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Brief paper

Protocol selection for second-order consensus against disturbance[☆]

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ABSTRACT

Noticing that both the absolute and the relative velocity protocols can solve the second-order consensus of multi-agent systems, this paper aims to investigate which of the above two protocols has better anti-disturbance capability, in which the anti-disturbance capability is measured by the $\mathcal{L}_2$ gains from disturbance to consensus errors. More specifically, by the orthogonal transformation technique, the analytic expression of the $\mathcal{L}_2$ gain of a second-order multi-agent system with the absolute velocity protocol is firstly derived, followed by the counterpart with the relative velocity protocol. It is shown that both the $\mathcal{L}_2$ gains for the absolute and the relative velocity protocols are determined only by the minimum non-zero eigenvalues of Laplacian matrices and the tunable gains of position-like and velocity-like states. Then, we establish the graph conditions to tell which protocol has better anti-disturbance capability. Moreover, we propose a two-step scheme to improve the anti-disturbance capability of second-order multi-agent systems. Finally, numerical tests are given for different types of interaction graphs.

1. Introduction

Over the past decades, the distributed coordination of multi-agent systems (MASs) has been extensively investigated in control community. As a fundamental issue in multi-agent coordination, consensus problems have attracted tremendous attention. Roughly speaking, consensus means that a group of agents reach an agreement regarding a common quantity of interest by designing an appropriate communication protocol (Olfati-Saber, Fax, & Murray, 2007).

In retrospect, in the study of consensus problems, agents are assumed to take first-order dynamics in early pioneering works (Jadbabaie, Lin, & Morse, 2003; Olfati-Saber & Murray, 2004; Ren & Beard, 2005). However, the first-order dynamics model is hardly capable of describing many mechanical systems such as holonomic mobile robots, unmanned aerial vehicles and underwater vehicles. Specifically, the dynamic models of the holonomic mobile robots can be feedback linearized as double integrators. Moreover, the unmanned aerial vehicles and the underwater vehicles adjust their motions directly by accelerations rather than speeds. Therefore, numerous researchers have investigated the consensus protocols for second-order MASs to overcome this shortcoming. Subsequently, two classic second-order consensus protocols emerged and employed different treatments to velocity information. One protocol proposed in Xie and Wang (2007) introduced the absolute velocity information of agents themselves as local feedbacks. Another protocol devised in Ren and Atkins (2007) required each agent to use the relative velocity measurements with respect to its neighbors. Then, a series of researches on second-order consensus sprang up based on these two classic protocols. More general forms of second-order consensus protocols were studied in Mei, Ren, and Chen (2015), Yu, Chen, and Cao (2010) and Zhu, Tian, and Kuang (2009). Some results took the communication delays into consideration (Hou, Fu, Zhang, & Wu, 2017; Lin & Jia, 2009; Qin, Gao, & Zheng, 2011). The authors in Ai, Song, and You (2016) addressed the consensus of second-order MASs with limited interaction ranges. In order to reduce the resource consumption, an event-based consensus protocol was developed for second-order MASs in Zhu, Pu, Wang, and Li (2017). The resilient consensus was studied in Dibaji and Ishii (2017) for second-order MASs with

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faulty or malicious agents. The authors in Zhao, Zheng, Liu, and Liu (2021), Zhao, Zheng, and Zhu (2020), Zheng, Zhao, Ma, and Wang (2019), Zheng, Zhu, and Wang (2011) investigated the consensus of heterogeneous and hybrid MASs, in which agents have different dynamics behaviors. Moreover, the game-based consensus of second-order hybrid MASs was considered in Zhou, Liu, Zheng, Xiao, and Xi (2023).

Since the disturbance is rife in reality, it is of great significance to investigate the anti-disturbance capability for second-order MASs with absolute or relative velocity protocols. In literature, the $\mathcal{L}_2$ gain is popular in MAS community to characterize the influence of disturbance on the consensus. For example, by linear matrix inequalities (LMIs) technique, authors in Chen, Zhang, and Zheng (2020), Han, Zhang, and Jiang (2016), Huang, Huang, and Chen (2018), Li, Duan, and Chen (2011), Li, Qin, and Shi (2015), Lin and Jia (2010), Lin, Jia, and Li (2008), Liu and Jia (2012) and Wang, Duan, Li, and Wen (2014) aimed at reaching consensus with a desired $\mathcal{L}_2$ gain. In a separate direction, some studies tapped into the role of networks on the $\mathcal{L}_2$ gain of MASs (Pirani, Sandberg, & Johansson, 2019; Pirani, Shahrivar, Fidan, & Sundaram, 2018; Siami & Motee, 2014; Yang, Wang, & Tan, 2012). In Siami and Motee (2014), the authors built the relation between the $\mathcal{L}_2$ gains of first-order MASs and the minimum non-zero eigenvalues of the Laplacian matrices associated with undirected graphs. As pointed out in Pirani et al. (2018), the $\mathcal{L}_2$ gains of first-order leader–follower MASs on undirected graphs relies on the minimum eigenvalue of the grounded Laplacian matrices. For first-order leader–follower MASs on directed graphs, it was shown in Pirani et al. (2019) that the $\mathcal{L}_2$ gains of first-order leader–follower MASs on directed graphs depends on the minimum singular values of the grounded Laplacian matrices. Besides, authors in Yang et al. (2012) analyzed the properties of $\mathcal{L}_2$ gains for second-order leader–follower MASs with only the relative velocity protocol in the presence of communication errors and measurement errors rather than disturbances. They aimed to obtain the most robust communication topology and optimal scaling factor in a special class of directed tree graphs.

Note that the second-order consensus protocols in all aforementioned literature were founded on the basic structure of the classic absolute velocity protocol (Xie & Wang, 2007) and relative velocity protocol (Ren & Atkins, 2007). Therefore, it is of great significance to study the analytic expressions of $\mathcal{L}_2$ gains for the second-order MASs with general absolute and relative velocity protocols and tell which structure of the above two protocols has better anti-disturbance capability for consensus of second-order MASs. As far as we know, no previous study has investigated this issue.

Motivated by the above observations, this paper aims to develop the protocol selection criteria between the general absolute and relative velocity protocols for better anti-disturbance capability of the second-order consensus. The considered problem is challenging as the general communication topology results in difficulty in establishing the quantitative relations between weighted adjacency matrices, tunable gains and the anti-disturbance capability. In this paper, the anti-disturbance capability is measured by the $\mathcal{L}_2$ gain from disturbance to consensus error, and we intend to establish the quantitative relations between weighted adjacency matrices, tunable gains and anti-disturbance capability for the second-order MASs with absolute and relative velocity protocols, respectively. Furthermore, on the basis of the established quantitative relations, we give the graph conditions of protocol selection for better anti-disturbance capability. Our contributions are summarized as follows:

(1) By the orthogonal transformation technique, we establish the quantitative relations between the weighted adjacency matrices, tunable gains, and the $\mathcal{L}_2$ gains for second-order MASs with general absolute and relative velocity protocols, respectively. It is shown that the $\mathcal{L}_2$ gains are monotonically decreasing with respect to the minimum non-zero eigenvalues of the Laplacian matrices and the tunable state gains, and non-increasing with respect to the tunable velocity gains.
(2) A protocol selection criteria is developed for second-order MASs. We give the graph conditions to tell which one of the absolute and relative velocity protocols has better anti-disturbance capability.
(3) For any given connected undirected graph, we present a two-step scheme to improve the anti-disturbance capability of second-order MASs. It is tractable and highly efficient when networks are unable to rearrange or expand.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and formulate the problem. The quantitative relations between weighted adjacency matrices, tunable gains and $\mathcal{L}_2$ gains, and the graph conditions for better anti-disturbance capability are given in Section 3. In Section 4, numerical tests are given for several different types of communication graphs. We conclude our work in Section 5.

Notations. Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ is the n-dimensional real column vector space, $\mathbb{R}^{m\times n}$ represents the $m \times n$ real matrix space. Denote the all-one and all-zero matrices with appropriate dimensions by $\mathbf{1}$ and $\mathbf{0}$, respectively. Specifically, $\mathbf{1}_n$ and $\mathbf{0}_n$ refer to the $n \times 1$ all ones and all zeros column vectors, respectively. Let I_n be the n-dimensional identity matrix. $\mathbf{j}$ stands for the imaginary unit. For a matrix X, $X^\top$ labels its transpose, X^H denotes its conjugate transpose, and $\sigma_{\max}(X)$ represents its maximum singular value. For a Hermitian matrix X, $\lambda_{\max}(X)$ denotes its maximum eigenvalue. $X \in \mathbb{R}^{n\times n}$ is orthogonal if $X^\top X = XX^\top = I_n$. $\mathrm{diag}\{a_1, a_2, \ldots, a_n\}$ designates a diagonal matrix, where a_i is the i_{th} diagonal element. Define a set $\mathcal{I}_n = \{1, 2, \ldots, n\}$. Null set is represented by $\emptyset$. For given sets $\mathcal{R}_1$ and $\mathcal{R}_2$, $\mathcal{R}_1 \cup \mathcal{R}_2$ and $\mathcal{R}_1 \cap \mathcal{R}_2$ indicate the set union and set intersection, respectively. $\mathcal{L}_2\,[0, \infty)$ dictates the space of square-integrable vector functions, i.e., $f(t) \in \mathcal{L}_2\,[0, \infty)$ if and only if $\int_0^\infty f^\top(t)f(t)\,\mathrm{d}t < \infty$.

