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Global inverse optimality for a class of recurrent neural networks with multiple proportional delays

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ABSTRACT

This paper formulates two novel theoretical designs of input-to-state stabilizing control for a class of recurrent neural networks with multiple proportional delays. The analysis tool developed in this paper is based on Lyapunov function and inverse optimality method, which does not require solving Hamilton-Jacobi-Bellman equations. Two inverse optimal feedback laws are constructed via the dimensions of state and input, which ensure the input-state stability for the considered system. When the dimensions of state and input are different, we establish a scalar function and give one of the control laws by Sontag's formula. Furthermore, the designs of inverse optimal control reach both global inverse optimality and global asymptotic stability of the system for some meaningful cost functional. Four numerical examples are provided to show the effectiveness of the inverse optimal control.

1. Introduction

Recurrent neural networks (RNNs) have notably attracted attentions since they are widely used in combinatorial optimization, pattern recognition, and associative memory [1]. However, these applications are largely dependent on the stability of the systems, namely, the long-time asymptotic behavior of the solutions to RNNs. For example, Akhmet, Aruğaslanc and Yılmaz [2] obtained the global exponential stability of RNNs with piecewise constant argument in terms of Lyapunov functions. For the Lipschitz continuous and monotone increasing activation functions, the global both asymptotic and exponential stability of the RNNs were discussed in Hu and Wang [3]. Fan and Zhu [4] investigated the mean square exponential stability of discrete-time stochastic neural networks (NNs) with partially unstable subsystems and mixed delays, which utilizes the approaches of Lyapunov-Krasovskii functional and stationary distribution of Markov chain. There is an intensive literature in this area, for example, [5,6], a few to name. Indeed, for a given RNNs, it is very important to determine whether the system is stable.

Time delay is another important issue and one of the potential causes of oscillation or instability and poor performance in neural networks. In particular, proportional delay [7–10] is a type of unbounded time-varying delays that is not same as other kinds of delays [4,11–16]. The research on the stability of NNs with proportional delay has attracted the attention of many scholars. For instance, by applying linear matrix inequality (LMI), Zhou [17] discussed the global asymptotic stability (GAS) of cellular NNs with

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proportional delays. Jia [18] investigated the multiple proportional delays of fuzzy cellular NNs by using differential inequality techniques and finite-time stability theory. For more results we refer the reader to [19,20] and the references therein.

Note that the aforementioned works only focus on the case of the NNs with proportional delays under the same dimensions of state and input. These restrictions have huge impact on the application of NNs in automatic control and system identification. Therefore, NNs systems with different dimensions of state and input, represent a wide class of important nonlinear systems [21,22].

At present, there is a wealth of works [9,23,24] on optimal control for nonlinear systems. Among them, it is worth pointing out that this optimization problem [24], requires solving the nonlinear two-point boundary value problem, or the nonlinear Hamilton-Jacobi-Bellman (HJB) equation. Nevertheless, it is difficult to get an analytical solution to the HJB equation for optimal control of nonlinear systems. Particularly, the problem of optimal control for most delayed nonlinear systems can only be solved numerically [25,26]. It is well known that the numerical methods also have some defects such as slow convergence rate and selection of the appropriate initial values. To conquer this difficulty, many scholars put forward the inverse optimal control approach, which is involved in linear systems [27], nonlinear systems [28–36], and delay systems [37,38]. Up to now, there are no related research results for nonlinear systems with proportional delay by using inverse optimal control.

In general, in many fields such as control and engineering, the dimensions of state and input for nonlinear systems may be the same or different [39,40]. Moreover, it is difficult to design feedback control law for optimal stabilization of such systems. Therefore, it is very interesting and challenging to investigate a control design method for delayed nonlinear systems such that desired system stability is achieved and a cost functional is minimized. The main contribution of this paper is twofold. First, two inverse optimal controllers are developed for input-to-state stability (ISS) of RNNs with multiple proportional delays by using the inverse optimality, Lyapunov technique and HJB equation. The presented approach is simpler than the existing LMI method. Second, for the different dimensions of state and input, a scalar function is constructed by using Sontag's formula, and the control law is derived that can ensure the ISS of the system. This paper generalizes the previous results to the general nonlinear system with multiple proportional delays.

The remainder of this paper is proposed as follows. In Section 2, we present some preliminaries. In Section 3, we present the main results of this paper. In Section 4, numerical simulations are provided to illustrate the effectiveness of the proposed methods. Finally, a conclusion is given in Section 5.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ be the set of all $n\times m$ real matrices, the superscript "T" be the transpose of a matrix or vector, and $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^T x}$ be the Euclidean norm of a vector x. If A is a matrix, then $\|\mathrm{A}\|$ denotes the Frobenius matrix norm, i.e., $\|\mathrm{A}\| = \sqrt{\sum_{ij} |a_{ij}|^2} = \sqrt{\mathrm{tr}\left(\mathrm{A}^T\mathrm{A}\right)}$, where $\mathrm{tr}(\cdot)$ is the trace of a matrix.

In this paper, we consider the following RNN with multiple proportional delays

$$\begin{cases} \dot{x}(t) = -Ax(t) + Bf(x(t)) + Cg(x(pt)) + Dh(x(qt)) + Ev, \ t \geq 1, \\ x(t) = x_0, \ t \in [q', 1], \end{cases} \tag{1}$$

where $x(t) = (x_1(t), ..., x_n(t))^T \in R^n$ is the state vector associated with n neurons, $A = \mathrm{diag}(a_1, a_2, ..., a_n)$ is a diagonal matrix with $a_i > 0$ being the firing rate, B, C and D are the connection weight matrix and delayed connection weight matrices, respectively, E is a given control input matrix with appropriate dimension, v is the control input, $f(x(t)) = (f_1(x_1(t)), ..., f_n(x_n(t)))^T$, $g(x(pt)) = (g_1(x_1(pt)), ..., g_n(x_n(pt)))^T$ and $h(x(qt)) = (h_1(x_1(qt)), ..., h_n(x_n(qt)))^T$ are the activation functions. The constants p, q are proportional delay factors and satisfy $0 < p,\ q \leq 1$, $pt = t - (1-p)t$ and $qt = t - (1-q)t$, where $(1-p)t$ and $(1-q)t$ are time-varying continuous functions satisfying $(1-p)t \to +\infty$ and $(1-q)t \to +\infty$ as $t \to +\infty$ and $p \neq 1$, $q \neq 1$, $q' = \min\{p, q\}$. $x_0 = (x_{10}, ..., x_{n0})^T \in \mathbb{R}^n$ is the initial value of $x(t)$ at $t \in [q', 1]$.

By defining the transformation for the variable $\zeta(t) = x(e^t)$, system (1) is reformulated as

$$\begin{cases} \dot{\zeta}(t) = e^t\Big[-A\zeta(t) + Bf(\zeta(t)) + Cg(\zeta(t-\varsigma)) + Dh(\zeta(t-\xi)) + Ev\Big], \ t \geq 0, \\ \zeta(t) = \zeta_0(t), \ t \in [-\tau, 0], \end{cases} \tag{2}$$

where $\varsigma = -\log p$, $\xi = -\log q$, $\tau = \max\{\varsigma, \xi\}$, $\zeta_0(t) = (\zeta_{10}(t), \zeta_{20}(t), ..., \zeta_{n0}(t))^T \in C([-\tau, 0]; \mathbb{R}^n)$.

