1, \dots, m, j = 1, \dots, q.$$

In this paper, matrix $A = [a_{ij}]_{n\times n} \in \mathbb{R}^{n\times n}_{max}$ signifies the adjacency matrix of the graph G_A. For any $i, j \in V$, if $(j, i) \in \mathcal{E}$, $a_{ij} = e$, otherwise $a_{ij} = \varepsilon$. Specifically, $a_{ij} = e$ indicates that agent i, j is connected and $a_{ij} = \varepsilon$ indicates disconnection. Therefore, the adjacency matrix $A(k) = [a_{ij}(k)] \in \mathbb{R}^{n\times n}_{max}$ corresponding to the random graph $G(k)$ can be referred to as

$$a_{ij}(k) = \begin{cases} e, & with \quad probability \quad p_{ij}, \\ \varepsilon, & with \quad probability \quad 1 - p_{ij}, \end{cases} \tag{1}$$

where $i \neq j$. We consider the case that the self-loops always exist, i.e., $P(a_{ii}(k) = e) = 1,\ i \in V,\ k \in \mathbb{N}$.

The adjacency matrix $\overline{A} = E[A]$ corresponding to the expected graph $\overline{G}$ can be determined in the following way:

$$(E[A])_{ij} = \begin{cases} e, & for \quad p_{ij} > 0, \\ \varepsilon, & for \quad p_{ij} = 0. \end{cases} \tag{2}$$

If the expected graph $\overline{G}$ is strongly connected, then $\forall i, j \in V$, there must be a directed path $(i, i_1), (i_1, i_2), \dots, (i_s, j)$ such that $p_{i_1 i} > 0, p_{i_2 i_1} > 0, \dots, p_{j i_s} > 0$.

In this paper, we will prove our results using regular matrices and positive series, which are defined as follows.

Definition 1 (*[35]*). A matrix $A \in \mathbb{R}^{n\times n}_{max}$ is said to be a regular matrix if A has at least one finite element (not ε) in per row.

Definition 2 (*[37]*). For a series, if all of its terms are non-negative, it is called a positive series.

2.3. Some lemmas

This subsection introduces some lemmas needed in the paper.

Lemma 1 (*Borel–Cantelli Lemma [38]*). *Let $X_1, X_2, \cdots$ be an event sequence in a space of probability. If $\sum_{k=1}^{\infty} P(X_k) < \infty$, then $P(\limsup_{k\to\infty} X_k) = 0$.*

Lemma 2 (*Dominated Convergence Theorem [38]*). *Suppose that in probability $X_n \to X$ and $|X_n| \le Y$, $EY < \infty$. Then $EX_n \to EX$.*

Lemma 3 (*[35]*). *Let $A \in \mathbb{R}^{m\times n}_{max}$ be a regular matrix, then for any vector $u, v \in \mathbb{R}^n$,*

$$\|A \otimes u - A \otimes v\|_\infty \le \|u - v\|_\infty . \tag{3}$$

3. Problem statement

A discrete-time MAS with n agents in a directed random network is considered, which purpose is to achieve max-consensus almost surely by using the max-plus algebra theory. The algorithm known as max-consensus is represented as:

$$x_i(k+1) = \max_{j \in N_{i,k}} \left\{ x_j(k) \right\}, i = 1, \dots, n, \tag{4}$$

where $x_i(k+1) \in \mathbb{R}$, $k \in \mathbb{N}$. It demonstrates that the state of agent i at time $k+1$ is determines by the maximum state of its neighbors at time k. Obviously, in conventional algebra, MAS (4) is nonlinear. In max-plus algebra, it is linear equation denoted by the following:

$$x_i(k+1) = \bigoplus_{j \in N_{i,k}} (x_j(k)), i = 1, \dots, n. \tag{5}$$

The compact form of (5) is

$$x(k+1) = A(k) \otimes x(k), \tag{6}$$

where $x(k) = (x_1(k), x_2(k), \dots, x_n(k))^T$, $x_i(k) \in \mathbb{R}$, and matrix $A(k) \in \mathbb{R}^{n\times n}_{max}$ is the adjacency matrix corresponding to the random graph $G(k)$. Due to MAS (6), the expression that follows can be obtained

$$x(k+1) = \bigotimes_{l=0}^{k} A(l) \otimes x(0). \tag{7}$$

Denote $x_{max} = (x_{max_1}, x_{max_2}, \dots, x_{max_n})^T$, where $x_{max_i} = \max_{j \in V} \left\{ x_j(0) \right\}, i \in V$. Then $x_{max} = I \otimes x(0)$.

Remark 1. To address the problem of max-consensus in MASs, researchers have proposed many max-consensus algorithms [15, 19,21,22]. Max-consensus algorithms are a class of approaches to achieve the maximum value of the initial state for all agents through interaction and coordination among agents. These algorithms can base on different protocols, communication networks, and decision rules. In this paper, a common protocol is used. It is based on iteration and information exchange, where an agent updates its state in each iteration and exchanges information with other agents to promote the system toward the maximum state.

The following definition introduces max-consensus in different sense of probability, respectively.