2. Preliminaries and problem statement

2.1. Preliminaries

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted undirected graph with n vertices, where $\mathcal{V} = \{s_1, s_2, \ldots, s_n\}$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the set of edges, and $\mathcal{A} = [a_{ij}]_{n\times n}$ is the weighted adjacency matrix with $a_{ij} = a_{ji} \geq 0$. $\varepsilon_{ij} = (s_i, s_j) \in \mathcal{E}$ if and only if there exist information exchanges between vertices s_i and s_j. The adjacency element associated with the edge ε_{ij} is a_{ij}, and $a_{ij} > 0$ if and only if $\varepsilon_{ij} \in \mathcal{E}$. Moreover, suppose that $\mathcal{G}$ has no self-cycles for every node, i.e., $a_{ii} = 0$. A *path* between two distinct vertices v_i and v_j is a finite-ordered sequence of distinct edges in $\mathcal{G}$ with the form $(v_i, v_{k_1}), (v_{k_1}, v_{k_2}), \ldots, (v_{k_l}, v_j)$. An undirected graph is called *connected* if there exists a path between any two distinct vertices of the graph. The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ associated with graph $\mathcal{G}$ is defined as $l_{ii} = \sum_{j=1,j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}, j \neq i$.

The following definitions and lemmas will be utilized to establish our main results.

Definition 1 (*Silva & Zhao, 2016*)**.** For an undirected graph $\mathcal{G}$ with n vertices, the network density d is defined as $d = \frac{\epsilon}{\frac{1}{2}n(n-1)}$, where ϵ represents the total number of undirected edges in the graph $\mathcal{G}$ and $\frac{1}{2}n(n-1)$ is the maximum theoretical number of undirected edges between the n vertices.

Lemma 1 (Jadbabaie et al., 2003)**.** *The Laplacian matrix $L \in \mathbb{R}^{n\times n}$ associated with the undirected graph $\mathcal{G}$ possesses a simple zero eigenvalue with eigenvector $\mathbf{1}_n$ if and only if $\mathcal{G}$ is connected. In addition, all the other non-zero eigenvalues are positive.*

Denote the eigenvalues of Laplacian matrix L associated with graph $\mathcal{G}$ by λ_i, $i \in \mathcal{I}_n$. For convenience, if $\mathcal{G}$ is undirected and connected, suppose that $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n$ and let $\Gamma = \{\lambda_2, \ldots, \lambda_n\}$ be the set of all non-zero eigenvalues of L.

Lemma 2 (Liu & Jia, 2012)**.** *Let $\mathcal{G}$ be a connected undirected graph with a Laplacian matrix $L \in \mathbb{R}^{n\times n}$ and $\Phi_n = [\varphi_{ij}] \in \mathbb{R}^{n\times n}$ be a symmetric matrix whose elements are given as $\varphi_{ii} = \frac{n-1}{n}$ and $\varphi_{ij} = -\frac{1}{n}$, $j \ne i$. Then, there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ with the last column being $\frac{\mathbf{1}_n}{\sqrt{n}}$ such that*

$$Q^\top \Phi_n Q = \bar{\Phi}_n = \begin{bmatrix} I_{n-1} & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^\top & 0 \end{bmatrix} \tag{1}$$

and

$$Q^\top L Q = \bar{L} = \begin{bmatrix} \bar{L}_1 & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^\top & 0 \end{bmatrix}, \tag{2}$$

where $\bar{L}_1 \in \mathbb{R}^{(n-1)\times(n-1)}$ is positive definite.

Definition 2 (Chen, 2000)**.** The H_∞ norm of an asymptotically stable continuous-time transfer matrix $T(s)$ is defined as $\|T(s)\|_\infty = \sup_{\upsilon\in\mathbb{R}} \sigma_{\max}\big[T(\mathbf{j}\upsilon)\big]$.

2.2. Problem statement

In this paper, we consider a MAS consisting of n agents with double-integrator dynamics

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t) + \omega_i(t), i \in \mathcal{I}_n, \end{aligned} \tag{3}$$

where $x_i(t) \in \mathbb{R}$, $v_i(t) \in \mathbb{R}$, $u_i(t) \in \mathbb{R}$, and $\omega_i(t) \in \mathbb{R}$ are the position-like, the velocity-like, the control input and the external disturbance of the i_{th} agent, respectively. In addition, we suppose that $\omega_i(t) \in \mathcal{L}_2[0,\infty)$.

The multi-agent system (3) is said to reach second-order consensus if and only if $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$ and $\lim_{t\to\infty} \|v_i(t) - v_j(t)\| = 0$, $\forall i, j \in \mathcal{I}_n$. In the absence of disturbance, i.e., $\omega_i(t) = 0$, as one can observe from Yu et al. (2010) and Zhu et al. (2009), the general absolute velocity protocol (4) and the general relative velocity protocol (5) can both solve the second-order consensus under the condition that $\mathcal{G}$ is a connected undirected graph.

$$u_i(t) = \alpha \sum_{j=1}^{n} a_{ij}\big[x_j(t) - x_i(t)\big] - \beta v_i(t), \tag{4}$$

$$u_i(t) = \alpha \sum_{j=1}^{n} a_{ij}\big[x_j(t) - x_i(t)\big] + \beta \sum_{j=1}^{n} a_{ij}\big[v_j(t) - v_i(t)\big], \tag{5}$$

where a_{ij} is the $(i,j)_{\text{th}}$ entry of the weighted adjacency matrix $\mathcal{A}$ associated with the undirected graph $\mathcal{G}$, and the positive constants α and β are the tunable state gain and the tunable velocity gain, respectively.

Remark 1. Classic absolute velocity protocol and relative velocity protocol (see Ren & Atkins, 2007 and Xie & Wang, 2007) only considered the velocity gain β. Protocols (4) and (5) maintain their fundamental structure but take the gains α and β both into consideration. Furthermore, (4) and (5) are the most fundamental forms of protocols extracted from existing extensive researches of second-order MASs. For instance, protocols in Ai et al. (2016), Dibaji and Ishii (2017), Han et al. (2016), Hou et al. (2017), Huang et al. (2018), Li et al. (2015), Lin and Jia (2009, 2010), Lin et al. (2008), Mei et al. (2015), Qin et al. (2011), Yang et al. (2012), Yu et al. (2010), Zhao et al. (2021, 2020), Zheng et al. (2019, 2011), Zhou et al. (2023) and Zhu et al. (2017, 2009) were all based on the structure of (4) and (5), and appropriately modified according to different scenarios. Therefore, protocols (4) and (5) are general and representative. We start with a preliminary attempt on protocol selection between the two fundamental protocols, rather than comparing all protocols to them. It will provide the possibility of performance comparison between many different protocols.

Different from existing works, we restrict our attention to investigate which one of the above protocols has better anti-disturbance capability. And we aim to bring forward simple graph conditions for protocol selection.

