Neurocomputing 572 (2024) 127199

Contents lists available at ScienceDirect

Neurocomputing

journal homepage: www.elsevier.com/locate/neucom

Enclosing control of hybrid multi-agent systems

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ARTICLE INFO

Communicated by H. Zhang

Keywords:
Enclosing control
Hybrid multi-agent systems
Directed networks

ABSTRACT

This paper investigates the enclosing control of hybrid multi-agent systems under directed networks. Firstly, we establish some criteria for continuous-time and discrete-time multi-agent systems to achieve enclosing control. Secondly, according to the communication modes among agents, four distributed enclosing control protocols are proposed for hybrid multi-agent systems. Then, under the proposed protocols, we give the corresponding sufficient conditions to guarantee enclosing control. Finally, the correctness of our results is verified by simulations.

1. Introduction

In the decades, coordinated control of multi-agent systems (MASs) has received widespread concern, including consensus [1–3], flocking [4,5], formation control [6,7], coverage control [8,9], etc. As a basic one among these control problems, consensus means a group of autonomous agents reach an agreement on a consistent quantity of interest based on local information. According to whether leaders exist in the MASs and the number of leaders, consensus problems are divided into leaderless consensus [10,11], leader-following consensus [12], and multi-leader consensus-like tracking [13].

Some practical applications may require multiple leaders with better perception and decision abilities, for example, environmental quality monitoring, transporting vital cargo, and flying drones in formation. Enclosing control and containment control are fundamental multi-leader consensus-like tracking problems. The essential differences between them are the different targets being surrounded and the different relationships between leaders and followers. In containment control, the followers actively enter and stay in the convex hull formed by the leaders, so the relationship between the followers and the leaders is cooperative. However, enclosing control requires that leaders are passively located in the convex hull spanned by followers, and the relationship between the followers and the leaders can be cooperative or antagonistic. Although the above two control problems have different objectives, the analytical methods related to containment control are referable for studying the enclosing control problem.

In recent years, the problems related to containment control have achieved numerous results. In Dimarogonas et al. [14], a time-varying feedback control strategy was designed for the multi-unicycle system to achieve containment control. Under time-varying undirected networks, Notarstefano et al. [15] considered the intermittent communication among agents and solved the containment control problem for continuous-time (CT) MASs. Based on state feedback and observer design, Ma et al. [16] proposed two control protocols for general linear discrete-time (DT) MASs to guarantee containment control. Zhu et al. [17] investigated the containment control of switched MASs consisting of CT and DT subsystems under arbitrary switching. In [18], Liu et al. studied the containment control of first- and second-order MASs. Zheng and Wang [19] proposed effective distributed protocols for heterogeneous MASs to achieve containment control in finite time. For higher-order MASs with nonlinear CT dynamics, a distributed adaptive protocol was designed to achieve containment control in [20].

For specific scenarios, such as wolves rounding up prey and bodyguards protecting the leaders, the targets/leaders are enclosed in a convex hull. In these scenarios, containment control cannot meet the control requirements. In addition, enclosing control can enclose the targets/leaders without setting the desired shape of the convex hull in advance, which is necessary for formation-containment control [21,22]. Therefore, some researchers are interested in the enclosing control problem. Wang et al. [23] proposed two distributed enclosing control protocols, and the effectiveness of the protocols was verified through a multi-unmanned vehicle system in [24]. By regarding the leaders as an external system, the enclosing control of second-order MASs was further studied in [25]. For multiple unicycle-type vehicle systems, Yu et al. [26] designed a distributed protocol for vehicles to orbit around

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https://doi.org/10.1016/j.neucom.2023.127199
Received 8 August 2023; Received in revised form 14 November 2023; Accepted 26 December 2023
Available online 29 December 2023

the target. Ju et al. [27] presented a discontinuously controller for achieving the enclosing control of MASs. To localize and enclose the target, an estimator with bearing measurements was proposed in [28].