Definition 3. Discrete-time MAS (6) converges to max-consensus

(1) in expectation if for any initial state value $x(0) \in \mathbb{R}^n$, it holds that

$$\lim_{k\to\infty} E[\left\| x(k) - x_{max} \right\|_\infty] = 0; \tag{8}$$

(2) in mean square sense if for any initial state value $x(0) \in \mathbb{R}^n$, it holds that

$$\lim_{k\to\infty} E[\left\| x(k) - x_{max} \right\|_\infty^2] = 0; \tag{9}$$

(3) almost surely sense if any initial state value $x(0) \in \mathbb{R}^n$, it holds that

$$P(\lim_{k\to\infty} \left\| x(k) - x_{max} \right\|_\infty = 0) = 1; \tag{10}$$

(4) in probability if $\forall \epsilon > 0$ and for any initial state value $x(0) \in \mathbb{R}^n$, it holds that

$$\lim_{k\to\infty} P(\left\| x(k) - x_{max} \right\|_\infty > \epsilon) = 0. \tag{11}$$

Remark 2. Max-consensus of MASs has been widely concerned since it is practicable. A classic example is leader election. Consider each agent as an elector, and each elector's comprehensive ability as its own state. By interacting with their neighbors, the states of each elector are compared and the elector with the maximal state is elected as the leader.

4. Main results

We present the main results on max-consensus of discrete-time MAS (6) in a directed random network at this section. Without causing confusion, in the following we will refer to $\oplus$ as an additive operation and $\otimes$ as a multiplicative operation, respectively. For proofing all agents in MAS (6) almost surely converge to the maximal value of the initial states of all agents, we first prove the following lemma.

Lemma 4 is aim to build the conditions that each agent can obtain information from the other agents in random communication networks. Inspired by [39], we set that the probabilities of edges in the random network are different and independent of each other. We consider from a statistical perspective, statistics all possible cases where there exists a directed path of length $k+1$ between agents i and j, $\forall i, j \in V$.

Lemma 4. *Consider a sequence of graphs $G(k) = (V, \mathcal{E}(k))$ for $k \in \mathbb{N}$ and let $A(k)$ be the adjacency matrix associated to each graph $G(k)$. Then for any $i, j \in V$, $(\bigotimes_{l=0}^{k} A(l))_{ij}$ almost surely converges to e if and only if the expected graph $\overline{G}$ is strongly connected.*

The following Theorem provides a description of the criterion for MAS (6) achieving max-consensus.

Theorem 1. *MAS (6) can achieve max-consensus almost surely if and only if the expected graph $\overline{G}$ is strongly connected.*

Proof (*Sufficiency*). Let $A_k = \bigotimes_{l=0}^{k} A(l)$. It is known that $\|x(k+1) - x_{max}\|_\infty = \|A_k \otimes x(0) - I \otimes x(0)\|_\infty$, and $A_k \otimes x(0), I \otimes x(0) \in \mathbb{R}^n$ are both finite. There exist $i_0 \in V$, such that

$$\|A_k \otimes x(0) - I \otimes x(0)\|_\infty = \left|(A_k \otimes x(0) - I \otimes x(0))_{i_0}\right|.$$

Since $A_k \otimes x(0) - I \otimes x(0) \le 0$, then

$$\begin{aligned}\|x(k+1) - x_{max}\|_\infty =&(I \otimes x(0) - A_k \otimes x(0))_{i_0}\\ =&\max_{j\in V}(I_{i_0 j} + x_j(0)) - \max_{m\in V}(A_{k_{i_0 m}} + x_m(0))\\ =&(I_{i_0 j_0} + x_{j_0}(0)) - \max_{m\in V}(A_{k_{i_0 m}} + x_m(0))\\ =&(e + x_{j_0}(0)) - \max_{m\in V}(A_{k_{i_0 m}} + x_m(0))\\ =&x_{j_0}(0) - \max_{m\in V}(A_{k_{i_0 m}} + x_m(0)),\end{aligned}$$

where $x_{j_0}(0) = \max_{j\in V}\{x_j(0)\}$, and $I_{ij} = \varepsilon$, for $\forall i, j \in V, i \neq j$.

By Lemma 4, $P(\lim_{k\to\infty} A_{k_{ij}} = e) = 1$ holds for all $i, j \in V, i \neq j$, since that expected graph $\overline{G}$ is strongly connected. Then $\exists i_0 \in V$, $P(\lim_{k\to\infty} A_{k_{i_0 m}} = e) = 1$ holds $\forall m \in V, m \neq i_0$. Hence $\exists i_0 \in V$, one has

$$\begin{aligned}&P(\lim_{k\to\infty}\max_{m\in V}(A_{k_{i_0 m}} + x_m(0)) = \max_{m\in V}(e + x_m(0))) = 1\\ \Rightarrow& P[\lim_{k\to\infty}(x_{j_0}(0) - \max_{m\in V}(A_{k_{i_0 m}} + x_m(0))) = x_{j_0}(0) - \max_{m\in V}(x_m(0))] = 1\\ \Rightarrow& P[\lim_{k\to\infty}(x_{j_0}(0) - \max_{m\in V}(A_{k_{i_0 m}} + x_m(0))) = x_{j_0}(0) - x_{j_0}(0)] = 1\\ \Rightarrow& P[\lim_{k\to\infty}(x_{j_0}(0) - \max_{m\in V}(A_{k_{i_0 m}} + x_m(0))) = 0] = 1,\end{aligned}$$

for all $m \in V$, $m \neq i_0$, i.e., $P(\lim_{k\to\infty}\|x(k+1) - x_{max}\|_\infty = 0) = 1$. It implies that MAS (6) can achieve max-consensus almost surely.