Define $y_i^x(t) = x_i(t) - \frac{1}{n}\sum_{j=1}^{n} x_j(t)$ and $y_i^v(t) = v_i(t) - \frac{1}{n}\sum_{j=1}^{n} v_j(t)$, $i \in \mathcal{I}_n$, for each agent to measure the consensus errors of position and velocity, respectively. Aggregating the outputs of all agents into a vector $y(t) \in \mathbb{R}^{2n}$ gives rise to $y(t) = [y^{x\top}(t), y^{v\top}(t)]^\top$, where the agglomerate vectors $y^x(t) = [y_1^x(t), \ldots, y_n^x(t)]^\top$ and $y^v(t) = [y_1^v(t), \ldots, y_n^v(t)]^\top$ denote the collective position error and the collective velocity error, respectively. Here, the nominal output $y(t)$ is termed as collective consensus error. Substituting the absolute velocity protocol (4) into (3) and taking $y(t)$ into consideration yield the closed-loop system

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I_n \\ -\alpha L & -\beta I_n \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ I_n \end{bmatrix} \omega(t), \\ y(t) = \begin{bmatrix} \Phi_n & \mathbf{0} \\ \mathbf{0} & \Phi_n \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}, \end{cases} \tag{6}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^\top$, $v(t) = [v_1(t), \ldots, v_n(t)]^\top$, $\omega(t) = [\omega_1(t), \ldots, \omega_n(t)]^\top$, and L is the Laplacian matrix associated with the graph $\mathcal{G}$. Similarly, substituting the relative velocity protocol (5) into (3) and taking $y(t)$ into consideration give the closed-loop system

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I_n \\ -\alpha L & -\beta L \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ I_n \end{bmatrix} \omega(t), \\ y(t) = \begin{bmatrix} \Phi_n & \mathbf{0} \\ \mathbf{0} & \Phi_n \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}. \end{cases} \tag{7}$$

In this paper, we respectively use the $\mathcal{L}_2$ gains from disturbance $\omega(t)$ to collective consensus error $y(t)$ of the systems (6) and (7) to measure the anti-disturbance capabilities of the MAS (3)–(4) and MAS (3)–(5). As shown in Pirani, Nekouei, Sandberg, and Johansson (2022), the $\mathcal{L}_2$ gains of systems (6) and (7) are defined by

$$\sup_{\substack{\omega(t)\ne 0 \\ \omega(t)\in\mathcal{L}_2[0,\infty)}} \sqrt{\frac{\int_0^\infty y^\top(t) y(t)\,\mathrm{d}t}{\int_0^\infty \omega^\top(t)\omega(t)\,\mathrm{d}t}}. \tag{8}$$

However, it is impossible to enumerate innumerable disturbances in $\mathcal{L}_2[0,\infty)$. Therefore, we cannot directly use the $\mathcal{L}_2$ gains to analyze the anti-disturbance capabilities of the considered MASs. Let $T_1(s)$ and $T_2(s)$ be the transfer matrices from the disturbance $\omega(t)$ to the collective consensus error $y(t)$ of the systems (6) and (7), respectively. It follows from Chen (2000) that the $\mathcal{L}_2$ gains of systems (6) and (7) are equal to $\|T_1(s)\|_\infty$ and $\|T_2(s)\|_\infty$, respectively, where $\|T_i(s)\|_\infty$ refers to the H_∞ norm of $T_i(s)$ ($i = 1, 2$). As shown in Definition 2, we can compute $\|T_i(s)\|_\infty$ ($i = 1, 2$) by using frequency-domain analysis. Clearly, smaller H_∞ norm of transfer matrix means better anti-disturbance capability.

In the following, for the connected undirected graph $\mathcal{G}$, we directly use $\|T_1(s)\|_\infty$ and $\|T_2(s)\|_\infty$ to characterize the anti-disturbance capabilities of the MAS (3)–(4) and MAS (3)–(5), respectively. We say that the protocol (4) outperforms the protocol (5) if $\|T_1(s)\|_\infty \le \|T_2(s)\|_\infty$, and the protocol (5) outperforms the protocol (4) if $\|T_2(s)\|_\infty \le \|T_1(s)\|_\infty$. If $\|T_1(s)\|_\infty \equiv \|T_2(s)\|_\infty$, the protocol (4) is said to perform as well as the protocol (5).

3. Main results

In this section, we will establish the graph conditions to tell which protocol has better anti-disturbance for MAS (3).

3.1. Anti-disturbance capability of the second-order MAS using absolute velocity information

Theorem 1 gives the analytic expression of the anti-disturbance capability of the second-order MAS (3) with the absolute velocity protocol (4).

Theorem 1. *Consider the MAS (3)–(4) in which $\mathcal{G}$ is a connected undirected graph with the Laplacian matrix L. Then, we obtain*

$$\|T_1(s)\|_\infty = \begin{cases} \dfrac{1}{\sqrt{(\alpha\lambda_2)^2 - \left[\sqrt{(\alpha\lambda_2+1)^2-\beta^2}-1\right]^2}}, \\ \qquad \text{if } 0<\beta<\sqrt{(\alpha\lambda_2)^2+2\alpha\lambda_2}, \\ \dfrac{1}{\alpha\lambda_2}, \qquad \text{if } \beta \ge \sqrt{(\alpha\lambda_2)^2+2\alpha\lambda_2}. \end{cases} \tag{9}$$

Proof. Since the undirected graph $\mathcal{G}$ is connected, it follows from Lemma 2 that there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that (1) and (2) hold. Then introducing the following orthogonal transformation

$$\begin{bmatrix}\hat{x}(t)\\ \hat{v}(t)\end{bmatrix} = \begin{bmatrix}Q^\top & \mathbf{0}\\ \mathbf{0} & Q^\top\end{bmatrix}\begin{bmatrix}x(t)\\ v(t)\end{bmatrix}, \hat{y}(t) = \begin{bmatrix}Q^\top & \mathbf{0}\\ \mathbf{0} & Q^\top\end{bmatrix} y(t), \quad \hat{\omega}(t) = Q^\top\omega(t), \tag{10}$$

for system (6) gives rise to

$$\begin{cases}\begin{bmatrix}\dot{\hat{x}}(t)\\ \dot{\hat{v}}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{0} & I_n\\ -\alpha\bar{L} & -\beta I_n\end{bmatrix}\begin{bmatrix}\hat{x}(t)\\ \hat{v}(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ I_n\end{bmatrix}\hat{\omega}(t),\\ \hat{y}(t) = \begin{bmatrix}\bar{\Phi}_n & \mathbf{0}\\ \mathbf{0} & \bar{\Phi}_n\end{bmatrix}\begin{bmatrix}\hat{x}(t)\\ \hat{v}(t)\end{bmatrix},\end{cases} \tag{11}$$

where $\bar{L} = \begin{bmatrix}\bar{L}_1 & \mathbf{0}_{n-1}\\ \mathbf{0}_{n-1}^\top & 0\end{bmatrix}$. As evidenced from Lemma 1 and (2), the positive definite matrix $\bar{L}_1$ possesses the same non-zero eigenvalues of L which implies that $-\bar{L}_1$ is Hurwitz stable.

As stated in Definition 2, $\|T_1(s)\|_\infty$ is only defined for the asymptotically stable system. But the system (6) is marginally stable since its system matrix contains the eigenvalue 0. We need to further confirm the existence of $\|T_1(s)\|_\infty$. Clearly, system (11) is composed of an asymptotically stable subsystem of order $2n-2$ and a marginally stable subsystem of order 2. Since the asymptotically stable modes is observable and the marginally stable modes is unobservable from $y(t)$ such that $\int_0^\infty y^\top(t)y(t)\,dt < \infty$, the $\mathcal{L}_2$ gain (8) still exists. Thus, $\|T_1(s)\|_\infty$ still exists and is completely determined by the asymptotically stable subsystem. Consider the asymptotically stable subsystem of (11) taking the form of

$$\begin{cases}\begin{bmatrix}\dot{\hat{x}}^1(t)\\ \dot{\hat{v}}^1(t)\end{bmatrix} = \begin{bmatrix}\mathbf{0} & I_{n-1}\\ -\alpha\bar{L}_1 & -\beta I_{n-1}\end{bmatrix}\begin{bmatrix}\hat{x}^1(t)\\ \hat{v}^1(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ I_{n-1}\end{bmatrix}\hat{\omega}^1(t),\\ \hat{y}^1(t) = \begin{bmatrix}I_{n-1} & \mathbf{0}\\ \mathbf{0} & I_{n-1}\end{bmatrix}\begin{bmatrix}\hat{x}^1(t)\\ \hat{v}^1(t)\end{bmatrix},\end{cases} \tag{12}$$

where $\hat{x}^1(t) = [I_{n-1}\ \mathbf{0}_{n-1}]\hat{x}(t)$, $\hat{v}^1(t) = [I_{n-1}\ \mathbf{0}_{n-1}]\hat{v}(t)$ and $\hat{\omega}^1(t) = [I_{n-1}\ \mathbf{0}_{n-1}]\hat{\omega}(t)$. Since $\bar{L}_1$ is positive definite and possesses the same non-zero eigenvalues of L, according to the spectral theorem (Horn & Johnson, 2012), there exists an orthogonal matrix $V \in \mathbb{R}^{(n-1)\times(n-1)}$ such that $V^\top\bar{L}_1V = \Lambda$, where $\Lambda = \mathrm{diag}\{\lambda_2, \lambda_3, \ldots, \lambda_n\}$ is composed of the non-zero eigenvalues of L. Then performing the following orthogonal transformation

$$\begin{bmatrix}\tilde{x}(t)\\ \tilde{v}(t)\end{bmatrix} = \begin{bmatrix}V^\top & \mathbf{0}\\ \mathbf{0} & V^\top\end{bmatrix}\begin{bmatrix}\hat{x}^1(t)\\ \hat{v}^1(t)\end{bmatrix}, \tilde{y}(t) = \begin{bmatrix}V^\top & \mathbf{0}\\ \mathbf{0} & V^\top\end{bmatrix}\hat{y}^1(t), \quad \tilde{\omega}(t) = V^\top\hat{\omega}^1(t), \tag{13}$$

for the system (12) provides

$$\begin{cases}\begin{bmatrix}\dot{\tilde{x}}(t)\\ \dot{\tilde{v}}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{0} & I_{n-1}\\ -\alpha\Lambda & -\beta I_{n-1}\end{bmatrix}\begin{bmatrix}\tilde{x}(t)\\ \tilde{v}(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ I_{n-1}\end{bmatrix}\tilde{\omega}(t),\\ \tilde{y}(t) = \begin{bmatrix}I_{n-1} & \mathbf{0}\\ \mathbf{0} & I_{n-1}\end{bmatrix}\begin{bmatrix}\tilde{x}(t)\\ \tilde{v}(t)\end{bmatrix}.\end{cases} \tag{14}$$