Remark 2.1. The system (2) is a recurrent neural network with unbounded time-varying coefficients and multiple constant delays. If $p = q = 1$, then the system (1) is a RNN without delay in [11]. The system (1) is also completely different from the cellular neural network with bounded delays and time-varying coefficients [12]. Moreover, it is shown that the expression of control input is the major difference between the system (2) and the system in [12]. Therefore, system (1) is a more general nonlinear RNN with multiple proportional delays than those systems given in [10–12,17,34].

Lemma 2.2. *[41,42] For any vectors $y, z \in R^n$, the matrix inequality holds*

$$2y^T z \leq y^T y + z^T z. \tag{3}$$

Lemma 2.3. *[32,35] Assume that there exists a positive semidefinite function $U(t,\zeta)\in C^1$ associated with the following nonlinear system*

$$\dot{\zeta}(t)=\mathcal{P}(\zeta)+\mathcal{Q}(\zeta)v, \tag{4}$$

the HJB equation satisfies as follows

$$U_t(t,\zeta)+U_\zeta(t,\zeta)\mathcal{P}(\zeta)-\frac{1}{4}U_\zeta(t,\zeta)\mathcal{Q}(\zeta)\mathcal{W}^{-1}(\zeta)\mathcal{Q}^T(\zeta)U_\zeta^T(t,\zeta)+\mathcal{V}(\zeta)=0. \tag{5}$$

Moreover, there is a feedback control

$$v^*(\zeta)=-\frac{1}{2}\mathcal{W}^{-1}(\zeta)\big(U_\zeta(t,\zeta)\mathcal{Q}(\zeta)\big)^T \tag{6}$$

reaches the GAS for system (4) at the origin by minimizing the cost functional

$$\mathcal{L}(v)=\int_0^{+\infty}\big(\mathcal{V}(\zeta)+v^T\mathcal{W}(\zeta)v\big)dt \tag{7}$$

where $\mathcal{V}(\zeta)\geq 0$ and $\mathcal{W}(\zeta)>0$ for all ζ. Then v^* is the optimal stabilizing control, and $U(t,\zeta)$ is the optimal value function.

Assumption 2.1. Assume that activation functions $f_j(\cdot)$, $g_j(\cdot)$ and $h_j(\cdot)$ satisfy the following conditions:

(i) $f_j(0)=0,\ g_j(0)=0,\ h_j(0)=0$;
(ii) $\left|f_j(w_j)-f_j(\zeta_j)\right|\leq k_j\left|w_j-\zeta_j\right|,\ \left|g_j(w_j)-g_j(\zeta_j)\right|\leq l_j\left|w_j-\zeta_j\right|,\ \left|h_j(w_j)-h_j(\zeta_j)\right|\leq m_j\left|w_j-\zeta_j\right|$

where $\forall w_j,\zeta_j\in\mathbb{R}$, k_j, l_j and $m_j (j=1,2,...,n)$ are nonnegative constants, and $|\cdot|$ denotes the absolute value.

Assumption 2.2. Assume that the weight matrix E of system (2) satisfies the following conditions:

(i) if $E\in\mathbb{R}^{n\times n}$ is a square matrix, then E is invertible, i.e., $\det(E)\neq 0$;
(ii) if $E\in\mathbb{R}^{n\times m}$ is not a square matrix, then E is selected to satisfy the inequality $EE^T\geq 0$.

Remark 2.4. The activation functions in this paper are no longer needed to be monotonous, bounded and differentiable (see, e.g. [4–6,16,37,38]). For example, the sigmoid function is selected as the activation function in [5], which is a strictly monotonically increasing bounded function.

3. The main results

In this section, the input-to-state stabilizing control for system (2) based on the inverse optimality method and Lyapunov function is investigated under Assumptions 2.1 and 2.2. According to the same and different dimensions of input and state, we will give two inverse optimal feedback laws. Before that, we calculate the Lyapunov function to facilitate the discussion of the following two cases.

We now build a candidate Lyapunov function as

$$U(t,\zeta)=\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t)+\int_{t-\varsigma}^{t}\big(Cg(\zeta(s))\big)^T\big(Cg(\zeta(s))\big)\mathrm{d}s+\int_{t-\xi}^{t}\big(Dh(\zeta(s))\big)^T\big(Dh(\zeta(s))\big)\mathrm{d}s. \tag{8}$$

Differentiating $U(t,\zeta)$ gives

$$\begin{aligned}\dot{U}(t,\zeta)=&-\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t)\\&+e^{-t}e^{t}\zeta^T(t)\Big[-A\zeta(t)+Bf(\zeta(t))+Cg(\zeta(t-\varsigma))+Dh(\zeta(t-\xi))+Ev\Big]\\&+\big(Cg(\zeta(t))\big)^T\big(Cg(\zeta(t))\big)+\big(Dh(\zeta(t))\big)^T\big(Dh(\zeta(t))\big)\\&-\big(Cg(\zeta(t-\varsigma))\big)^T\big(Cg(\zeta(t-\varsigma))\big)-\big(Dh(\zeta(t-\xi))\big)^T\big(Dh(\zeta(t-\xi))\big).\end{aligned} \tag{9}$$

Since $e^{-t}\zeta^T(t)\zeta(t)\geq 0$, one obtains

$$\begin{aligned}\dot{U}(t,\zeta)\leq&\,\zeta^T(t)\Big[-A\zeta(t)+Bf(\zeta(t))+Cg(\zeta(t-\varsigma))+Dh(\zeta(t-\xi))+Ev\Big]\\&+\big(Cg(\zeta(t))\big)^T\big(Cg(\zeta(t))\big)+\big(Dh(\zeta(t))\big)^T\big(Dh(\zeta(t))\big)\\&-\big(Cg(\zeta(t-\varsigma))\big)^T\big(Cg(\zeta(t-\varsigma))\big)-\big(Dh(\zeta(t-\xi))\big)^T\big(Dh(\zeta(t-\xi))\big)\\=&-\zeta^T(t)A\zeta(t)+\zeta^T(t)Bf(\zeta(t))+\zeta^T(t)Cg(\zeta(t-\varsigma))+\zeta^T(t)Dh(\zeta(t-\xi))\end{aligned} \tag{10}$$

$$+\zeta^T(t)Ev+\left(Cg(\zeta(t))\right)^T\left(Cg(\zeta(t))\right)+\left(Dh(\zeta(t))\right)^T\left(Dh(\zeta(t))\right)$$
$$-\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)-\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right).$$

From Lemma 2.2, the following three inequalities hold:

$$\begin{aligned}\zeta^T(t)Bf\left(\zeta(t)\right)\le&\frac{1}{2}\left\|Bf\left(\zeta(t)\right)\right\|^2+\frac{1}{2}\zeta^T(t)\zeta(t)\\ \le&\frac{1}{2}\left\|B\right\|^2\left\|f\left(\zeta(t)\right)\right\|^2+\frac{1}{2}\zeta^T(t)\zeta(t),\end{aligned}\tag{11}$$

$$\zeta^T(t)Cg\left(\zeta(t-\varsigma)\right)\le\frac{1}{2}\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)+\frac{1}{2}\zeta^T(t)\zeta(t)\tag{12}$$

and

$$\zeta^T(t)Dh\left(\zeta(t-\xi)\right)\le\frac{1}{2}\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right)+\frac{1}{2}\zeta^T(t)\zeta(t).\tag{13}$$

Note from Assumption 2.1 that

$$\left\|f(\zeta(t))\right\|^2\le(K\zeta(t))^T(K\zeta(t))=\zeta^T(t)K^2\zeta(t)\le k^2\zeta^T(t)\zeta(t),\tag{14}$$

$$\left\|g(\zeta(t))\right\|^2\le(L\zeta(t))^T(L\zeta(t))=\zeta^T(t)L^2\zeta(t)\le l^2\zeta^T(t)\zeta(t)\tag{15}$$

and

$$\left\|h(\zeta(t))\right\|^2\le(M\zeta(t))^T(M\zeta(t))=\zeta^T(t)M^2\zeta(t)\le m^2\zeta^T(t)\zeta(t),\tag{16}$$

where $K=\mathrm{diag}(k_1,k_2,...,k_n)$, $k=\max\limits_{1\le j\le n}\left\{k_j\right\}$, $L=\mathrm{diag}(l_1,l_2,...,l_n)$, $l=\max\limits_{1\le j\le n}\left\{l_j\right\}$, $M=\mathrm{diag}(m_1,m_2,...,m_n)$, and $m=\max\limits_{1\le j\le n}\left\{m_j\right\}$.