However, the previously mentioned works only considered the MASs consisting of CT or DT dynamic agents. In some human–machine cooperation/confrontation systems, humans can make calculations and decisions continuously, while machines can only do it at discrete moments. Therefore, hybrid multi-agent systems (HMASs) composed of CT and DT dynamic agents are of great application value. By analyzing the communication modes among agents, Zheng et al. [29] presented three effective distributed protocols for achieving the consensus of HMASs. As an extension of [29], a unified framework was established for CT and DT MASs to achieve consensus in [30]. For the second-order HMASs, Zheng et al. [31] designed two effective distributed protocols to guarantee consensus. As a more general case of [29,31], the consensus of heterogeneous HMASs was investigated in [32]. For the HMASs with malicious agents, Shang [33] proposed a resilient consensus strategy to achieve consensus. Ma et al. [34] solved the consensus problem for HMASs by adopting a game-theoretic approach. Under different communication modes among agents, Chen et al. [35] gave some criteria for HMASs to achieve containment control.

In the real world, enclosing control of HMASs may be applied to solve some practical application problems, including dogs herding sheep, drones driving flocks of birds, robot fish encircling schools of fish, and so forth. However, there is little relevant literature on enclosing control of HMASs, and the existing control methods cannot solve this problem well. Since hybrid multi-agent systems are composed of CT and DT dynamic agents, the information communication between different agents may occur in real time or at the sampling time. In addition, some agents can obtain their states in real time, while the others can only get their states at the sampling time. The information obtained by agent i from its neighbors is also of different types, including CT information and DT information. Therefore, it is difficult to design effective enclosing control protocols for HMASs. Moreover, it is also a challenge to analyze and obtain the conditions for solving the enclosing control problem under different communication modes. The contributions of this paper are threefold. Firstly, we analyze the enclosing control problem for MASs and further relax the graphic conditions of literature [24]. Second, we propose four distributed protocols for different communication modes of HMASs. Third, we give some sufficient conditions to guarantee enclosing control by constructing an error vector and analyzing the error system.

The remaining parts of this paper are structured as follows. Section 2 lists some preliminaries to be used. Section 3 solves the enclosing control problem for CT and DT MASs. In Sections 4 and 5, we propose four distributed protocols and obtain sufficient conditions for HMASs to achieve enclosing control. In Section 6, simulations validate that the proposed protocols are effective. Section 7 gives the conclusion.

Notations: Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ be the set of natural numbers, real numbers, and complex numbers. $\mathbf{1}_n$ is the column vector that all elements are one. $0_{m\times n}$ is the $m \times n$ all-zero matrix. I_n is the $n \times n$ identity matrix. For a complex number $\lambda \in \mathbb{C}$, its module, real part and imaginary part are denoted as $|\lambda|$, $\mathrm{Re}(\lambda)$, and $\mathrm{Im}(\lambda)$. $\det(X)$ and $\|X\|$ are the determinant and the Euclidian norm of matrix X. Moreover, $\mathrm{diag}\{a_1, a_2, \ldots, a_n\}$ denotes a diagonal matrix whose diagonal elements are $a_1, a_2, \ldots, a_n$. $\mathcal{I}_m = \{1, 2, \ldots, m\}$, $\mathcal{I}_n/\mathcal{I}_m = \{m+1, m+2, \ldots, n\}$.

2. Preliminaries

Graph theory: Denote by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ a weighted digraph with n vertices, where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$, $\mathcal{E} = \{e_{ij} = (v_i, v_j)\} \subseteq \mathcal{V} \times \mathcal{V}$, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ are the vertex set, the edge set, and the weighted adjacency matrix. If $e_{ij} \in \mathcal{E}$, then v_i is a neighbor of v_j, and v_j can get the information of v_i directly. $\mathcal{N}_i = \{v_j \in \mathcal{V} : e_{ji} \in \mathcal{E}\}$ is the neighbor set of v_i. If $v_j \in \mathcal{N}_i$, then $a_{ij} > 0$, otherwise $a_{ij} = 0$. Moreover, $a_{ii} = 0$ for $i = 1, 2, \ldots, n$. A directed path in $\mathcal{G}$ is a sequence $v_{i_1}, v_{i_2}, \ldots, v_{i_s} \in \mathcal{V}$ such that $e_{i_k i_{k+1}} \in \mathcal{E}$ for $k = 1, \ldots, s-1$. A digraph $\mathcal{G}$ is strongly connected if there is a directed path between any two distinct vertices. The digraph corresponding to matrix $B = [b_{ij}] \in \mathbb{R}^{m\times m}$ is denoted as $\mathcal{G}(B)$, and there is an edge e_{ji} in $\mathcal{G}(B)$ if $b_{ij} \neq 0$. The matrix B is irreducible if and only if $\mathcal{G}(B)$ is strongly connected.