(Necessity). If the expected graph is not strongly connected, then there exists $i_0, j_0 \in V, i_0 \neq j_0$, such that $P((\bigotimes_{l=0}^{k} A(l))_{i_0 j_0} = \varepsilon) = 1$. This implies that there is no path between agent j_0 and i_0. Hence, if $x_{i_0}(0) = \max_{j\in V}\{x_j(0)\}$, then $x_{j_0}(k) < x_{i_0}(0)$ for $\forall k \in \mathbb{N}$. i.e., discrete-time MAS (6) cannot reach max-consensus for any initial condition. The necessity of proof has been completed. □

Remark 3. For reaching consensus of MASs in directed graphs, it is commonly required that the directed graph contains a spanning tree. A spanning tree provides a connected sub-graph that connects all agents and does not form loops, thus ensuring that information can flow throughout the network to achieve consensus. However, in our paper, if we only require the directed graph to contain a spanning tree, a part of the agents will not be able to achieve max-consensus when the agent with the maximal initial state is not the root node. Thus, all agents can achieve max-consensus only if the directed graph contains a directed spanning tree and the agent with the maximal initial state is the root node. However, this condition is too artificial. We further require a strongly connected directed communication graph, which implies the existence of a directed path between any two nodes in the network.

The following theorem indicates that the four cases in Definition 3 are equivalent when the expected graph is strongly connected.

Theorem 2. *The four statements that follow are equivalent if expected graph $\overline{G}$ is strongly connected.*

(a) Discrete-time MAS (6) reaches max-consensus almost surely;
(b) Discrete-time MAS (6) reaches max-consensus in probability;
(c) Discrete-time MAS (6) reaches max-consensus in expectation;
(d) Discrete-time MAS (6) reaches max-consensus in mean square.

Proof. $(a) \Rightarrow (b)$, Eq. (10) can be equated to $\forall\epsilon > 0$, there is a positive integer $N \in \mathbb{N}$, such that for all $k \ge N$, there is $\|x(k) - x_{max}\|_\infty < \epsilon$.

i.e, $\forall\epsilon > 0$,

$$\lim_{k_0\to\infty} P(\bigcup_{k\ge k_0}\|x(k) - x_{max}\|_\infty > \epsilon) = 0.$$

Let $\{\|x(k) - x_{max}\|_\infty > \epsilon\}$ be denoted as an event, then it is obviously established that

$$\{\|x(k) - x_{max}\|_\infty > \epsilon\} \subseteq \bigcup_{k\ge k_0}\{\|x(k) - x_{max}\|_\infty > \epsilon\},$$

we can get

$$\lim_{k\to\infty} P(\|x(k)-x_{max}\|_\infty > \epsilon) \le \lim_{k_0\to\infty} P(\bigcup_{k>k_0} \|x(k)-x_{max}\|_\infty > \epsilon) = 0.$$

Due to $\lim_{k\to\infty} P(\|x(k)-x_{max}\|_\infty > \epsilon) \ge 0$, then it is shown that $\forall\epsilon > 0$,

$$\lim_{k\to\infty} P(\|x(k)-x_{max}\|_\infty > \epsilon) = 0,$$

i.e., $(a) \Rightarrow (b)$ holds.

$(b) \Rightarrow (c)$, for all $k \in \mathbb{N}$, $P(a_{ii}(k) = e) = 1$, then $A(k)$ is a regular matrix. By Lemma 3, we derive

$$\begin{aligned} \|x(k+1)-x_{max}\|_\infty &= \|A(k)\otimes x(k) - A(k)\otimes x_{max}\|_\infty \\ &\le \|x(k)-x_{max}\|_\infty . \end{aligned} \tag{12}$$

According to Eq. (12), we get that the sequence $\{\|x(k)-x_{max}\|_\infty\}$ is bounded and monotonic non-increasing, then $\|x(k)-x_{max}\|_\infty \le \|x(0)-x_{max}\|_\infty$. Since $E\left[\|x(0)-x_{max}\|_\infty\right] = \|x(0)-x_{max}\|_\infty < \infty$, from Lemma 2 (Dominated Convergence Theorem), we can obtain that discrete-time MAS (6) achieves max-consensus in expectation.

$(c) \Rightarrow (d)$, by Eq. (12), $\|x(k)-x_{max}\|_\infty$ is non-increasing, then one has

$$\begin{aligned} E\left[\|x(k)-x_{max}\|_\infty^2\right] &\le E\left[\|x(0)-x_{max}\|_\infty \|x(k)-x_{max}\|_\infty\right] \\ &= \|x(0)-x_{max}\|_\infty E\left[\|x(k)-x_{max}\|_\infty\right]. \end{aligned}$$

Since $\lim_{k\to\infty} E[\|x(k)-x_{max}\|_\infty] = 0$, one has

$$\lim_{k\to\infty} E[\|x(k)-x_{max}\|_\infty^2] \le \lim_{k\to\infty} \|x(0)-x_{max}\|_\infty E\left[\|x(k)-x_{max}\|_\infty\right] = 0.$$

Due to $\lim_{k\to\infty} E[\|x(k)-x_{max}\|_\infty^2] \ge 0$, i.e., $(c) \Rightarrow (d)$ holds.