Denote the transfer matrices of the systems (11), (12) and (14) by $T_3(s)$, $T_4(s)$ and $T_5(s)$, respectively. Then, we can obtain

$$T_3(s) = \begin{bmatrix}Q^\top & \mathbf{0}\\ \mathbf{0} & Q^\top\end{bmatrix}T_1(s)Q = \begin{bmatrix}\frac{1}{s}\Psi & \mathbf{0}_{n-1}\\ \mathbf{0}_{n-1}^\top & 0\\ \Psi & \mathbf{0}_{n-1}\\ \mathbf{0}_{n-1}^\top & 0\end{bmatrix}$$

and $T_4(s) = \begin{bmatrix}V & \mathbf{0}\\ \mathbf{0} & V\end{bmatrix}T_5(s)V^\top = \begin{bmatrix}\frac{1}{s}\Psi\\ \Psi\end{bmatrix}$, where $\Psi = [(s+\beta)I_{n-1} + \frac{\alpha}{s}\bar{L}_1]^{-1}$. It follows from $T_3^H(\mathbf{j}\upsilon)T_3(\mathbf{j}\upsilon) = Q^\top T_1^H(\mathbf{j}\upsilon)T_1(\mathbf{j}\upsilon)Q$, $T_4^H(\mathbf{j}\upsilon)T_4(\mathbf{j}\upsilon) = VT_5^H(\mathbf{j}\upsilon)T_5(\mathbf{j}\upsilon)V^\top$ and $T_3^H(\mathbf{j}\upsilon)T_3(\mathbf{j}\upsilon) = \begin{bmatrix}T_4^H(\mathbf{j}\upsilon)T_4(\mathbf{j}\upsilon) & \mathbf{0}_{n-1}\\ \mathbf{0}_{n-1}^\top & 0\end{bmatrix}$ that $\lambda_{\max}[T_1^H(\mathbf{j}\upsilon)T_1(\mathbf{j}\upsilon)] = \lambda_{\max}[T_3^H(\mathbf{j}\upsilon)T_3(\mathbf{j}\upsilon)] = \lambda_{\max}[T_4^H(\mathbf{j}\upsilon)T_4(\mathbf{j}\upsilon)] = \lambda_{\max}[T_5^H(\mathbf{j}\upsilon)T_5(\mathbf{j}\upsilon)]$. Therefore, according to Definition 2, we can deduce that $\|T_1(s)\|_\infty = \|T_3(s)\|_\infty = \|T_4(s)\|_\infty = \|T_5(s)\|_\infty$.

Then, we turn to compute $\|T_5(s)\|_\infty$. The transfer matrix $T_5(s)$ is shown as $T_5(s) = [\Xi^\top\ \Upsilon^\top]^\top$, where $\Xi = \mathrm{diag}\{\Xi_2, \ldots, \Xi_n\}$, $\Upsilon = \mathrm{diag}\{\Upsilon_2, \ldots, \Upsilon_n\}$, $\Xi_i = \frac{1}{s^2+\beta s+\alpha\lambda_i}$ and $\Upsilon_i = \frac{s}{s^2+\beta s+\alpha\lambda_i}$, $i = 2, \ldots, n$. It can be obtained that $T_5^H(\mathbf{j}\upsilon)T_5(\mathbf{j}\upsilon) = \mathrm{diag}\{\delta_2(\upsilon), \ldots, \delta_n(\upsilon)\}$, where $\delta_i(\upsilon) = \frac{1+\upsilon^2}{(\alpha\lambda_i-\upsilon^2)^2+(\beta\upsilon)^2} > 0$, $i = 2, \ldots, n$, $\upsilon \in \mathbb{R}$. According to Definition 2 and $\|T_1(s)\|_\infty = \|T_5(s)\|_\infty$, $\|T_1(s)\|_\infty$ can be treated as

$$\begin{aligned}\|T_1(s)\|_\infty &= \sup_{\upsilon\in\mathbb{R}}\sqrt{\lambda_{\max}\left[T_5^H(\mathbf{j}\upsilon)T_5(\mathbf{j}\upsilon)\right]}\\ &= \max_{i=2,\ldots,n}\sqrt{\sup_{\upsilon\in\mathbb{R}}\delta_i(\upsilon)}.\end{aligned} \tag{15}$$

It follows from $\frac{d\delta_i(\upsilon)}{d\upsilon} = \frac{-2\upsilon\left[\upsilon^4+2\upsilon^2+\beta^2-(\alpha\lambda_i)^2-2\alpha\lambda_i\right]}{\left[\upsilon^4+(\beta^2-2\alpha\lambda_i)\upsilon^2+(\alpha\lambda_i)^2\right]^2} = 0$ that

$$\upsilon\left[\upsilon^4 + 2\upsilon^2 + \beta^2 - (\alpha\lambda_i)^2 - 2\alpha\lambda_i\right] = 0. \tag{16}$$

Obviously, $\sup_{\upsilon\in\mathbb{R}}\delta_i(\upsilon)$ depends on $\beta^2 - (\alpha\lambda_i)^2 - 2\alpha\lambda_i$. If $\beta \ge \sqrt{(\alpha\lambda_i)^2+2\alpha\lambda_i}$, by solving (16), we can obtain $\sup_{\upsilon\in\mathbb{R}}\delta_i(\upsilon) = \delta_i(\upsilon_{i,1}^*) = g_1(\lambda_i) = \frac{1}{(\alpha\lambda_i)^2}$, where $\upsilon_{i,1}^* = 0$. If $\beta < \sqrt{(\alpha\lambda_i)^2+2\alpha\lambda_i}$, by solving (16), it can be concluded that $\sup_{\upsilon\in\mathbb{R}}\delta_i(\upsilon) = \delta_i(\upsilon_{i,2}^*) = \delta_i(\upsilon_{i,3}^*) = g_2(\lambda_i) = \frac{1}{(\alpha\lambda_i)^2-(\sqrt{\Delta_i}-1)^2}$, where $\upsilon_{i,2}^*, \upsilon_{i,3}^* = \pm\sqrt{\sqrt{\Delta_i}-1}$, and $\Delta_i = (\alpha\lambda_i+1)^2 - \beta^2$.

Building on these preliminary observations, we refer to $\mathcal{R}_1 = \{r \in \Gamma \mid \beta \ge \sqrt{(\alpha r)^2+2\alpha r}\}$ and $\mathcal{R}_2 = \{r \in \Gamma \mid \beta < \sqrt{(\alpha r)^2+2\alpha r}\}$ as two sets of non-zero eigenvalues of L. Recall that $\Gamma = \{\lambda_2, \ldots, \lambda_n\}$ is the set of all the non-zero eigenvalues of L. Obviously, $\mathcal{R}_1 \cup \mathcal{R}_2 = \Gamma$ and $\mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset$ hold. We will complete the proof by enumeration.

(I) $\mathcal{R}_1 = \Gamma$ and $\mathcal{R}_2 = \emptyset$;

In this case, we have $\beta \geq \sqrt{(\alpha\lambda_i)^2 + 2\alpha\lambda_i}, \forall i \in \{2, \ldots, n\}$ which leads to $\sup_{\upsilon\in\mathbb{R}} \delta_i(\upsilon) = g_1(\lambda_i), \forall i \in \{2, \ldots, n\}$. According to Lemma 3 in Appendix A, $g_1(t)$ is decreasing on $(0, +\infty)$. Combining with $0 < \lambda_2 \leq \cdots \leq \lambda_n$, one can deduce that $\max_{i=2,\ldots,n} g_1(\lambda_i) = g_1(\lambda_2)$. Therefore, (15) can be written as $\|T_1(s)\|_\infty = \max_{i=2,\ldots,n} \sqrt{g_1(\lambda_i)} = \sqrt{g_1(\lambda_2)} = \frac{1}{\alpha\lambda_2}$.