Lemma 1 (*[36]*). *The matrix $B = [b_{ij}] \in \mathbb{C}^{m\times m}$ is nonsingular if B is irreducible and $|b_{ii}| \geq \sum_{j=1,j\neq i}^{m} |b_{ij}|$ for all i with the inequality strict for at least one i.*

Lemma 2 (*[34]*). *For matrix $B = [b_{ij}] \in \mathbb{C}^{m\times m}$, its all eigenvalues locate in the union of m discs $\cup_{i=1}^{m}\{z \in \mathbb{C} : |z - b_{ii}| \leq \sum_{j=1,j\neq i}^{m} |b_{ij}|\}$.*

Definition 1 (*[37]*). The set $S \subset \mathbb{R}^n$ is a convex set if $\xi x + (1-\xi)y \in S$ for any $x \in S$, $y \in S$, and $\xi \in [0, 1]$. For a finite set $Y = \{y_1, y_2, \ldots, y_n\} \in \mathbb{R}^m$, denoted by $\mathrm{Co}\{Y\} = \{\sum_{i=1}^{n} \eta_i y_i | \eta_i \in \mathbb{R}, \eta_i \geq 0, \sum_{i=1}^{n} \eta_i = 1\}$ its convex hull.

Definition 2. If $\mathcal{N}_i \neq \emptyset$, then agent i is called a follower, and a leader otherwise. Moreover, $\mathcal{F} = \mathcal{I}_{n-m}$ and $\mathcal{R} = \mathcal{I}_n/\mathcal{I}_{n-m}$ represent the set of followers and leaders.

Definition 3. The MAS achieves enclosing control if the followers dynamically span a convex hull containing all leaders.

Definition 4. If there is at least one leader in the set $\mathcal{N}_i$, then $i \in \mathcal{F}_1$, otherwise $i \in \mathcal{F}_2$. Especially, agent $i \in \mathcal{F}_1^{(s)}$ if the leader s is the only leader in the set $\mathcal{N}_i$. Moreover, $\mathcal{F}_1^{(s)} \subset \mathcal{F}_1$, $\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}$, and $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$.

Assumption 1. The communication network of followers is strongly connected.

3. Enclosing control of MASs

3.1. Convergence analysis for CT protocol

The dynamics of agents are given as follows

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{I}_n, \tag{1}$$

where $x_i(t) \in \mathbb{R}$, $u_i(t) \in \mathbb{R}$ are the state and the control input of agent i at time t. For the system (1), the enclosing control protocol is designed as

$$\begin{cases} u_i(t) = \alpha \sum\limits_{j\in\mathcal{F}} a_{ij}\big(x_j(t) - x_i(t)\big) + \sum\limits_{j\in\mathcal{R}} a_{ij}\big(x_j(t) - x_i(t)\big), & i \in \mathcal{F}_1, \\ u_i(t) = \sum\limits_{j\in\mathcal{F}} a_{ij}\big(x_j(t) - x_i(t)\big), & i \in \mathcal{F}_2, \\ u_i(t) = 0, & i \in \mathcal{R}, \end{cases} \tag{2}$$

where the parameter $\alpha < 0$. Let $x(t) = [x_{\mathcal{F}}^T(t), x_{\mathcal{R}}^T(t)]^T$, $x_{\mathcal{F}}(t) = [x_1(t), x_2(t), \ldots, x_{n-m}(t)]^T$, and $x_{\mathcal{R}}(t) = [x_{n-m+1}(t), x_{n-m+2}(t), \ldots, x_n(t)]^T$. Then we have

$$\dot{x}(t) = -Fx(t), \tag{3}$$

where $F = \begin{bmatrix} F_{\mathcal{F}\mathcal{F}} & F_{\mathcal{F}\mathcal{R}} \\ 0_{m\times(n-m)} & 0_{m\times m} \end{bmatrix}$, $F_{\mathcal{F}\mathcal{F}} \in \mathbb{R}^{(n-m)\times(n-m)}$, $F_{\mathcal{F}\mathcal{R}} \in \mathbb{R}^{(n-m)\times m}$,