$(d) \Rightarrow (a)$, by Chebyshev's inequality [40], $\forall\epsilon > 0$,

$$P(\|x(k)-x_{max}\|_\infty \ge \epsilon) \le \frac{E[\|x(k)-x_{max}\|_\infty^2]}{\epsilon^2}. \tag{13}$$

It is known that Eq. (9) holds, hence

$$\lim_{k\to\infty} P(\|x(k)-x_{max}\|_\infty \ge \epsilon) = 0.$$

By Eq. (12), the sequence $\{\|x(k)-x_{max}\|_\infty\}$ is bounded and monotonic non-increasing, then

$$\bigcup_{k=k_0}^{\infty} \{\|x(k)-x_{max}\|_\infty > \epsilon\} = \{\|x(k)-x_{max}\|_\infty > \epsilon\}. \tag{14}$$

Thus

$$\lim_{k_0\to\infty} P(\bigcup_{k>k_0} \|x(k)-x_{max}\|_\infty > \epsilon) = \lim_{k\to\infty} P(\|x(k)-x_{max}\|_\infty > \epsilon) = 0, \tag{15}$$

i.e., $(d) \Rightarrow (a)$ holds. □

According to the results in Theorem 2, we can also obtain the following conclusions.

Remark 4. A sufficient and necessary criterion for MAS (6) to reach max-consensus in mean square, expectation and probability is that the expected graph $\overline{G}$ satisfies the condition of strong connected.

5. Simulation

We consider a directed network with $n = 15$ agents, where the probability of the presence of an edge is $P(a_{ij} = e) = 0.1$, $i, j = 1, \dots, 15$, and $i \ne j$. Each agent has a fixed self-loop, i.e., $P(a_{ii} = e) = 1$, $i = 1, \dots, 15$, as shown in Fig. 1, and the network evolves from 0 to 8 s. In Fig. 1, we find that the graphs for each time interval have different patterns and are irregular, which reflects the randomness. We give the initial state as $x(0) = [7, 3, 9, 11, 8, 6, 4, 1, 1.7, 3.2, 5.3, 9.9, 7.4, 3.5, 1.4]^\top$. The agents' dynamics are described in (4). Fig. 2 displays each agent's state trajectories in MASs. We can observe that all agents' state eventually reaches the maximum value of the initial state. Therefore, we can obtain that discrete-time MAS (6) reaches max-consensus in the directed random network, which conforms to the result stated in Theorem 1.

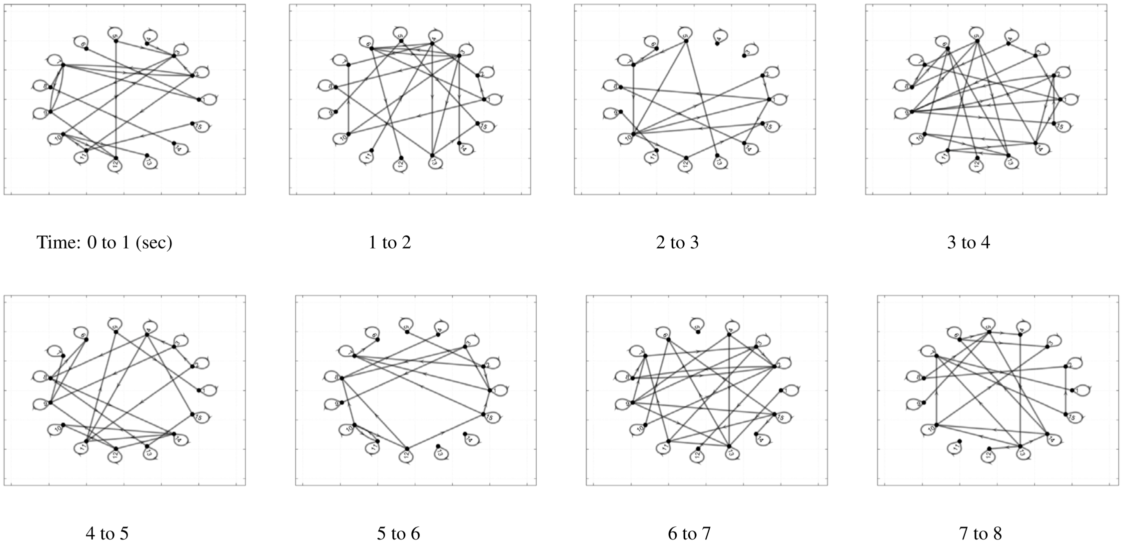

Fig. 1. Directed random graph: $n = 15, p = 0.1$.