(II) $\mathcal{R}_1 = \emptyset$ and $\mathcal{R}_2 = \Gamma$;

Under this circumstance, we can obtain that $\beta < \sqrt{(\alpha\lambda_i)^2 + 2\alpha\lambda_i}, \forall i \in \{2, \ldots, n\}$ which leads to $\lambda_i > \frac{\sqrt{1+\beta^2}-1}{\alpha} > \frac{\beta-1}{\alpha}, \forall i \in \{2, \ldots, n\}$ and $\sup_{\upsilon\in\mathbb{R}} \delta_i(\upsilon) = g_2(\lambda_i), \forall i \in \{2, \ldots, n\}$. According to Lemma 3 in Appendix A, $g_2(t)$ is decreasing on $(\frac{\beta-1}{\alpha}, +\infty)$. Combining with $\frac{\beta-1}{\alpha} < \lambda_2 \leq \cdots \leq \lambda_n$, we have $\max_{i=2,\ldots,n} g_2(\lambda_i) = g_2(\lambda_2)$. Thus, (15) becomes $\|T_1(s)\|_\infty = \max_{i=2,\ldots,n} \sqrt{g_2(\lambda_i)} = \sqrt{g_2(\lambda_2)} = \frac{1}{\sqrt{(\alpha\lambda_2)^2 - (\sqrt{\Delta_2}-1)^2}}$.

(III) $\mathcal{R}_1 \neq \emptyset$ and $\mathcal{R}_2 \neq \emptyset$;

In this case, there must exist an eigenvalue λ_l $(2 \leq l \leq n-1)$ of L such that $\mathcal{R}_1 = \{\lambda_2, \ldots, \lambda_l\}$, $\mathcal{R}_2 = \{\lambda_{l+1}, \ldots, \lambda_n\}$ and $\lambda_2 < \lambda_{l+1}$. Thus, we can get $\beta \geq \sqrt{(\alpha\lambda_i)^2 + 2\alpha\lambda_i}, \forall i \in \{2, \ldots, l\}$ and $\beta < \sqrt{(\alpha\lambda_i)^2 + 2\alpha\lambda_i}, \forall i \in \{l+1, \ldots, n\}$ which respectively imply that $\sup_{\upsilon\in\mathbb{R}} \delta_i(\upsilon) = g_1(\lambda_i), \forall i \in \{2, \ldots, l\}$ and $\sup_{\upsilon\in\mathbb{R}} \delta_i(\upsilon) = g_2(\lambda_i), \forall i \in \{l+1, \ldots, n\}$. Moreover, it is inferred from $\beta < \sqrt{(\alpha\lambda_i)^2 + 2\alpha\lambda_i}, \forall i \in \{l+1, \ldots, n\}$ that $\lambda_i > \frac{\sqrt{1+\beta^2}-1}{\alpha} > \frac{\beta-1}{\alpha}, \forall i \in \{l+1, \ldots, n\}$. Since $0 < \lambda_2 \leq \cdots \leq \lambda_l$ and $\frac{\beta-1}{\alpha} < \lambda_{l+1} \leq \cdots \leq \lambda_n$, it follows from Lemma 3 in Appendix A that $\max_{i=2,\ldots,l} g_1(\lambda_i) = g_1(\lambda_2)$ and $\max_{i=l+1,\ldots,n} g_2(\lambda_i) = g_2(\lambda_{l+1})$. Therefore, based on above facts, (15) can be written as $\|T_1(s)\|_\infty = \max\{\max_{i=2,\ldots,l} \sqrt{g_1(\lambda_i)}, \max_{i=l+1,\ldots,n} \sqrt{g_2(\lambda_i)}\} = \max\{\sqrt{g_1(\lambda_2)}, \sqrt{g_2(\lambda_{l+1})}\}$.

Let $\xi = \sqrt{(\alpha\lambda_{l+1}+1)^2 - (\alpha\lambda_2+1)^2 + 1}$. Since $\lambda_2 < \lambda_{l+1}$ and $\alpha > 0$, we have $2\alpha\lambda_{l+1} - \alpha\lambda_2 > \alpha\lambda_2$ which gives rise to

$$
\begin{aligned}
2\xi &= 2\sqrt{(\alpha\lambda_{l+1})^2 - (\alpha\lambda_2)^2 + 2\alpha\lambda_{l+1} - 2\alpha\lambda_2 + 1} \\
&> 2\sqrt{(\alpha\lambda_{l+1})^2 - \alpha\lambda_2(2\alpha\lambda_{l+1} - \alpha\lambda_2) + 2\alpha\lambda_{l+1} - 2\alpha\lambda_2 + 1} \\
&= 2\sqrt{(\alpha\lambda_{l+1} - \alpha\lambda_2 + 1)^2} = 2(\alpha\lambda_{l+1} - \alpha\lambda_2 + 1) > 0.
\end{aligned}
$$

Therefore, it follows from $\beta \geq \sqrt{(\alpha\lambda_2)^2 + 2\alpha\lambda_2}$ that

$$
\begin{aligned}
g_2(\lambda_{l+1}) &\leq \frac{1}{(\alpha\lambda_{l+1})^2 - (\xi-1)^2} \\
&= \frac{1}{(\alpha\lambda_2)^2 - 2(\alpha\lambda_{l+1} - \alpha\lambda_2 + 1) + 2\xi} \\
&< \frac{1}{(\alpha\lambda_2)^2} = g_1(\lambda_2).
\end{aligned}
$$

Thus, we get $\|T_1(s)\|_\infty = \max\{\sqrt{g_1(\lambda_2)}, \sqrt{g_2(\lambda_{l+1})}\} = \sqrt{g_1(\lambda_2)} = \frac{1}{\alpha\lambda_2}$.

To sum up the above cases, we have $\|T_1(s)\|_\infty = \frac{1}{\alpha\lambda_2}$ as long as $\lambda_2 \in \mathcal{R}_1$, i.e., $\beta \geq \sqrt{(\alpha\lambda_2)^2 + 2\alpha\lambda_2}$. Otherwise, $\|T_1(s)\|_\infty = \frac{1}{\sqrt{(\alpha\lambda_2)^2 - (\sqrt{\Delta_2}-1)^2}}$. Namely, (9) is derived. □

3.2. Anti-disturbance capability of the second-order MAS using relative velocity information

The following theorem gives the analytic expression of the anti-capability of the second-order MAS (3) with the relative velocity protocol (5).

Theorem 2. *Consider the MAS (3)–(5) in which G is a connected undirected graph with L being its Laplacian matrix. Then, we obtain*

$$
\|T_2(s)\|_\infty = \begin{cases} \dfrac{1}{\sqrt{(\alpha\lambda_2)^2 - \left[\sqrt{(\alpha\lambda_2+1)^2 - (\beta\lambda_2)^2} - 1\right]^2}}, & \text{if } 0 < \beta < \sqrt{\alpha^2 + \dfrac{2\alpha}{\lambda_2}}, \\ \dfrac{1}{\alpha\lambda_2}, & \text{if } \beta \geq \sqrt{\alpha^2 + \dfrac{2\alpha}{\lambda_2}}. \end{cases} \quad (17)
$$

Proof. The proof is provided in Appendix B. □

Remark 2. It is observed from Theorems 1 and 2 that $\|T_1(s)\|_\infty$ and $\|T_2(s)\|_\infty$ are both monotonically decreasing with respect to λ_2 for any given tunable gains, which is consistent with the corresponding result for first-order MASs (Siami & Motee, 2014). In other words, optimizing networks to generate a larger λ_2 is still valid to improve the anti-disturbance capability for second-order MASs.

3.3. Protocol selection for better anti-disturbance capability

According to the analytic expressions presented in Theorems 1 and 2, in Theorem 3 we will establish the graph conditions of protocol selection for better anti-disturbance capability.

Theorem 3. *Consider the second-order MAS (3) on a connected undirected graph $\mathcal{G}$ with the Laplacian matrix L. Then, we conclude that*

(1) Protocol (4) outperforms protocol (5) if $\lambda_2 < 1$;
(2) Protocol (5) outperforms protocol (4) if $\lambda_2 > 1$;
(3) Protocol (4) performs as well as protocol (5) if $\lambda_2 = 1$.