$$(F_{\mathcal{F}\mathcal{F}})_{ii} = \begin{cases} \alpha \sum\limits_{j\in\mathcal{F}} a_{ij} + \sum\limits_{j\in\mathcal{R}} a_{ij}, & i \in \mathcal{F}_1, \\ \sum\limits_{j\in\mathcal{F}} a_{ij}, & i \in \mathcal{F}_2, \end{cases}$$

$$(F_{\mathcal{F}\mathcal{F}})_{ij} = \begin{cases} -\alpha a_{ij}, & i \in \mathcal{F}_1, j \in \mathcal{F}, \textit{and } i \neq j, \\ -a_{ij}, & i \in \mathcal{F}_2, j \in \mathcal{F}, \textit{and } i \neq j, \end{cases}$$

$$(F_{\mathcal{F}\mathcal{R}})_{ij} = F_{i,j+n-m} = -a_{i,j+n-m}, \quad i \in \mathcal{F} \textit{ and } j \in \mathcal{R}.$$

Remark 1. In this paper, λ_i denotes the eigenvalue of matrix $F_{\mathcal{F}\mathcal{F}}$, where $i \in \mathcal{F}$.

Lemma 3. *Under Assumption 1, system (1) with protocol (2) can achieve enclosing control if the set $\mathcal{F}_1^{(s)}$ is nonempty for any $s \in \mathcal{R}$, and the parameter satisfies*

$$0 > \alpha > \max_{i\in\mathcal{F}_1} \frac{-\sum_{j\in\mathcal{R}} a_{ij}}{2\sum_{j\in\mathcal{F}} a_{ij}}. \tag{4}$$

Proof. *Step 1:* From the expression of the matrix F, we have

$$\begin{cases}(F_{\mathcal{F}\mathcal{F}})_{ii} = \alpha \sum\limits_{j\in\mathcal{F},j\neq i} a_{ij} + \sum\limits_{j\in\mathcal{R}} a_{ij}, & i \in \mathcal{F}_1,\\ \sum\limits_{j\in\mathcal{F},j\neq i} (F_{\mathcal{F}\mathcal{F}})_{ij} = \sum\limits_{j\in\mathcal{F},j\neq i} -\alpha a_{ij}, & i \in \mathcal{F}_1,\\ (F_{\mathcal{F}\mathcal{F}})_{ii} = \sum\limits_{j\in\mathcal{F},j\neq i} a_{ij} = \sum\limits_{j\in\mathcal{F},j\neq i} -(F_{\mathcal{F}\mathcal{F}})_{ij}, & i \in \mathcal{F}_2.\end{cases} \tag{5}$$

Due to $0 > \alpha > \max_{i\in\mathcal{F}_1} \frac{-\sum_{j\in\mathcal{R}} a_{ij}}{2\sum_{j\in\mathcal{F}} a_{ij}}$ and $a_{ij} \geq 0$, then

$$(F_{\mathcal{F}\mathcal{F}})_{ii} > \sum_{j\in\mathcal{F},j\neq i} -\alpha a_{ij} = \sum_{j\in\mathcal{F},j\neq i} (F_{\mathcal{F}\mathcal{F}})_{ij}, \quad i \in \mathcal{F}_1. \tag{6}$$

Since the set $\mathcal{F}_1^{(s)}$ is nonempty for any $s \in \mathcal{R}$, we can get that $(F_{\mathcal{F}\mathcal{F}})_{ii} \geq \sum_{j\in\mathcal{F},j\neq i} |(F_{\mathcal{F}\mathcal{F}})_{ij}|$ for $i \in \mathcal{F}$, and at least one i makes the inequality strict hold. From Assumption 1, we know that the matrix $F_{\mathcal{F}\mathcal{F}}$ is irreducible. By Lemmas 1 and 2, the matrix $F_{\mathcal{F}\mathcal{F}}$ is nonsingular and all eigenvalues have positive real parts. Therefore, $\lim_{t\to\infty} e^{-F_{\mathcal{F}\mathcal{F}}t} = 0_{(n-m)\times(n-m)}$. From (3) and by some calculations, we have

$$x(t) = e^{-Ft}x(0), \tag{7}$$

where

$$e^{-Ft} = \begin{bmatrix} e^{-F_{\mathcal{F}\mathcal{F}}t} & F_{\mathcal{F}\mathcal{F}}^{-1}(e^{-F_{\mathcal{F}\mathcal{F}}t} - I_{n-m})F_{\mathcal{F}\mathcal{R}} \\ 0_{m\times(n-m)} & I_m \end{bmatrix}.$$