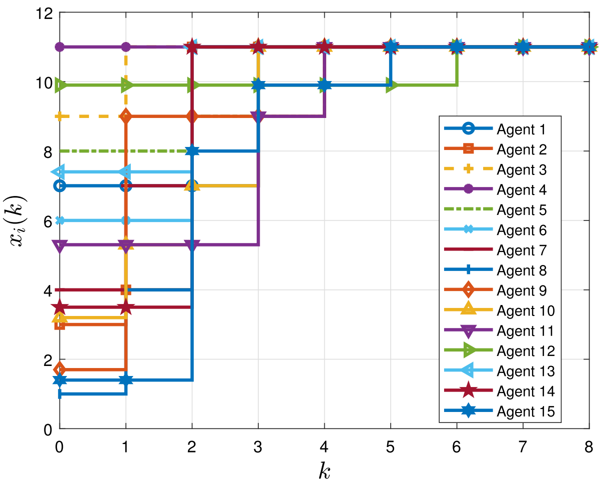

Fig. 2. All agents' state trajectories in MAS (6).

6. Conclusions

This paper studied the max-consensus of discrete-time MASs in random networks. The max-plus algebra was used to describe the dynamical structure and to solve max-consensus algorithm's convergence problems. Then, we derived sufficient and necessary conditions for reaching max-consensus almost surely of MASs. And the results shown that the max-consensus is equivalent in different probabilistic senses, which include almost surely, in probability, expectation, and mean square when the expected graph is strongly connected. In numerical simulation, the MASs achieve consensus when directed edges of the network are randomly connected. It indicate that the offered consensus algorithm is effective and the max-plus algebraic algorithm can be applied in a probabilistic sense. In consideration of more general network systems in the real world, our future work will consider the max- (or min-) consensus problem for random networks with non-independence assumption.

CRediT authorship contribution statement

Jianing Yang: Conceptualization, Software, Writing – original draft. **Liqi Zhou:** Methodology, Writing – review & editing, Software. **Bohui Wang:** Writing – review & editing. **Yuanshi Zheng:** Conceptualization, Methodology, Writing – review & editing, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

Proof of Lemma 4. (Sufficiency). Firstly, we need to obtain an explicit form to represent the specific operation of each element a_{ij} of the matrix obtained after multiplying $k+1$ matrices, that is, $A(k)\otimes\cdots\otimes A(1)\otimes A(0)$ satisfies

$$
\begin{aligned}
(\bigotimes_{l=0}^{k} A(l))_{ij} &= (A(k)\otimes A(k-1)\otimes\cdots\otimes A(0))_{ij}\\
&= \bigoplus_{r_{k-1}=1}^{n}\bigoplus_{r_{k-2}=1}^{n}\cdots\bigoplus_{r_0=1}^{n}(a_{ir_{k-1}}(k)\otimes a_{r_{k-1}r_{k-2}}(k-1)\otimes\cdots\otimes a_{r_1r_0}(1)\otimes a_{r_0j}(0)),
\end{aligned}
\tag{16}
$$

where $r_z\in V, z=1,2,\dots,k-1$.

In the following, we use mathematical induction to prove it. For $k=1$, one has

$$
\begin{aligned}
(\bigotimes_{l=0}^{1} A(l))_{ij} &= (A(1)\otimes A(0))_{ij} = \bigoplus_{r_0=1}^{n}(a_{ir_0}(1)\otimes a_{r_0j}(0))\\
&= (a_{i1}(1)\otimes a_{1j}(0))\oplus(a_{i2}(1)\otimes a_{2j}(0))\oplus\cdots\oplus(a_{in}(1)\otimes a_{nj}(0)).
\end{aligned}
$$

For $k=2$, one has

$$
\begin{aligned}
(\bigotimes_{l=0}^{2} A(l))_{ij} &= (A(2)\otimes A(1)\otimes A(0))_{ij}\\
&= \bigoplus_{r_1=1}^{n}(a_{ir_1}(2)\otimes(\bigoplus_{r_0=1}^{n}(a_{r_1r_0}(1)\otimes a_{r_0j}(0))))\\
&= \bigoplus_{r_1=1}^{n}[a_{ir_1}(2)\otimes((a_{r_11}(1)\otimes a_{1j}(0))\oplus(a_{r_12}(1)\otimes a_{2j}(0))\oplus\cdots\oplus(a_{r_1n}(1)\otimes a_{nj}(0)))]\\
&= \bigoplus_{r_1=1}^{n}[(a_{ir_1}(2)\otimes a_{r_11}(1)\otimes a_{1j}(0))\oplus(a_{ir_1}(2)\otimes a_{r_12}(1)\otimes a_{2j}(0))\oplus\cdots\oplus(a_{ir_1}(2)\otimes a_{r_1n}(1)\otimes a_{nj}(0))]\\
&= \bigoplus_{r_1=1}^{n}\bigoplus_{r_0=1}^{n}(a_{ir_1}(2)\otimes a_{r_1r_0}(1)\otimes a_{r_0j}(0)).
\end{aligned}
$$

Next, we suppose that Eq. (16) holds, then for $k=k+1$,

$$
\begin{aligned}
(\bigotimes_{l=0}^{k+1} A(l))_{ij} &=(A(k+1)\otimes A(k)\otimes\cdots\otimes A(1)\otimes A(0))_{ij}\\
&= \bigoplus_{r_k=1}^{n}a_{ir_k}(k+1)\otimes(\bigoplus_{r_{k-1}=1}^{n}\bigoplus_{r_{k-2}=1}^{n}\cdots\bigoplus_{r_0=1}^{n}(a_{r_kr_{k-1}}(k)\otimes a_{r_{k-1}r_{k-2}}(k-1)\otimes\cdots\otimes a_{r_1r_0}(1)\otimes a_{r_0j}(0))).
\end{aligned}
$$