In addition, from (11) and (14), we can get

$$\begin{aligned}\zeta^T(t)Bf(\zeta(t))\le&\frac{1}{2}k^2\|B\|^2\zeta^T(t)\zeta(t)+\frac{1}{2}\zeta^T(t)\zeta(t)\\ =&\zeta^T(t)\left(\frac{1+k^2\|B\|^2}{2}\right)\zeta(t).\end{aligned}\tag{17}$$

Applying Assumption 2.1 yields

$$\left(Cg(\zeta(t))\right)^T\left(Cg(\zeta(t))\right)\le l^2\|C\|^2\zeta^T(t)\zeta(t)\tag{18}$$

and

$$\left(Dh(\zeta(t))\right)^T\left(Dh(\zeta(t))\right)\le m^2\|D\|^2\zeta^T(t)\zeta(t).\tag{19}$$

Substituting (12), (13), (17), (18) and (19) into (10) results in

$$\begin{aligned}\dot{U}(t,\zeta)\le&-\zeta^T(t)A\zeta(t)+\zeta^T(t)Ev+\zeta^T(t)\left(\frac{3+k^2\|B\|^2+2l^2\|C\|^2+2m^2\|D\|^2}{2}\right)\zeta(t)\\ &-\frac{1}{2}\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)-\frac{1}{2}\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right).\end{aligned}\tag{20}$$

Since $\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)\ge0$ and $\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right)\ge0$, (20) is guaranteed by

$$\dot{U}(t,\zeta)\le-\zeta^T(t)A\zeta(t)+\zeta^T(t)\left(\frac{3+k^2\|B\|^2+2l^2\|C\|^2+2m^2\|D\|^2}{2}\right)\zeta(t)+\zeta^T(t)Ev.\tag{21}$$

The control design is carried out using the dimensions of state ζ and input v, and split into two cases.

Case 1: $\zeta\in\mathbb{R}^n$, $v\in\mathbb{R}^n$, and $E\in\mathbb{R}^{n\times n}$.

Considering

$$\zeta^T(t)\left(\frac{3+k^2\|B\|^2+2l^2\|C\|^2+2m^2\|D\|^2}{2}\right)\zeta(t)+\zeta^T(t)Ev=0,$$

then it is admissible to choose a control law from Assumption 2.2 (i)

$$v=-E^{-1}\left(\frac{3+k^2\|B\|^2+2l^2\|C\|^2+2m^2\|D\|^2}{2}\right)\zeta(t)\tag{22}$$

namely $\dot{U}(t,\zeta)\le-\zeta^T(t)A\zeta(t)\le0$, (22) as a stabilizing control.

The three derivatives of $U(t,\zeta)$ (8) along the trajectory of the system (2) are

$$\begin{aligned} U_t(t,\zeta) = &-\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t) + \big(Cg(\zeta(t))\big)^T\big(Cg(\zeta(t))\big) + \big(Dh(\zeta(t))\big)^T\big(Dh(\zeta(t))\big) \\ &- \big(Cg(\zeta(t-\varsigma))\big)^T\big(Cg(\zeta(t-\varsigma))\big) - \big(Dh(\zeta(t-\xi))\big)^T\big(Dh(\zeta(t-\xi))\big), \end{aligned} \tag{23}$$

$$\begin{aligned} U_\zeta(t,\zeta)\mathcal{P}(\zeta) = &-\zeta^T(t)A\zeta(t) + \zeta^T(t)Bf\big(\zeta(t)\big) \\ &+ \zeta^T(t)Cg(\zeta(t-\varsigma)) + \zeta^T(t)Dh(\zeta(t-\xi)) \end{aligned} \tag{24}$$

and

$$U_\zeta(t,\zeta)\mathcal{Q}(\zeta) = \zeta^T(t)E. \tag{25}$$

Substituting (23), (24) and (25) into (5) leads to

$$\begin{aligned} 0 = &-\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t) + \mathcal{V}(\zeta) - \zeta^T(t)A\zeta(t) + \zeta^T(t)Bf(\zeta(t)) \\ &+ \zeta^T(t)Cg(\zeta(t-\varsigma)) + \zeta^T(t)Dh(\zeta(t-\xi)) - \frac{1}{4}\zeta^T(t)E\mathcal{W}^{-1}(\zeta)E^T\zeta(t) \\ &+ \big(Cg(\zeta(t))\big)^T\big(Cg(\zeta(t))\big) + \big(Dh(\zeta(t))\big)^T\big(Dh(\zeta(t))\big) \\ &- \big(Cg(\zeta(t-\varsigma))\big)^T\big(Cg(\zeta(t-\varsigma))\big) - \big(Dh(\zeta(t-\xi))\big)^T\big(Dh(\zeta(t-\xi))\big). \end{aligned} \tag{26}$$

After making a modification of the control law (22), one has

$$\begin{aligned} v &= -\theta E^{-1}\left(\frac{3+k^2\|B\|^2+2l^2\|C\|^2+2m^2\|D\|^2}{2}\right)\zeta(t) \\ &= -\theta E^{-1}\left(\frac{3+k^2\|B\|^2+2l^2\|C\|^2+2m^2\|D\|^2}{2}\right)(E^T)^{-1}E^T\zeta(t) \\ &= -\theta(E^TE)^{-1}\left(\frac{3+k^2\|B\|^2+2l^2\|C\|^2+2m^2\|D\|^2}{2}\right)E^T\zeta(t) \\ &= -\frac{\theta}{2}(E^TE)^{-1}\left(3+k^2\|B\|^2+2l^2\|C\|^2+2m^2\|D\|^2\right)E^T\zeta(t) \end{aligned} \tag{27}$$

where $\theta > 2$ is a constant.

Substituting (25) into (6), and noting (27), one can choose

$$\mathcal{W}(\zeta) = \theta^{-1}E^TE\left(3+k^2\|B\|^2+2l^2\|C\|^2+2m^2\|D\|^2\right)^{-1}. \tag{28}$$

Then, one can get from (26) that

$$\begin{aligned} \mathcal{V}(\zeta) = &\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t) + \zeta^T(t)A\zeta(t) - \zeta^T(t)Bf(\zeta(t)) \\ &- \zeta^T(t)Cg(\zeta(t-\varsigma)) - \zeta^T(t)Dh(\zeta(t-\xi)) + \frac{1}{4}\zeta^T(t)E\mathcal{W}^{-1}(\zeta)E^T\zeta(t) \\ &- \big(Cg(\zeta(t))\big)^T\big(Cg(\zeta(t))\big) - \big(Dh(\zeta(t))\big)^T\big(Dh(\zeta(t))\big) \\ &+ \big(Cg(\zeta(t-\varsigma))\big)^T\big(Cg(\zeta(t-\varsigma))\big) + \big(Dh(\zeta(t-\xi))\big)^T\big(Dh(\zeta(t-\xi))\big). \end{aligned} \tag{29}$$

The following theorem gives the design of inverse optimal control to the same dimensions of state and input.