Proof. Firstly, we intend to prove the conclusion (3). By substituting $\lambda_2 = 1$ into (9) and (17), respectively, we can verify that $\|T_1(s)\|_\infty \equiv \|T_2(s)\|_\infty$ for any $\alpha > 0$ and $\beta > 0$. Therefore, we can say that the protocol (4) performs as well as the protocol (5).

Next, we will prove the conclusion (1). Let $p = \sqrt{(\alpha\lambda_2)^2 + 2\alpha\lambda_2}$ and $q = \sqrt{\alpha^2 + \frac{2\alpha}{\lambda_2}}$. Note that $p = \lambda_2 q$. Recall that $\Delta_2 = (\alpha\lambda_2+1)^2 - \beta^2$ and $\Theta_2 = (\alpha\lambda_2+1)^2 - (\beta\lambda_2)^2$. It follows from $\lambda_2 < 1$ that $p < q$ and $\Delta_2 < \Theta_2$. We will complete the proof of conclusion (1) by enumeration.

(I) $\beta < p < q$;

As seen in Theorems 1 and 2, we can obtain $\|T_1(s)\|_\infty = \frac{1}{\sqrt{(\alpha\lambda_2)^2 - (\sqrt{\Delta_2}-1)^2}}$ and $\|T_2(s)\|_\infty = \frac{1}{\sqrt{(\alpha\lambda_2)^2 - (\sqrt{\Theta_2}-1)^2}}$. Since $\beta < p$ and $\Delta_2 < \Theta_2$, we have $1 < \Delta_2 < \Theta_2$. It follows that $\sqrt{(\alpha\lambda_2)^2 - (\sqrt{\Theta_2}-1)^2} < \sqrt{(\alpha\lambda_2)^2 - (\sqrt{\Delta_2}-1)^2}$ which means that $\|T_1(s)\|_\infty < \|T_2(s)\|_\infty$.

(II) $p \leq \beta < q$;

In terms of Theorems 1 and 2, one can obtain that $\|T_1(s)\|_\infty = \frac{1}{\alpha\lambda_2}$ and $\|T_2(s)\|_\infty = \frac{1}{\sqrt{(\alpha\lambda_2)^2 - (\sqrt{\Theta_2}-1)^2}}$. It is inferred from $\sqrt{(\alpha\lambda_2)^2 - (\sqrt{\Theta_2}-1)^2} < \sqrt{(\alpha\lambda_2)^2}$ that $\|T_1(s)\|_\infty < \|T_2(s)\|_\infty$.

(III) $p < q \leq \beta$.

It follows from Theorems 1 and 2 that $\|T_1(s)\|_\infty = \|T_2(s)\|_\infty = \frac{1}{\alpha\lambda_2}$.

In summary, $\|T_1(s)\|_\infty \leq \|T_2(s)\|_\infty$ holds for any $\alpha > 0$ and $\beta > 0$ when $\lambda_2 < 1$. Therefore, we can say that the protocol (4) outperforms the protocol (5) if $\lambda_2 < 1$. Similarly, if $\lambda_2 > 1$, we can also conclude that the protocol (5) outperforms the protocol (4). For saving space, we omit it. □

As one can see, the protocol selection for better anti-disturbance capability exclusively relies on the graph-theoretic feature λ_2. It means that we do not have to execute complicated calculations and comparisons on $\|T_1(s)\|_\infty$ and $\|T_2(s)\|_\infty$ for all positive tunable gains α and β, the graph conditions about λ_2 are more concise and tractable. Furthermore, better protocol structure implies more or at least the same performance improvement by equally tuning gains.

Additionally, our results embody a twofold approach for improving the anti-disturbance capability of the MAS (3). In a fixed communication network scenario, we can first opt for a better communication protocol in terms of Theorem 3. Then the importance of the tunable gains α and β now comes to the fore. As shown in (9) and (17), $\|T_1(s)\|_\infty$ and $\|T_2(s)\|_\infty$ can be viewed as the functions of tunable gains α and β. The partial derivatives of $\|T_1(s)\|_\infty$ and $\|T_2(s)\|_\infty$ with respect to α and β are all continuous, and we have $\frac{\partial\|T_1(s)\|_\infty}{\partial\alpha} < 0$, $\frac{\partial\|T_1(s)\|_\infty}{\partial\beta} \le 0$, $\frac{\partial\|T_2(s)\|_\infty}{\partial\alpha} < 0$ and $\frac{\partial\|T_2(s)\|_\infty}{\partial\beta} \le 0$, where the equations $\frac{\partial\|T_1(s)\|_\infty}{\partial\beta} = 0$ and $\frac{\partial\|T_2(s)\|_\infty}{\partial\beta} = 0$ hold if and only if $\beta \ge \sqrt{(\alpha\lambda_2)^2 + 2\alpha\lambda_2}$ and $\beta \ge \sqrt{\alpha^2 + \frac{2\alpha}{\lambda_2}}$, respectively. Therefore, we are able to further improve the anti-disturbance capability by increasing the tunable gain α or β. Nevertheless, on account of $\lim_{\alpha\to\infty}\|T_1(s)\|_\infty = \frac{1}{\beta}$, $\lim_{\beta\to\infty}\|T_1(s)\|_\infty = \frac{1}{\alpha\lambda_2}$, $\lim_{\alpha\to\infty}\|T_2(s)\|_\infty = \frac{1}{\beta\lambda_2}$ and $\lim_{\beta\to\infty}\|T_2(s)\|_\infty = \frac{1}{\alpha\lambda_2}$, adjusting single tunable gain to improve the anti-disturbance capability is limited. Consequently, the optimal anti-disturbance capability is obtained when both tunable gains are sufficiently large.

Remark 3. In this paper, we use $\mathcal{L}_2$ gain as the measure of anti-disturbance capability. It is proved that apart from α and β, the analytic expressions (9) and (17) of the $\mathcal{L}_2$ gains both depend on λ_2 rather than the whole spectrum of the Laplacian matrix L. The graph conditions for protocol selection are derived from comparing (9) and (17) under the same tunable gains. Therefore, the protocol selection only depends on λ_2 rather than the whole spectrum of L. If we use other performance indices, the results may be different.

Remark 4. In addition, for a connected undirected graph $\mathcal{G}$, directly comparing the partial derivatives yields

$$\begin{cases} \frac{\partial\|T_i(s)\|_\infty}{\partial\alpha} < \frac{\partial\|T_i(s)\|_\infty}{\partial\beta} = 0, & \text{if } \phi_i(\alpha,\beta) \ge 0 \\ \frac{\partial\|T_i(s)\|_\infty}{\partial\alpha} < \frac{\partial\|T_i(s)\|_\infty}{\partial\beta} < 0, & \text{if } \phi_i(\alpha,\beta) < 0,\ \psi_i(\alpha,\beta) < 0 \\ \frac{\partial\|T_i(s)\|_\infty}{\partial\alpha} = \frac{\partial\|T_i(s)\|_\infty}{\partial\beta} < 0, & \text{if } \phi_i(\alpha,\beta) < 0,\ \psi_i(\alpha,\beta) = 0 \\ \frac{\partial\|T_i(s)\|_\infty}{\partial\beta} < \frac{\partial\|T_i(s)\|_\infty}{\partial\alpha} < 0, & \text{if } \phi_i(\alpha,\beta) < 0,\ \psi_i(\alpha,\beta) > 0 \\ i = 1, 2, \end{cases}$$

where $\psi_1(\alpha,\beta) = \alpha\lambda_2^2 - (\lambda_2+\beta)[\sqrt{(\alpha\lambda_2+1)^2-\beta^2} - 1]$, $\psi_2(\alpha,\beta) = \alpha\lambda_2^2 - (\lambda_2+\beta\lambda_2^2)[\sqrt{(\alpha\lambda_2+1)^2-(\beta\lambda_2)^2} - 1]$, $\phi_1(\alpha,\beta) = \beta - \sqrt{(\alpha\lambda_2)^2+2\alpha\lambda_2}$ and $\phi_2(\alpha,\beta) = \beta - \sqrt{\alpha^2+\frac{2\alpha}{\lambda_2}}$. According to $\phi_i(\alpha,\beta)$ and $\psi_i(\alpha,\beta)$, we can judge which gain is more critical. For instance, when $\phi_i(\alpha,\beta) < 0$ and $\psi_i(\alpha,\beta) > 0$ hold for given tunable gains, increasing gain β producing higher performance improvement than equally increasing gain α since $\frac{\partial\|T_i(s)\|_\infty}{\partial\beta} < \frac{\partial\|T_i(s)\|_\infty}{\partial\alpha} < 0$. It means that the gain β is more critical than the gain α in this case. In other words, one can be guided by $\phi_i(\alpha,\beta)$ and $\psi_i(\alpha,\beta)$ to prioritize increasing more critical gain.