Since the formula $\bigoplus_{r_{k-1}=1}^{n}\bigoplus_{r_{k-2}=1}^{n}\cdots\bigoplus_{r_0=1}^{n}(a_{r_kr_{k-1}}(k)\otimes a_{r_{k-1}r_{k-2}}(k-1)\otimes\cdots\otimes a_{r_1r_0}(1)\otimes a_{r_0j}(0))$ is too long to understand, we analyzed it separately. Using the distributivity of two binary operations in max-plus algebra, we can get

$$
\begin{aligned}
&\bigoplus_{r_{k-1}=1}^{n}\bigoplus_{r_{k-2}=1}^{n}\cdots\bigoplus_{r_0=1}^{n}(a_{r_kr_{k-1}}(k)\otimes a_{r_{k-1}r_{k-2}}(k-1)\otimes\cdots\otimes a_{r_1r_0}(1)\otimes a_{r_0j}(0))\\
&= \bigoplus_{r_{k-1}=1}^{n}\bigoplus_{r_{k-2}=1}^{n}\cdots\bigoplus_{r_1=1}^{n}[a_{r_kr_{k-1}}(k)\otimes a_{r_{k-1}r_{k-2}}(k-1)\otimes\cdots\otimes a_{r_2r_1}(2)\otimes(a_{r_11}(1)\otimes a_{1j}(0))\oplus a_{r_kr_{k-1}}(k)\otimes a_{r_{k-1}r_{k-2}}(k-1)\\
&\quad\otimes\cdots\otimes a_{r_2r_1}(2)\otimes(a_{r_12}(1)\otimes a_{2j}(0))\oplus\cdots\oplus a_{r_kr_{k-1}}(k)\otimes a_{r_{k-1}r_{k-2}}(k-1)\otimes\cdots\otimes a_{r_2r_1}(2)\otimes(a_{r_1n}(1)\otimes a_{nj}(0))]\\
&= \bigoplus_{r_{k-1}=1}^{n}\bigoplus_{r_{k-2}=1}^{n}\cdots\bigoplus_{r_1=1}^{n}[(a_{r_kr_{k-1}}(k)\otimes a_{r_{k-1}r_{k-2}}(k-1)\otimes\cdots\otimes a_{r_2r_1}(2))\otimes(a_{r_11}(1)\otimes a_{1j}(0)\oplus a_{r_12}(1)\otimes a_{2j}(0)\oplus\cdots\\
&\quad\oplus a_{r_1n}(1)\otimes a_{nj}(0))]\\
&= \bigoplus_{r_{k-1}=1}^{n}\bigoplus_{r_{k-2}=1}^{n}\cdots\bigoplus_{r_1=1}^{n}[(a_{r_kr_{k-1}}(k)\otimes a_{r_{k-1}r_{k-2}}(k-1)\otimes\cdots\otimes a_{r_2r_1}(2))\otimes(\bigoplus_{r_0=1}^{n}(a_{r_1r_0}(1)\otimes a_{r_0j}(0)))]
\end{aligned}
$$

$$= \cdots$$
$$= \bigoplus_{r_{k-1}=1}^{n} [a_{r_k r_{k-1}}(k) \otimes [\bigoplus_{r_{k-2}=1}^{n} [a_{r_{k-1} r_{k-2}}(k-1) \otimes [\cdots \otimes [\bigoplus_{r_1=1}^{n} [a_{r_2 r_1}(2) \otimes [\bigoplus_{r_0=1}^{n} (a_{r_1 r_0}(1) \otimes a_{r_0 j}(0))]]] \cdots]]]].$$