Theorem 3.1. *There exist positive definite and strictly positive functions $\mathcal{V}(\zeta)$ (29) and $\mathcal{W}(\zeta)$ (28), respectively, such that the system (2) with the feedback control law*

$$v = v^* = -\frac{1}{2}\mathcal{W}^{-1}(\zeta)E^T\zeta(t) \tag{30}$$

reaches global inverse optimality at the origin by minimizing the cost functional

$$\mathcal{L}(v) = \int_0^{+\infty}\big(\mathcal{V}(\zeta) + v^T\mathcal{W}(\zeta)v\big)dt. \tag{31}$$

Therefore, the optimal control law (i.e., control input) (30) reaches both ISS and GAS for the system (2).

Proof. Construct a Lyapunov function $U(t,\zeta)$ described by (8) whose derivative along system (2) is given as

$$\begin{aligned}\dot{U}(t,\zeta)=&-\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t)-\zeta^T(t)A\zeta(t)+\zeta^T(t)Bf(\zeta(t))\\&+\zeta^T(t)Cg(\zeta(t-\varsigma))+\zeta^T(t)Dh(\zeta(t-\xi))+\zeta^T(t)Ev\\&+\left(Cg(\zeta(t))\right)^T\left(Cg(\zeta(t))\right)+\left(Dh(\zeta(t))\right)^T\left(Dh(\zeta(t))\right)\\&-\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)-\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right).\end{aligned}\tag{32}$$

Under the control law (30), we have

$$\begin{aligned}\dot{U}(t,\zeta)=&-\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t)-\zeta^T(t)A\zeta(t)+\zeta^T(t)Bf(\zeta(t))\\&+\zeta^T(t)Cg(\zeta(t-\varsigma))+\zeta^T(t)Dh(\zeta(t-\xi))\\&-\frac{\theta}{2}\zeta^T(t)E(E^TE)^{-1}\left(3+k^2\|B\|^2+2l^2\|C\|^2+2m^2\|D\|^2\right)E^T\zeta(t)\\&+\left(Cg(\zeta(t))\right)^T\left(Cg(\zeta(t))\right)+\left(Dh(\zeta(t))\right)^T\left(Dh(\zeta(t))\right)\\&-\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)-\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right)\\=&-\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t)-\zeta^T(t)A\zeta(t)+\zeta^T(t)Bf(\zeta(t))\\&+\zeta^T(t)Cg(\zeta(t-\varsigma))+\zeta^T(t)Dh(\zeta(t-\xi))\\&-\frac{\theta}{2}\left(3+k^2\|B\|^2+2l^2\|C\|^2+2m^2\|D\|^2\right)\zeta^T(t)\zeta(t)\\&+\left(Cg(\zeta(t))\right)^T\left(Cg(\zeta(t))\right)+\left(Dh(\zeta(t))\right)^T\left(Dh(\zeta(t))\right)\\&-\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)-\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right).\end{aligned}\tag{33}$$

For $\forall\zeta\neq 0$, substituting (12), (13), (17), (18), (19) into (33) and noting the fact $\theta>2$ results in

$$\begin{aligned}\dot{U}(t,\zeta)\leq&-\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t)-\zeta^T(t)A\zeta(t)\\&-\frac{(\theta-1)}{2}\left(3+k^2\|B\|^2+2l^2\|C\|^2+2m^2\|D\|^2\right)\zeta^T(t)\zeta(t)\\&-\frac{1}{2}\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)-\frac{1}{2}\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right),\end{aligned}\tag{34}$$

which implies that $\dot{U}(t,\zeta)<0$. Therefore, the feedback control law (30) reaches GAS for the system (2) at the origin $\zeta=0$ and ensures $\lim_{t\to+\infty}\zeta(t)=0$.

Next, we will discuss $\mathcal{V}(\zeta)$ and $\mathcal{W}(\zeta)$.

Based on (28) and (29), one has

$$\begin{aligned}\mathcal{V}(\zeta)=&\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t)+\frac{1}{4}\zeta^T(t)E\mathcal{W}^{-1}(\zeta)E^T\zeta(t)+\zeta^T(t)A\zeta(t)\\&-\zeta^T(t)Bf(\zeta(t))-\zeta^T(t)Cg(\zeta(t-\varsigma))-\zeta^T(t)Dh(\zeta(t-\xi))\\&-\left(Cg(\zeta(t))\right)^T\left(Cg(\zeta(t))\right)-\left(Dh(\zeta(t))\right)^T\left(Dh(\zeta(t))\right)\\&+\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)+\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right)\\=&\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t)+\zeta^T(t)A\zeta(t)-\zeta^T(t)Bf(\zeta(t))\\&-\zeta^T(t)Cg(\zeta(t-\varsigma))-\zeta^T(t)Dh(\zeta(t-\xi))\\&+\frac{\theta}{4}\left(3+k^2\|B\|^2+2l^2\|C\|^2+2m^2\|D\|^2\right)\zeta^T(t)\zeta(t)\\&-\left(Cg(\zeta(t))\right)^T\left(Cg(\zeta(t))\right)-\left(Dh(\zeta(t))\right)^T\left(Dh(\zeta(t))\right)\\&+\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)+\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right).\end{aligned}\tag{35}$$

Substituting (12), (13), (17), (18), (19) into (35), one obtains

$$\begin{aligned}\mathcal{V}(\zeta)\geq&\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t)+\zeta^T(t)A\zeta(t)\\&+\frac{(\theta-2)}{2}\left(\frac{3+k^2\|B\|^2+2l^2\|C\|^2+2m^2\|D\|^2}{2}\right)\zeta^T(t)\zeta(t)\\&+\frac{1}{2}\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)+\frac{1}{2}\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right)\\\geq&0.\end{aligned}\tag{36}$$

(36) implies that $\mathcal{V}(\zeta)$ is a positive definite and radially unbounded function.

Recalling $\det(E) \neq 0$ in Assumption 2.2 (i), we can then obtain $E^T E > 0$ such that

$$\mathcal{W}(\zeta) = \theta^{-1} E^T E \left(3 + k^2 \|B\|^2 + 2l^2 \|C\|^2 + 2m^2 \|D\|^2\right)^{-1} > 0. \tag{37}$$

From (28) and (29), it holds that

$$\dot{U}(t,\zeta) = -\mathcal{V}(\zeta) - v^T \mathcal{W}(\zeta) v + (v - v^*)^T \mathcal{W}(\zeta)(v - v^*). \tag{38}$$

Hence, one obtains

$$\dot{U}(t,\zeta) + \mathcal{V}(\zeta) + v^T \mathcal{W}(\zeta) v = (v - v^*)^T \mathcal{W}(\zeta)(v - v^*) \geq 0. \tag{39}$$

Namely, $v = v^*$ fulfills

$$\min_v \left[\mathcal{V}(\zeta) + v^T \mathcal{W}(\zeta) v + \dot{U}(t,\zeta)\right] = 0.$$

Furthermore, $U(t,\zeta)$ (8) fulfills the following HJB equation

$$U_t(t,\zeta) + U_\zeta(t,\zeta)\mathcal{P}(\zeta) - \frac{1}{4} U_\zeta(t,\zeta)\mathcal{Q}(\zeta)\mathcal{W}^{-1}(\zeta)\mathcal{Q}^T(\zeta) U_\zeta^T(t,\zeta) + \mathcal{V}(\zeta) = 0, \tag{40}$$

where $\mathcal{P}(\zeta) = e^t \Big[-A\zeta(t) + Bf(\zeta(t)) + Cg(\zeta(t-\varsigma)) + Dh(\zeta(t-\xi))\Big]$ and $\mathcal{Q}(\zeta) = e^t E$.