Then we can get

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{-Ft}x(0) = \begin{bmatrix} -F_{\mathcal{F}\mathcal{F}}^{-1}F_{\mathcal{F}\mathcal{R}}x_{\mathcal{R}}(0) \\ x_{\mathcal{R}}(0) \end{bmatrix}. \tag{8}$$

From (8), we have

$$\begin{cases}\lim\limits_{t\to\infty} F_{\mathcal{F}\mathcal{F}}x_{\mathcal{F}}(t) = -F_{\mathcal{F}\mathcal{R}}x_{\mathcal{R}}(0),\\ \lim\limits_{t\to\infty} x_{\mathcal{R}}(t) = x_{\mathcal{R}}(0).\end{cases} \tag{9}$$

Step 2: We note that $F\mathbf{1}_n = 0_{n\times1}$, then

$$F_{\mathcal{F}\mathcal{F}}\mathbf{1}_{n-m} = -F_{\mathcal{F}\mathcal{R}}\mathbf{1}_m. \tag{10}$$

Since the set $\mathcal{F}_1^{(s)}$ is nonempty for any $s \in \mathcal{R}$, without loss of generality, let $k \in \mathcal{F}_1^{(s)}$, then

$$\sum_{j=1}^{n-m} (F_{\mathcal{F}\mathcal{F}})_{kj} = \sum_{j=1}^{m} (-F_{\mathcal{F}\mathcal{R}})_{kj} = \sum_{j=1}^{m} a_{k,j+n-m} = a_{ks} > 0. \tag{11}$$

According to (9), we have

$$\lim_{t\to\infty} \sum_{j=1}^{n-m} \frac{1}{a_{ks}} (F_{\mathcal{F}\mathcal{F}})_{kj} x_j(t) = x_s(0), \tag{12}$$

where $\sum_{j=1}^{n-m} \frac{1}{a_{ks}} (F_{\mathcal{F}\mathcal{F}})_{kj} = 1$, and $\frac{1}{a_{ks}}(F_{\mathcal{F}\mathcal{F}})_{kj} \geq 0$ for $j \in \mathcal{I}_{n-m}$. By Definition 1, the leader s ultimately locates in the convex hull spanned by the follower set $k \cup \{\mathcal{N}_k/s\}$, where $\{\mathcal{N}_k/s\}$ represents the agents remaining in the neighbor set of the follower k except for the leader s. Repeating the above analysis, we know that followers dynamically span the convex hull enclosing all leaders. □

Remark 2. Note that the matrix $F_{\mathcal{F}\mathcal{F}}$ is strictly diagonally-dominant if $0 > \alpha > \max_{i\in\mathcal{F}_1} \frac{-\sum_{j\in\mathcal{R}} a_{ij}}{2\sum_{j\in\mathcal{F}} a_{ij}}$ and $\mathcal{F}_2 = \emptyset$. Under this condition, all eigenvalues of the matrix $F_{\mathcal{F}\mathcal{F}}$ have positive real parts without Assumption 1. Moreover, the result of literature [24] can be obtained if $\mathcal{F}_2 = \emptyset$ and the number of leaders in each follower's neighbor set is 1.

3.2. Convergence analysis for DT protocol

By discretizing system (1), the MAS with DT dynamics can be obtained as follows

$$x_i(t_{k+1}) = x_i(t_k) + hu_i(t_k), \quad i \in \mathcal{I}_n, \tag{13}$$

where $x_i(t_k) \in \mathbb{R}$, $u_i(t_k) \in \mathbb{R}$ are the state and the control input of agent i at the sampling time t_k, the sampling period $h = t_{k+1} - t_k > 0$, and $k \in \mathbb{N}$. For the system (13), the enclosing control protocol is designed as