Then we can get

$$\begin{aligned}
(\bigotimes_{l=0}^{k+1} A(l))_{ij} =& (A(k+1) \otimes A(k) \otimes \cdots \otimes A(1) \otimes A(0))_{ij} \\
=& \bigoplus_{r_k=1}^{n} a_{ir_k}(k+1) \otimes (\bigoplus_{r_{k-1}=1}^{n} \bigoplus_{r_{k-2}=1}^{n} \cdots \bigoplus_{r_0=1}^{n} (a_{r_k r_{k-1}}(k) \otimes a_{r_{k-1} r_{k-2}}(k-1) \otimes \cdots \otimes a_{r_1 r_0}(1) \otimes a_{r_0 j}(0))) \\
=& \bigoplus_{r_k=1}^{n} [a_{ir_k}(k+1) \otimes [\bigoplus_{r_{k-1}=1}^{n} [a_{r_k r_{k-1}}(k) \otimes [\bigoplus_{r_{k-2}=1}^{n} [a_{r_{k-1} r_{k-2}}(k-1) \otimes [\cdots \otimes [\bigoplus_{r_1=1}^{n} [a_{r_2 r_1}(2) \otimes \\
& [\bigoplus_{r_0=1}^{n} (a_{r_1 r_0}(1) \otimes a_{r_0 j}(0))]]] \cdots]]]]]] \\
=& \bigoplus_{r_k=1}^{n} [(a_{ir_k}(k+1) \otimes a_{r_k 1}(k) \otimes Q) \oplus (a_{ir_k}(k+1) \otimes a_{r_k 2}(k) \otimes Q) \oplus \cdots \oplus (a_{ir_k}(k+1) \otimes a_{r_k n}(k) \otimes Q)] \\
=& \bigoplus_{r_k=1}^{n} \bigoplus_{r_{k-1}=1}^{n} [a_{ir_k}(k+1) \otimes [a_{r_k r_{k-1}}(k) \otimes Q]] \\
=& \bigoplus_{r_k=1}^{n} \bigoplus_{r_{k-1}=1}^{n} [a_{ir_k}(k+1) \otimes [a_{r_k r_{k-1}}(k) \otimes [\bigoplus_{r_{k-2}=1}^{n} [a_{r_{k-1} r_{k-2}}(k-1) \otimes [\cdots \otimes [\bigoplus_{r_1=1}^{n} [a_{r_2 r_1}(2) \otimes [\bigoplus_{r_0=1}^{n} (a_{r_1 r_0}(1) \\
& \otimes a_{r_0 j}(0))]]] \cdots]]]]] \\
=& \cdots \\
=& \bigoplus_{r_k=1}^{n} \bigoplus_{r_{k-1}=1}^{n} \cdots \bigoplus_{r_0=1}^{n} (a_{ir_k}(k+1) \otimes a_{r_k r_{k-1}}(k) \otimes \cdots \otimes a_{r_0 j}(0)),
\end{aligned}$$

where

$$Q = \bigoplus_{r_{k-2}=1}^{n} [a_{r_{k-1} r_{k-2}}(k-1) \otimes [\cdots \otimes [\bigoplus_{r_1=1}^{n} [a_{r_2 r_1}(2) \otimes [\bigoplus_{r_0=1}^{n} (a_{r_1 r_0}(1) \otimes a_{r_0 j}(0))]]] \cdots]].$$

Hence, Eq. (16) holds.

According to Eq. (16), we obtain that $(\bigotimes_{l=0}^{k} A(l))_{ij}$ can be denoted as the sum of the n^k formulas. It is worth noting that $A(k) \otimes A(k-1) \otimes \cdots \otimes A(0) = I$ in max-plus algebra implies that there exists a directed path of length $k+1$ between any agents in the union graph consisting of $G(0), \ldots, G(k)$. And $(\bigotimes_{l=0}^{k} A(l))_{ij} = \varepsilon$ only if all these n^k formulas equal ε. The formula $a_{ir_{k-1}}(k) \otimes a_{r_{k-1} r_{k-2}}(k-1) \otimes \cdots \otimes a_{r_1 r_0}(1) \otimes a_{r_0 j}(0)$ obtained from Eq. (16) is equal to ε if at least one of these $k+1$ terms in $a_{ir_{k-1}}(k) \otimes a_{r_{k-1} r_{k-2}}(k-1) \otimes \cdots \otimes a_{r_1 r_0}(1) \otimes a_{r_0 j}(0)$ is equal to ε. Next, we will analyze the probability of each of these n^k formulas being equal to ε. We find that the most comprehensive case to consider is the complete graph, i.e., the probability of each edge $p_{ij} > 0$. Therefore, we intend to first consider all cases from the complete graph, and then scale the results obtained to get the final criterion condition. Since we assume that each agent has a self-loop with probability 1, we address all possible cases by categorizing them in terms of the number of self-loops. For $i \neq r_{k-1}, r_z \neq r_{z-1}, (z = 1, 2, \ldots, k-1), r_0 \neq j$, one has

$$P(a_{ir_{k-1}}(k) \otimes \cdots \otimes a_{r_0 j}(0) = \varepsilon) = 1 - p_{ir_{k-1}} \times \cdots \times p_{r_0 j} \leq 1 - (\min_{i,j \in V} p_{ij})^{k+1} < 1, \tag{17}$$

where $p_{ij} > 0$, $i, j \in V$. There are $\eta_{k+1} = \binom{k+1}{0} \times \theta_k, \theta_k \in ((n-1)^{k-1}, (n-1)^k)$ cases included in (17). If one and only one of formulas $i = r_{k-1}, r_z = r_{z-1}, (z = 1, 2, \ldots, k-1), r_0 = j$ holds, then

$$P(a_{ir_{k-1}}(k) \otimes \cdots \otimes a_{r_0 j}(0) = \varepsilon) \leq 1 - (\min_{i,j \in V} p_{ij})^{k} < 1. \tag{18}$$

There are $\eta_k = \binom{k+1}{1} \times \theta_{k-1}, \theta_{k-1} \in ((n-1)^{k-2}, (n-1)^{k-1})$ cases included in (18). And so on, if exactly $k+1-d$ equations hold in $i = r_{k-1}, r_z = r_{z-1}, (z = 1, 2, \ldots, k-1), r_0 = j$, then

$$P(a_{ir_{k-1}}(k) \otimes \cdots \otimes a_{r_0 j}(0) = \varepsilon) \leq 1 - (\min_{i,j \in V} p_{ij})^{d} < 1. \tag{19}$$