Thus, $U(t,\zeta)$ is the optimal value function of the cost function (31).

According to [43], it concludes that the input control (30) reaches ISS for the system (2). The proof is complete. □

Case 2: $\zeta \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, $E \in \mathbb{R}^{n\times m}$, and $m \neq n$ ($m < n$ or $m > n$).

Let

$$\alpha(\zeta) = \zeta^T(t) \left(\frac{3 + k^2 \|B\|^2 + 2l^2 \|C\|^2 + 2m^2 \|D\|^2}{2}\right) \zeta(t), \tag{41}$$

$$\beta^T(\zeta) = \zeta^T(t) E. \tag{42}$$

In view of Sontag's formula [44], we define a scalar function with respect to ζ

$$\gamma(\zeta) = \begin{cases} \dfrac{\alpha(\zeta) + \sqrt{\alpha^2(\zeta) + (\beta^T(\zeta)\beta(\zeta))^2}}{\beta^T(\zeta)\beta(\zeta)}, & \zeta \neq 0, \\ 0, & \zeta = 0. \end{cases} \tag{43}$$

Construct a control signal as

$$v = -\gamma(\zeta)\beta(\zeta).$$

Because of (42), we have

$$v = -\gamma(\zeta) E^T \zeta(t). \tag{44}$$

It is inferred from (6), (25), (26) and (44) that

$$\begin{aligned} \mathcal{V}(\zeta) = & \frac{1}{4}\beta^T(\zeta)\mathcal{W}^{-1}(\zeta)\beta(\zeta) + \frac{1}{2} e^{-t} \zeta^T(t)\zeta(t) + \zeta^T(t) A \zeta(t) \\ & - \zeta^T(t) B f(\zeta(t)) - \zeta^T(t) C g(\zeta(t-\varsigma)) - \zeta^T(t) D h(\zeta(t-\xi)) \\ & - \big(Cg(\zeta(t))\big)^T \big(Cg(\zeta(t))\big) - \big(Dh(\zeta(t))\big)^T \big(Dh(\zeta(t))\big) \\ & + \big(Cg(\zeta(t-\varsigma))\big)^T \big(Cg(\zeta(t-\varsigma))\big) + \big(Dh(\zeta(t-\xi))\big)^T \big(Dh(\zeta(t-\xi))\big), \end{aligned} \tag{45}$$

$$\mathcal{W}(\zeta) = \frac{1}{2}\gamma^{-1}(\zeta). \tag{46}$$

The following theorem states the inverse optimal control design with different dimensions of state and input.

Theorem 3.2. *There exist positive definite and strictly positive functions $\mathcal{V}(\zeta)$ (45) and $\mathcal{W}(\zeta)$ (46), respectively, such that the system (2) with the feedback control law*

$$v = v^* = -\frac{1}{2}\mathcal{W}^{-1}(\zeta) E^T \zeta(t) \tag{47}$$

reaches global inverse optimality at the origin by minimizing the cost functional

$$\mathcal{L}(v)=\int_0^{+\infty}\left(\mathcal{V}(\zeta)+v^T\mathcal{W}(\zeta)v\right)dt. \tag{48}$$

Therefore, the optimal control law (i.e., control input) (47) reaches both ISS and GAS for the system (2).

Proof. The derivative of $U(t,\zeta)$ (8) along the system (2) is

$$\begin{aligned}\dot{U}(t,\zeta)=&-\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t)-\zeta^T(t)A\zeta(t)+\zeta^T(t)Bf(\zeta(t))\\&+\zeta^T(t)Cg(\zeta(t-\varsigma))+\zeta^T(t)Dh(\zeta(t-\xi))+\zeta^T(t)Ev\\&+\left(Cg(\zeta(t))\right)^T\left(Cg(\zeta(t))\right)+\left(Dh(\zeta(t))\right)^T\left(Dh(\zeta(t))\right)\\&-\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)-\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right).\end{aligned} \tag{49}$$

For $\forall\zeta\neq 0$, substituting (12), (13), (17), (18), (19) and (47) into (49) leads to

$$\begin{aligned}\dot{U}(t,\zeta)\leq&-\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t)-\zeta^T(t)A\zeta(t)-\frac{1}{2}\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)\\&-\frac{1}{2}\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right)-\sqrt{\alpha^2(\zeta)+\left(\beta^T(\zeta)\beta(\zeta)\right)^2},\end{aligned} \tag{50}$$

which implies that $\dot{U}(t,\zeta)<0$. Therefore, the feedback control law (47) reaches GAS for the system (2) at the origin $\zeta=0$ and ensures $\lim_{t\to+\infty}\zeta(t)=0$.

Next, we will discuss $\mathcal{V}(\zeta)$ and $\mathcal{W}(\zeta)$.

It follows from (45) that

$$\begin{aligned}\mathcal{V}(\zeta)=&\frac{1}{2}\left(\alpha(\zeta)+\sqrt{\alpha^2(\zeta)+\left(\beta^T(\zeta)\beta(\zeta)\right)^2}\right)+\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t)\\&-\zeta^T(t)Bf(\zeta(t))-\zeta^T(t)Cg(\zeta(t-\varsigma))-\zeta^T(t)Dh(\zeta(t-\xi))\\&-\left(Cg(\zeta(t))\right)^T\left(Cg(\zeta(t))\right)-\left(Dh(\zeta(t))\right)^T\left(Dh(\zeta(t))\right)+\zeta^T(t)A\zeta(t)\\&+\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)+\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right).\end{aligned} \tag{51}$$

Substituting (12), (13), (17), (18) and (19) into (51) yields

$$\begin{aligned}\mathcal{V}(\zeta)\geq&\frac{1}{2}\left(\alpha(\zeta)+\sqrt{\alpha^2(\zeta)+\left(\beta^T(\zeta)\beta(\zeta)\right)^2}\right)+\zeta^T(t)A\zeta(t)\\&+\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t)-\zeta^T(t)\left(\frac{3+k^2\|B\|^2+2l^2\|C\|^2+2m^2\|D\|^2}{2}\right)\zeta(t)\\&+\frac{1}{2}\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)+\frac{1}{2}\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right)\\\geq&\frac{1}{2}\left(\sqrt{\alpha^2(\zeta)+\left(\beta^T(\zeta)\beta(\zeta)\right)^2}-\alpha(\zeta)\right)+\frac{1}{2}e^{-t}\zeta^T(t)\zeta(t)+\zeta^T(t)A\zeta(t)\\&+\frac{1}{2}\left(Cg(\zeta(t-\varsigma))\right)^T\left(Cg(\zeta(t-\varsigma))\right)+\frac{1}{2}\left(Dh(\zeta(t-\xi))\right)^T\left(Dh(\zeta(t-\xi))\right)\\\geq&0,\end{aligned} \tag{52}$$

then, this immediately implies that $\mathcal{V}(\zeta)$ is a positive definite and radially unbounded function.