Remark 5. The most fundamental criterion that distinguishes protocol (4) from (5) is the way of using velocity information rather than the tunable gains α and β. $v_i(t)$ and its counterpart $\sum_{j=1}^{n} a_{ij}[v_j(t) - v_i(t)]$ play the same role in dealing with velocity information. We only aim to reveal which is the better structure to use velocity information against disturbance. Therefore, we comply with the control variate method in the proof of Theorem 3. It inherently requires α and β to be the same for both protocols, and then study the effect of the way using velocity information on anti-disturbance capability. Although we can choose particular tunable gains such that a protocol outperforms the other one, the performance improvement is caused by the tunable gains rather than the structure of protocols. Tuning gains violates the rule of control variate method and covers the role of the structure of protocol. Furthermore, the tunable gains are usually implemented by proportional elements that are independent of the local measurements. Improving performance by tuning gains needs additional costs. In brief, protocols (4) and (5) should equally weight the state information and the velocity information in order to perform a fair performance comparison. Our result lays a foundation for designing better protocol from the view of underlying structure when facing more realistic and complex scenario. Furthermore, although it is intuitively true that the protocol selection may be related to the network topology because the two protocols differ in whether using the velocity information from neighbors, it is nontrivial to verify this fact from a theoretical perspective. However, Theorem 3 tells the overwhelming conclusion that the protocol selection indeed depends on the graph-theoretic feature λ_2.

Remark 6. Our results in this paper are valid only for connected undirected graphs. The difficulties that extending the results to the case of directed graphs are twofold. Firstly, most Laplacian matrices associated with directed graphs are not orthogonally diagonalizable such that the frequency-domain analysis used in this paper is hard to perform. Secondly, the asymmetry of the directed graph makes the $\mathcal{L}_2$ gain very sensitive to the real part, the imaginary part and the modulus of the complex eigenvalues of the Laplacian matrix. Therefore, it is difficult to obtain a general analytic expression of the $\mathcal{L}_2$ gain for all directed graphs. Then the protocol selection by quantitative comparison is also hard to realize, which may be solved by qualitative comparison in future works.

Remark 7. The protocol selection approach embodies potential advantages in practical applications. Specifically, if networks are unable to rearrange or expand, which means that modifying networks to improve the anti-disturbance capability is impracticable, we can still optimize the anti-disturbance capability by manipulating every agents to follow the identical optimal communication protocol. Moreover, for a certain control task of a MAS, selecting from existing practicable protocols rather than designing a new protocol is more tractable and highly efficient in the distributed scenario.

4. Numerical tests

We discuss the graph conditions for protocol selection on some different types of communication graphs. Consider the following well-known graphs with $n \ge 2$ vertices and 0-1 edge weights, which include the undirected complete graphs K_n, the undirected star graphs S_n, the undirected path graphs P_n, and the undirected $2k$-regular ring lattices $C_{k,n}$. It should be stressed that $C_{k,n}$ $(n \ge 2k+1)$ are highly structured networks with nodes placed on a ring, each connecting to its $2k$ nearest neighbors. For more details about those networks, one can refer to Lewis, Zhang, Hengster-Movric, and Das (2014) and Wu (2007). For above networks, Table 1 gives the analytic expressions of the minimum non-zero eigenvalue and the network density.

In addition, we summarize the relationship between the number of agents and our graph conditions for these graphs in Table 1. Then, we can directly select better protocol according to the

Table 1
The analytic expressions of the minimum non-zero eigenvalue and the network density.

Graph	Value of λ_2	Network density d	$\lambda_2 < 1$	$\lambda_2 = 1$	$\lambda_2 > 1$
K_n	$\lambda_2 = n$	$d = 1$	✗	✗	$n \geq 2$
S_n	$\lambda_2 = 1\ (n \geq 3)$ $\lambda_2 = 2\ (n = 2)$	$d = \frac{2}{n}$	✗	$n \geq 3$	$n = 2$
P_n	$\lambda_2 = 4\sin^2(\frac{\pi}{2n})$	$d = \frac{2}{n}$	$n > 3$	$n = 3$	$n = 2$
$C_{1,n}$	$\lambda_2 = 3 - \frac{\sin\frac{3\pi}{n}}{\sin\frac{\pi}{n}}$	$d = \frac{2}{n-1}$	$n \geq 7$	$n = 6$	$n \leq 5$
$C_{2,n}$	$\lambda_2 = 5 - \frac{\sin\frac{5\pi}{n}}{\sin\frac{\pi}{n}}$	$d = \frac{4}{n-1}$	$n \geq 14$	✗	$n \leq 13$
$C_{3,n}$	$\lambda_2 = 7 - \frac{\sin\frac{7\pi}{n}}{\sin\frac{\pi}{n}}$	$d = \frac{6}{n-1}$	$n \geq 24$	✗	$n \leq 23$

number of agents. Furthermore, our graph conditions seem to be consistent with the variation of the network density d. The network density d is diminishing with the growth in the number of nodes except the complete graphs which retain the highest density all through. It is obvious that absolute velocity protocol (4) can be viewed as the optimal selection when the network density d is not higher than a certain threshold, otherwise the relative velocity protocol (5) is always the best. And the threshold varies from different families of graphs.

5. Conclusion

In this paper, we investigated the anti-disturbance capability for the second-order MASs with the absolute velocity protocol and the relative velocity protocol, respectively, and gave the graph conditions to show which protocol owns better anti-disturbance capability. The anti-disturbance capability was characterized by the $\mathcal{L}_2$ gains from disturbance to consensus errors. Firstly, we built the analytic expression of the $\mathcal{L}_2$ gain of the MAS with absolute velocity protocol. Then the analytic expression of the $\mathcal{L}_2$ gain of the MAS with the relative velocity protocol was also established. It was shown that both the $\mathcal{L}_2$ gains for the absolute and the relative velocity protocols only depend on the minimum non-zero eigenvalue λ_2 of the Laplacian matrix L and the tunable gains α and β. Secondly, based on the analytic expressions of the $\mathcal{L}_2$ gain, we put forward the graph conditions related to λ_2 for protocol selection for better anti-disturbance capability. Moreover, we presented a two-step method for improving anti-disturbance capability. Finally, we provided numerical tests for some well-known graphs. Although it might be intuitively true that the network density is associated with protocol selection, this fact deserves to be further verified in future works.

Appendix A. A useful lemma

Lemma 3. *Let* $g_1(t) = \frac{1}{(\alpha t)^2}$, $g_2(t) = \frac{1}{(\alpha t)^2-(\sqrt{\Delta}-1)^2}$ *and* $g_3(t) = \frac{1}{(\alpha t)^2-(\sqrt{\Theta}-1)^2}$, *where* $\Delta = (\alpha t+1)^2-\beta^2$, $\Theta = (\alpha t+1)^2-(\beta t)^2$, *and the constants* α *and* β *are positive. Then the following statements hold*:

(1) $g_1(t)$ *is decreasing on* $(0, +\infty)$;
(2) $g_2(t)$ *is decreasing on* $(\frac{\beta-1}{\alpha}, +\infty)$;
(3) $g_3(t)$ *is decreasing on* $(0, +\infty)$ *if* $\beta \leq \alpha$;
(4) $g_3(t)$ *is decreasing on* $(0, \frac{2\alpha}{\beta^2-\alpha^2})$ *if* $\beta > \alpha$.

Proof. These can be proved by general analysis of the first derivations and the second derivations of $g_i(t)$, $i = 1, 2, 3$. □

Appendix B. Proof of Theorem 2

Similar to the proof of Theorem 1, by orthogonal transformation (10) and (13), we can get the asymptotically stable subsystem of (7). Denote the transfer matrix of this subsystem by $T_6(s)$. One can verify that

$$\begin{aligned}\|T_2(s)\|_\infty &= \sup_{\upsilon\in\mathbb{R}} \sqrt{\lambda_{\max}\left[T_6^H(\mathbf{j}\upsilon)T_6(\mathbf{j}\upsilon)\right]} \\ &= \max_{i=2,\ldots,n} \sqrt{\sup_{\upsilon\in\mathbb{R}} \theta_i(\upsilon)},\end{aligned} \tag{B.1}$$

where $\theta_i(\upsilon) = \frac{1+\upsilon^2}{(\alpha\lambda_i-\upsilon^2)^2+(\beta\lambda_i\upsilon)^2} > 0$, $i = 2, \ldots, n$, $\upsilon \in \mathbb{R}$. By solving $\frac{\mathrm{d}\theta_i(\upsilon)}{\mathrm{d}\upsilon} = 0$, we get $\sup_{\upsilon\in\mathbb{R}} \theta_i(\upsilon) = \theta_i(\upsilon_{i,1}^*) = g_1(\lambda_i) = \frac{1}{(\alpha\lambda_i)^2}$ if $\beta \geq \sqrt{\alpha^2 + \frac{2\alpha}{\lambda_i}}$, and $\sup_{\upsilon\in\mathbb{R}} \theta_i(\upsilon) = \theta_i(\upsilon_{i,2}^*) = \theta_i(\upsilon_{i,3}^*) = g_3(\lambda_i) = \frac{1}{(\alpha\lambda_i)^2-(\sqrt{\Theta_i}-1)^2}$ if $\beta < \sqrt{\alpha^2 + \frac{2\alpha}{\lambda_i}}$, where $\upsilon_{i,1}^* = 0$, $\upsilon_{i,2}^*, \upsilon_{i,3}^* = \pm\sqrt{\sqrt{\Theta_i} - 1}$ and $\Theta_i = (\alpha\lambda_i + 1)^2 - (\beta\lambda_i)^2$.