There are η_d cases included in (19), where $r_z \in \{1, 2, \ldots, n\}, (z = 0, 1, 2, \ldots, k-1)$, $\eta_d = \binom{k+1}{k+1-d} \times \theta_{d-1}, \theta_d \in \left((n-1)^{d-1}, (n-1)^d\right), (d = 1, 2, \ldots, k, k+1), \theta_0 = 1$ and $\sum_{d=1}^{k+1} \eta_d = n^k$. Then

$$P((\bigotimes_{l=0}^{k} A(l))_{ij} = \varepsilon) \leq \prod_{d=1}^{k+1} (1 - p^d)^{\eta_d}, \tag{20}$$

where $p = \min_{i,j \in V} p_{ij} \in (0, 1)$.

Since that the expected graph is strongly connected, there exist a directed path from j to i for any $i, j \in V, i \neq j$. Then, we denote the shortest length of path from j to i as d_{ij} for any $i, j \in V, i \neq j$. Thus, there exist $i_1, i_2, \ldots, i_{d_{ij}-1} \in V$, such that $\forall l \in \mathbb{N}, P(a_{ii_1}(l) = e) = p_{ii_1} > 0, P(a_{i_1 i_2}(l) = e) = p_{i_1 i_2} > 0, \ldots, P(a_{i_{d_{ij}-1} j}(l) = e) = p_{i_{d_{ij}-1} j} > 0$. From Eq. (20), for any $i, j \in V, i \neq j$, the probability of $(\bigotimes_{l=0}^{k} A(l))_{ij} = \varepsilon$ is

$$\begin{aligned} P((\bigotimes_{l=0}^{k} A(l))_{ij} = \varepsilon) &= \prod_{d=d_{ij}}^{k+1} (1 - p^d)^{\eta_d} \\ &\leq (1 - p^{d_{ij}})^{\binom{k+1}{k+1-d_{ij}}} \\ &\leq (1 - p^{d_{ij}})^{k+1}, \end{aligned} \tag{21}$$

where $1 \leq d_{ij} \leq n - 1 \leq k + 1$.

Then $\forall \epsilon > 0$, $\forall i, j \in V$, and $i \neq j$, we can get

$$\sum_{k=0}^{\infty} P\left(\left|(\bigotimes_{l=0}^{k} A(l))_{ij}\right| > \epsilon\right) = \sum_{k=0}^{\infty} P((\bigotimes_{l=0}^{k} A(l))_{ij} = \varepsilon). \tag{22}$$

According to Eq. (21), we obtain that $P((\bigotimes_{l=0}^{k} A(l))_{ij} = \varepsilon) \leq (1 - p^{d_{ij}})^{k+1}$. It follows from $\frac{(1-p^{d_{ij}})^{k+1}}{(1-p^{d_{ij}})^k} = 1 - p^{d_{ij}} < 1$ that the series $\sum_{k=0}^{\infty} (1 - p^{d_{ij}})^{k+1}$ is convergence. Since $\forall k \in \mathbb{N}, P((\bigotimes_{l=0}^{k} A(l))_{ij} = \varepsilon) > 0, (1 - p^{d_{ij}})^{k+1} > 0$, then $\sum_{k=0}^{\infty} P((\bigotimes_{l=0}^{k} A(l))_{ij} = \varepsilon)$ and $\sum_{k=0}^{\infty} (1 - p^{d_0})^{k+1}$ are both positive series. By the comparison principle of the positive series, we can obtain that the series $\sum_{k=0}^{\infty} P(\left|(\bigotimes_{l=0}^{k} A(l))_{ij}\right| > \epsilon)$ is convergence. And $\sum_{k=0}^{\infty} (1 - p^{d_{ij}})^{k+1} = \lim_{n\to\infty} \sum_{k=0}^{n} (1 - p^{d_{ij}})^{k+1} = \frac{1-p^{d_{ij}}}{p^{d_{ij}}} < \infty$, then $\sum_{k=0}^{\infty} P(\left|(\bigotimes_{l=0}^{k} A(l))_{ij}\right| > \epsilon) < \infty$. By Lemma 1 (Borel–Cantelli Lemma), $\forall \epsilon > 0$, and $\forall i, j \in V, i \neq j$, it is obtained that

$$P\left(\limsup_{k\to\infty} \left|(\bigotimes_{l=0}^{k} A(l))_{ij}\right| > \epsilon\right) = 0. \tag{23}$$

Ultimately, we conclude that $P(\lim_{k\to\infty} (\bigotimes_{l=0}^{k} A(l))_{ij} = e) = 1$.

(Necessity). If the expected graph is not strongly connected, there exists $i_0, j_0 \in V$ and $i_0 \neq j_0$, such that $P((\bigotimes_{l=0}^{k} A(l))_{i_0 j_0} = \varepsilon) = 1, \forall k \in \mathbb{N}$, which is conflict with the known condition $\forall \epsilon > 0$, $\forall i, j \in V, i \neq j$, $P(\lim_{k\to\infty} (\bigotimes_{l=0}^{k} A(l))_{ij} = \varepsilon) = 0$. The necessity is proved. □

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