According to (46), one has

$$\begin{aligned}\mathcal{W}(\zeta)=&\frac{1}{2}\gamma^{-1}(\zeta)\\=&\frac{1}{2}\left(\frac{\beta^T(\zeta)\beta(\zeta)}{\alpha(\zeta)+\sqrt{\alpha^2(\zeta)+\left(\beta^T(\zeta)\beta(\zeta)\right)^2}}\right)>0.\end{aligned} \tag{53}$$

In addition, from (45) and (46), one obtains

$$\dot{U}(t,\zeta)=-\mathcal{V}(\zeta)-v^T\mathcal{W}(\zeta)v+(v-v^*)^T\mathcal{W}(\zeta)(v-v^*). \tag{54}$$

Consequently

$$\dot{U}(t,\zeta)+\mathcal{V}(\zeta)+v^T\mathcal{W}(\zeta)v=(v-v^*)^T\mathcal{W}(\zeta)(v-v^*)\geq 0. \tag{55}$$

Namely, $v = v^*$ fulfills

$$\min_{v}\left[\mathcal{V}(\zeta) + v^T\mathcal{W}(\zeta)v + \dot{U}(t,\zeta)\right] = 0. \tag{56}$$

Meanwhile, $U(t,\zeta)$ (8) fulfills the following HJB equation

$$U_t(t,\zeta) + U_\zeta(t,\zeta)\mathcal{P}(\zeta) - \frac{1}{4}U_\zeta(t,\zeta)\mathcal{Q}(\zeta)\mathcal{W}^{-1}(\zeta)\mathcal{Q}^T(\zeta)U_\zeta^T(t,\zeta) + \mathcal{V}(\zeta) = 0, \tag{57}$$

where $\mathcal{P}(\zeta)$ and $\mathcal{Q}(\zeta)$ are the same as (40).

Thus, $U(t,\zeta)$ is the optimal value function of the cost function (48).

According to [43], it concludes that system (2) under the input control (47) reaches ISS. The proof is complete. □

Remark 3.3. We give two stabilization controls (30) and (47) in terms of constructing the appropriate Lyapunov function. Because the stabilization controls have much to do with the design parameters in the practical applications, by choosing suitable parameter θ, $\mathcal{W}(\zeta)$ of (28) becomes a constant under the conditions that all weights of NNs and Lipschitz coefficients of activation functions are constants. Then the control (30) becomes an implementable constant state feedback control.

Remark 3.4. As mentioned in the introduction, the proportional delay differs from other types of delays [11,12,16]. Thus the stabilization results in [11] can not be applied to system (1) or (2). The stabilization control for RNNs with constant delay in terms of LMI was investigated in [11]. It should be noted that, the LMI method has some difficulties in determining the constraint conditions of the network parameters, because it requires further verification of the positive definiteness of some higher-dimensional matrices. Moreover, the (inverse) optimality results in Theorems 3.1 and 3.2 are derived and they provide an optimal value function $U(t,\zeta)$ that is actually a Lyapunov functional for the system (2). The advantage of this approach is that it does not need to solve the HJB equation.

Remark 3.5. It is still an open problem to find the solution of HJB equation (5) for the general nonlinear system (4) [32,35]. Obviously, it is difficult or even impossible to solve HJB equations (40) and (57) for the RNN (2) with unbounded time-varying coefficients and multiple constant delays. In fact, in many cases, the HJB has no solution or the solution is not unique. Therefore, how to design suitable controllers to stabilize the considered system such that the cost function is optimized, is a very important topic for nonlinear systems.

4. Numerical examples

We will illustrate the inverse optimal control via numerical examples in this section.

Example 4.1. The RNN with multiple proportional delays is given

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= -\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 2\\ -3 & 4\end{bmatrix}\begin{bmatrix}\tanh(x_1)\\ \tanh(x_2)\end{bmatrix} + \begin{bmatrix}-2 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}\tanh(p_1x_1)\\ \tanh(p_2x_2)\end{bmatrix}\\ &+ \begin{bmatrix}0.1 & 0.5\\ 0.2 & 0.7\end{bmatrix}\begin{bmatrix}\tanh(q_1x_1)\\ \tanh(q_2x_2)\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}\end{aligned} \tag{58}$$

where $p_1 = p_2 = q_1 = q_2 = 0.4$, $x_{10} = 3, x_{20} = -3$, and $m = n = 2$.

The RNN with multiple proportional delays (58) is not global asymptotic stable under control $v = 0$ in Fig. 1. Moreover, it is easy to show that system (58) achieves ISS and GAS by feedback control v of Theorem 3.1 (Fig. 2). However, system (58) is restrict to the same dimensions of state and input.

Example 4.2. The RNN with multiple proportional delays is given

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= -\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 2\\ -3 & 4\end{bmatrix}\begin{bmatrix}\tanh(x_1)\\ \tanh(x_2)\end{bmatrix} + \begin{bmatrix}-2 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}\tanh(p_1x_1)\\ \tanh(p_2x_2)\end{bmatrix}\\ &+ \begin{bmatrix}0.1 & 0.5\\ 0.2 & 0.7\end{bmatrix}\begin{bmatrix}\tanh(q_1x_1)\\ \tanh(q_2x_2)\end{bmatrix} + \begin{bmatrix}1\\ -1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}\end{aligned} \tag{59}$$

where $x_{10} = 3, x_{20} = -3$, $m = 1$, and $n = 2$. The parameters are the same as in Example 4.1 except for the control input E.

Then system (59) under the feedback control v of Theorem 3.2 reaches ISS and GAS (Fig. 3).

Example 4.3. The RNN with multiple proportional delays is given by

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= -\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}4 & 5\\ -6 & 7\end{bmatrix}\begin{bmatrix}\sin(x_1)\\ \sin(x_2)\end{bmatrix} + \begin{bmatrix}-2 & 1\\ -2 & -1\end{bmatrix}\begin{bmatrix}2\sin\left(p_1x_1\right)\\ 2\sin\left(p_2x_2\right)\end{bmatrix}\\ &+ \begin{bmatrix}0.2 & 0.4\\ 0.2 & 1.8\end{bmatrix}\begin{bmatrix}\sin(q_1x_1)\\ \sin(q_2x_2)\end{bmatrix} + \begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}\end{aligned} \tag{60}$$

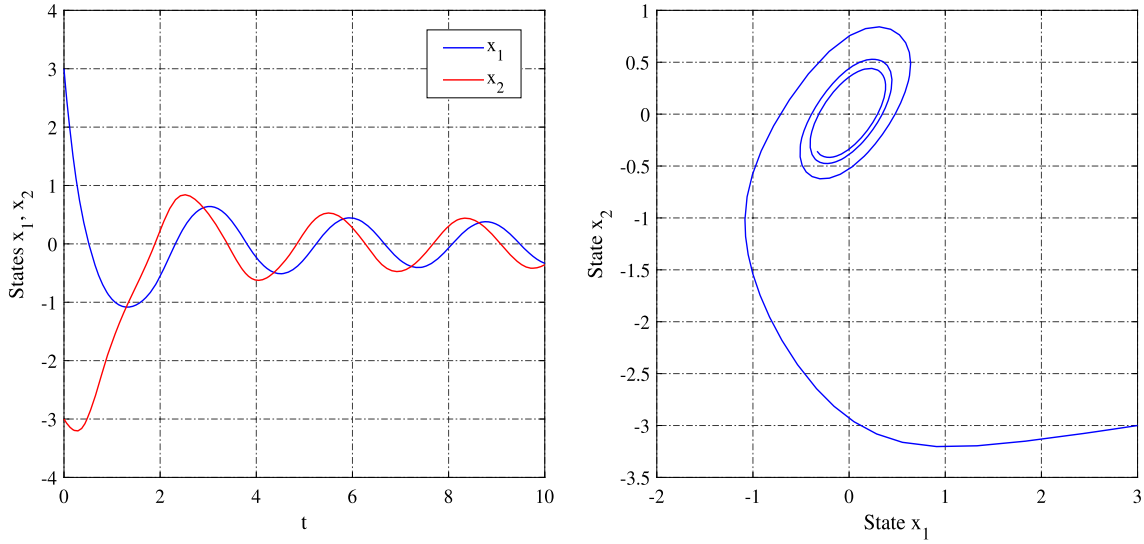

Fig. 1. The trajectories of time response (left), and phase plane (right) under control $v = 0$.