Aforementioned analysis prompts us to dictate two sets of non-zero eigenvalues of L, which are given as $\mathcal{K}_1 = \{k \in \Gamma \mid \beta \geq \sqrt{\alpha^2 + \frac{2\alpha}{k}}\}$ and $\mathcal{K}_2 = \{k \in \Gamma \mid \beta < \sqrt{\alpha^2 + \frac{2\alpha}{k}}\}$. Recall that $\Gamma = \{\lambda_2, \ldots, \lambda_n\}$ is the set of all non-zero eigenvalues of L. Obviously, we can get $\mathcal{K}_1 \cup \mathcal{K}_2 = \Gamma$ and $\mathcal{K}_1 \cap \mathcal{K}_2 = \emptyset$. We will complete the proof by enumeration.

(I) $\mathcal{K}_1 = \Gamma$ and $\mathcal{K}_2 = \emptyset$;

In this case, we have $\beta \geq \sqrt{\alpha^2 + \frac{2\alpha}{\lambda_i}}, \forall i \in \{2, \ldots, n\}$ which implies that $\sup_{\upsilon\in\mathbb{R}} \theta_i(\upsilon) = g_1(\lambda_i), \forall i \in \{2, \ldots, n\}$. According to Lemma 3 in Appendix A, $g_1(t)$ is decreasing on $(0, +\infty)$. Combining with the fact $0 < \lambda_2 \leq \cdots \leq \lambda_n$, one can obtain that $\max_{i=2,\ldots,n} g_1(\lambda_i) = g_1(\lambda_2)$. Then, (B.1) becomes $\|T_2(s)\|_\infty = \max_{i=2,\ldots,n} \sqrt{g_1(\lambda_i)} = \sqrt{g_1(\lambda_2)} = \frac{1}{\alpha\lambda_2}$.

(II) $\mathcal{K}_1 = \emptyset$ and $\mathcal{K}_2 = \Gamma$;

In this case, we get $\beta < \sqrt{\alpha^2 + \frac{2\alpha}{\lambda_i}}, \forall i \in \{2, \ldots, n\}$ which leads to $\sup_{\upsilon\in\mathbb{R}} \theta_i(\upsilon) = g_3(\lambda_i), \forall i \in \{2, \ldots, n\}$. If $\beta \leq \alpha$, it is inferred from $0 < \lambda_2 \leq \cdots \leq \lambda_n$ and Lemma 3 in Appendix A that $\max_{i=2,\ldots,n} g_3(\lambda_i) = g_3(\lambda_2)$. If $\beta > \alpha$, it follows from $0 < \lambda_2 \leq \cdots \leq \lambda_n$ and $\beta < \sqrt{\alpha^2 + \frac{2\alpha}{\lambda_i}}, \forall i \in \{2, \ldots, n\}$ that $0 < \lambda_2 \leq \cdots \leq \lambda_n < \frac{2\alpha}{\beta^2-\alpha^2}$. Then according to Lemma 3 in Appendix A, $\max_{i=2,\ldots,n} g_3(\lambda_i) = g_3(\lambda_2)$ still holds. Building on the above analysis, no matter $\beta \leq \alpha$ or $\beta > \alpha$, (B.1) can be written as $\|T_2(s)\|_\infty = \max_{i=2,\ldots,n} \sqrt{g_3(\lambda_i)} = \sqrt{g_3(\lambda_2)} = \frac{1}{\sqrt{(\alpha\lambda_2)^2-(\sqrt{\Theta_2}-1)^2}}$.

(III) $\mathcal{K}_1 \neq \emptyset$ and $\mathcal{K}_2 \neq \emptyset$.

Under this circumstance, there must exists an eigenvalue λ_m $(2 \leq m \leq n-1)$ of L such that $\mathcal{K}_1 = \{\lambda_{m+1}, \ldots, \lambda_n\}$, $\mathcal{K}_2 = \{\lambda_2, \ldots, \lambda_m\}$ and $\lambda_2 < \lambda_{m+1}$. Then, we have $\beta \geq \sqrt{\alpha^2 + \frac{2\alpha}{\lambda_i}}, \forall i \in \{m+1, \ldots, n\}$ and $\beta < \sqrt{\alpha^2 + \frac{2\alpha}{\lambda_i}}, \forall i \in \{2, \ldots, m\}$ which respectively lead to $\sup_{\upsilon\in\mathbb{R}} \theta_i(\upsilon) = g_1(\lambda_i), \forall i \in \{m+1,\ldots,n\}$ and $\sup_{\upsilon\in\mathbb{R}}\theta_i(\upsilon) = g_3(\lambda_i), \forall i\in\{2,\ldots,m\}$. According to Lemma 3 in Appendix A, $g_1(t)$ is decreasing on $(0,+\infty)$. Combining with $0<\lambda_{m+1}\le\cdots\le\lambda_n$, we can derive that $\max_{i=m+1,\ldots,n} g_1(\lambda_i) = g_1(\lambda_{m+1})$. Furthermore, it can be concluded that $\alpha < \sqrt{\alpha^2+\frac{2\alpha}{\lambda_n}} \le \cdots \le \sqrt{\alpha^2+\frac{2\alpha}{\lambda_{m+1}}} \le \beta$. Since $\alpha<\beta$, it follows from $0<\lambda_2\le\cdots\le\lambda_m$ and $\beta < \sqrt{\alpha^2+\frac{2\alpha}{\lambda_i}}, \forall i\in\{2,\ldots,m\}$ that $0<\lambda_2\le\cdots\le\lambda_m<\frac{2\alpha}{\beta^2-\alpha^2}$. According to Lemma 3 in Appendix A, we can obtain $\max_{i=2,\ldots,m} g_3(\lambda_i) = g_3(\lambda_2)$.

Based on the above analysis, (B.1) can be written as $\|T_2(s)\|_\infty = \max\{\max_{i=m+1,\ldots,n}\sqrt{g_1(\lambda_i)}, \max_{i=2,\ldots,m}\sqrt{g_3(\lambda_i)}\} = \max\{\sqrt{g_1(\lambda_{m+1})}, \sqrt{g_3(\lambda_2)}\}$. It follows from $\sup_{\upsilon\in\mathbb{R}}\theta_i(\upsilon) = g_3(\lambda_i)$, $\forall i\in\{2,\ldots,m\}$ and $\theta_i(\upsilon)>0$ that $g_3(\lambda_2)>0$. Then, according to $\lambda_2<\lambda_{m+1}$, we can deduce that $g_3(\lambda_2) = \frac{1}{(\alpha\lambda_2)^2-(\sqrt{\Theta_2}-1)^2} > \frac{1}{(\alpha\lambda_2)^2} > \frac{1}{(\alpha\lambda_{m+1})^2} = g_1(\lambda_{m+1})$. Therefore, we can obtain $\|T_2(s)\|_\infty = \sqrt{g_3(\lambda_2)} = \frac{1}{\sqrt{(\alpha\lambda_2)^2-(\sqrt{\Theta_2}-1)^2}}$.

To summarize the above cases, we can conclude that $\|T_2(s)\|_\infty = \frac{1}{\sqrt{(\alpha\lambda_2)^2-(\sqrt{\Theta_2}-1)^2}}$ as long as $\lambda_2\in\mathcal{K}_2$, i.e., $\beta<\sqrt{\alpha^2+\frac{2\alpha}{\lambda_2}}$. Otherwise, $\|T_2(s)\|_\infty = \frac{1}{\alpha\lambda_2}$. That is to say, (17) is obtained.

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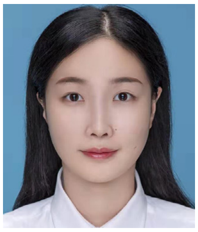

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