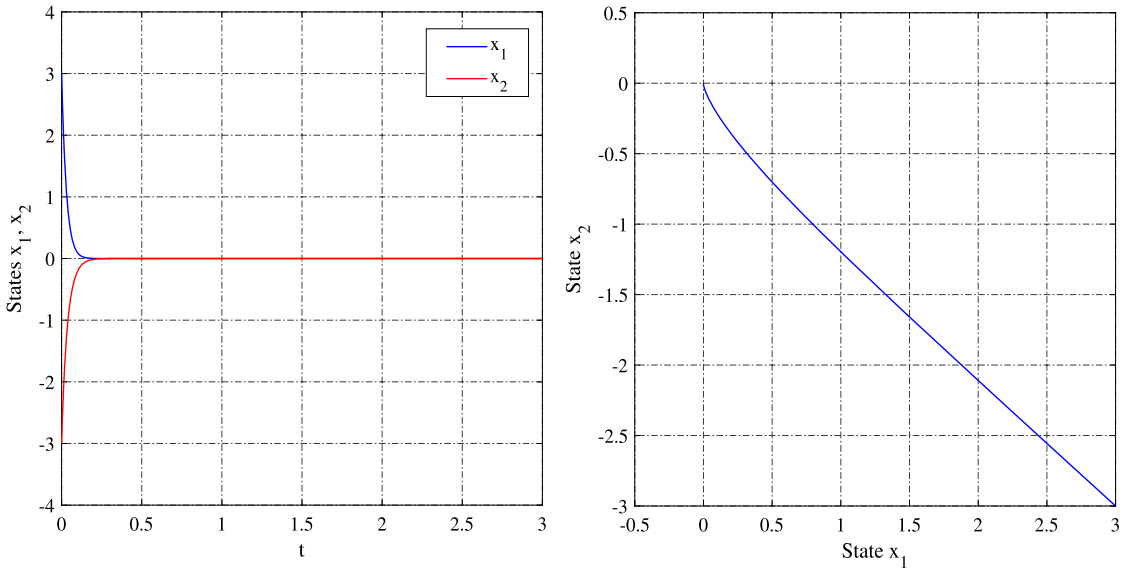

Fig. 2. The trajectories of time response (left), and phase plane (right) under control v of Theorem 3.1.

where $p_1 = p_2 = q_1 = q_2 = 0.5$, $x_{10} = -4, x_{20} = 4$, and $m = n = 2$.

Fig. 4 indicates that RNN with multiple proportional delays (60) is not globally asymptotically stable under control $v = 0$. However, it is easy to show that the system (60) achieves ISS and GAS by feedback control v of Theorem 3.1 (Fig. 5).

Example 4.4. The RNN with multiple proportional delays is given by

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= -\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}4 & 5\\ -6 & 7\end{bmatrix}\begin{bmatrix}\sin(x_1)\\ \sin(x_2)\end{bmatrix} + \begin{bmatrix}-2 & 1\\ -2 & -1\end{bmatrix}\begin{bmatrix}2\sin\left(p_1x_1\right)\\ 2\sin\left(p_2x_2\right)\end{bmatrix}\\ &+ \begin{bmatrix}0.2 & 0.4\\ 0.2 & 1.8\end{bmatrix}\begin{bmatrix}\sin(q_1x_1)\\ \sin(q_2x_2)\end{bmatrix} + \begin{bmatrix}4\\ -4\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}\end{aligned} \tag{61}$$

where $x_{10} = -4, x_{20} = 4, m = 1$, and $n = 2$. The parameters are the same as in Example 4.3 except for the control input E. By using feedback control v of Theorem 3.2, it is obvious that system (61) achieves ISS and GAS (Fig. 6).

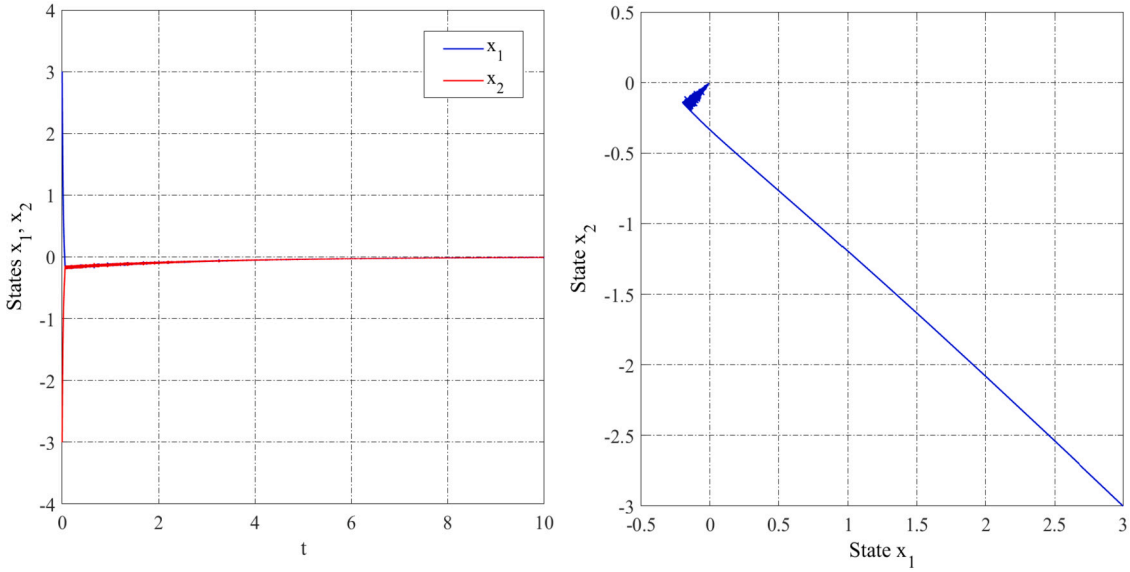

Fig. 3. The trajectories of time response (left), and phase plane (right) under control v of Theorem 3.2.

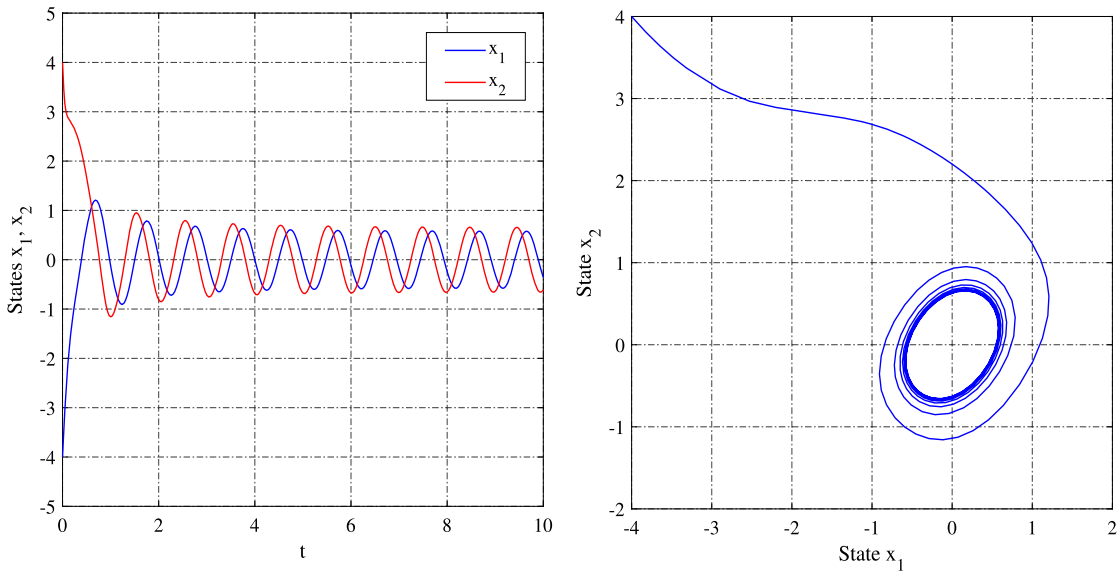

Fig. 4. The trajectories of time response (left), and phase plane (right) under control $v = 0$.

5. Conclusion

This paper has developed a new approach for input-to-state stabilizing control of RNNs with multiple proportional delays, which generalizes the previous results [5,11,34]. The presented designs have been derived from the nonlinear inverse optimal approach that does not require solving the HJB equation. Due to the difficulty in solving HJB equation, it is impossible to design a feedback control to reach optimal stabilization of nonlinear systems. However, inverse optimality provides us with a feasible method to solve such problems by applying the Lyapunov function. According to the Lyapunov function, Sontag's formula and inverse optimality, we have obtained two sufficient conditions of input-to-state stabilizing control for RNNs with multiple proportional delays, which depend on the dimensions of state and input. Several examples are given to show that our approach is simple and effective to be applied in real systems. The approach can also be extended to the more general stochastic nonlinear systems (including continuous and discrete systems) [22,39,45,46]. It is useful and interesting to consider the case where the weight matrixes B, C and D are time-varying in system (1) via convex optimization, inverse optimal control and reinforcement learning [15,47–49]. However, due to the page limit here, we will report these results elsewhere.

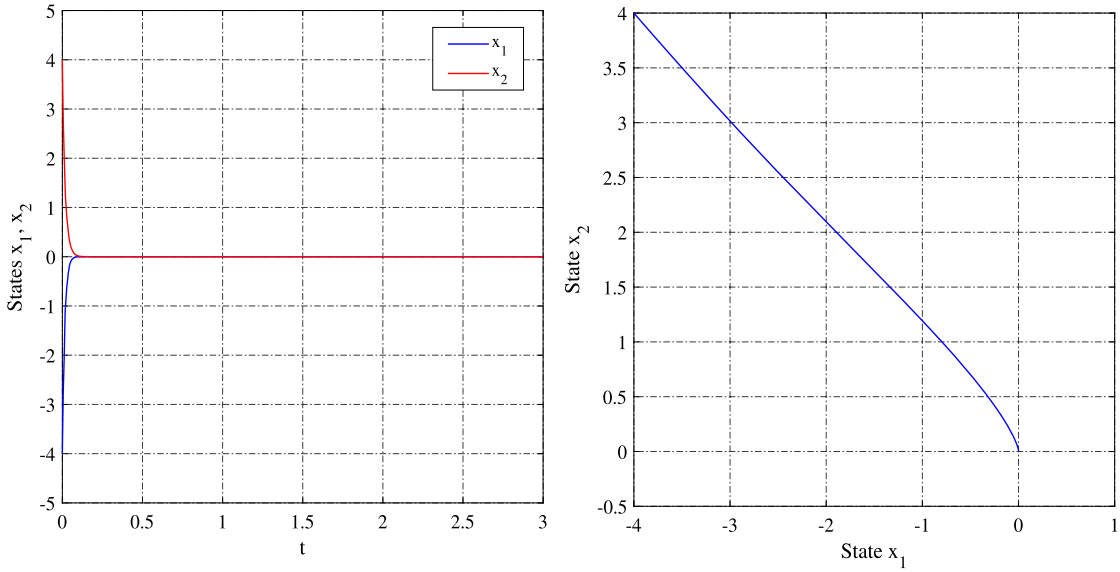

Fig. 5. The trajectories of time response (left), and phase plane (right) under control v of Theorem 3.1.

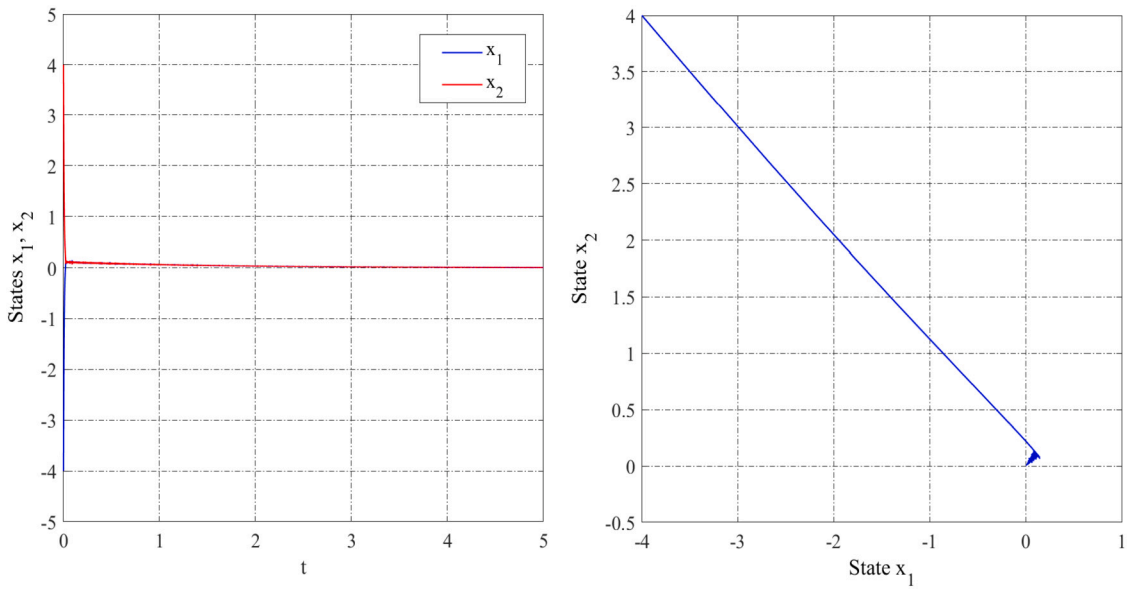

Fig. 6. The trajectories of time response (left), and phase plane (right) under control v of Theorem 3.2.

CRediT authorship contribution statement

Weijun Ma: Conceptualization, Methodology, Writing – original draft, Supervision. **Xuhui Guo:** Conceptualization, Methodology, Writing – original draft, Software. **Huaizhu Wang:** Conceptualization, Methodology. **Yuanshi Zheng:** Conceptualization, Methodology, Writing – reviewing, Visualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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