$$\begin{cases}u_i(t_k) = \alpha \sum\limits_{j\in\mathcal{F}} a_{ij}\big(x_j(t_k) - x_i(t_k)\big) + \sum\limits_{j\in\mathcal{R}} a_{ij}\big(x_j(t_k) - x_i(t_k)\big), & i \in \mathcal{F}_1,\\ u_i(t_k) = \sum\limits_{j\in\mathcal{F}} a_{ij}\big(x_j(t_k) - x_i(t_k)\big), & i \in \mathcal{F}_2,\\ u_i(t_k) = 0, & i \in \mathcal{R},\end{cases} \tag{14}$$

where the parameter $\alpha < 0$. Let $x(t_k) = [x_{\mathcal{F}}^T(t_k), x_{\mathcal{R}}^T(t_k)]^T$, $x_{\mathcal{F}}(t_k) = [x_1(t_k), x_2(t_k), \ldots, x_{n-m}(t_k)]^T$, and $x_{\mathcal{R}}(t_k) = [x_{n-m+1}(t_k), x_{n-m+2}(t_k), \ldots, x_n(t_k)]^T$.

Lemma 4. *Under Assumption 1, system (13) with protocol (14) can achieve enclosing control if the set $\mathcal{F}_1^{(s)}$ is nonempty for any $s \in \mathcal{R}$, and the parameter and the sampling period satisfy*

$$\begin{cases}0 > \alpha > \max\limits_{i\in\mathcal{F}_1} \dfrac{-\sum_{j\in\mathcal{R}} a_{ij}}{2\sum_{j\in\mathcal{F}} a_{ij}},\\ 0 < h < \min\limits_{i\in\mathcal{F}} \dfrac{2\mathrm{Re}(\lambda_i)}{|\lambda_i|^2}.\end{cases} \tag{15}$$

Proof. Using protocol (14) for system (13), we can obtain

$$\begin{cases}x_{\mathcal{F}}(t_{k+1}) = x_{\mathcal{F}}(t_k) - h[F_{\mathcal{F}\mathcal{F}}x_{\mathcal{F}}(t_k) + F_{\mathcal{F}\mathcal{R}}x_{\mathcal{R}}(t_k)],\\ x_{\mathcal{R}}(t_{k+1}) = x_{\mathcal{R}}(t_k).\end{cases} \tag{16}$$

Let $\delta_x(t_k) = F_{\mathcal{F}\mathcal{F}}x_{\mathcal{F}}(t_k) + F_{\mathcal{F}\mathcal{R}}x_{\mathcal{R}}(t_k)$, then

$$\delta_x(t_{k+1}) = (I_{n-m} - hF_{\mathcal{F}\mathcal{F}})\delta_x(t_k). \tag{17}$$

Due to $0 > \alpha > \max_{i\in\mathcal{F}_1} \frac{-\sum_{j\in\mathcal{R}} a_{ij}}{2\sum_{j\in\mathcal{F}} a_{ij}}$, from the *Step 1* of Lemma 3, we can obtain that $\mathrm{Re}(\lambda_i) > 0$ for $i \in \mathcal{F}$. Since $0 < h < \min_{i\in\mathcal{F}} \frac{2\mathrm{Re}(\lambda_i)}{|\lambda_i|^2}$, all eigenvalues of matrix $I_{n-m} - hF_{\mathcal{F}\mathcal{F}}$ are within the unit circle. Therefore, we have

$$\lim_{t_k\to\infty} \delta_x(t_k) = \lim_{k\to\infty} (I_{n-m} - hF_{\mathcal{F}\mathcal{F}})^k \delta_x(0) = 0_{(n-m)\times1}. \tag{18}$$

From (16), we know that $x_{\mathcal{R}}(t_k) = x_{\mathcal{R}}(0)$, then

$$\lim_{t_k\to\infty} F_{\mathcal{F}\mathcal{F}}x_{\mathcal{F}}(t_k) = -F_{\mathcal{F}\mathcal{R}}x_{\mathcal{R}}(0). \tag{19}$$

The remaining proof is similar to the *Step 2* of Lemma 3, which is omitted here. □

4. Followers with CT dynamics

Consider the HMAS consisting of DT dynamic leaders and CT dynamic followers. The dynamics of agents are given as follows

$$\begin{cases}\dot{x}_i(t) = u_i(t), & i \in \mathcal{F},\\ x_i(t_{k+1}) = x_i(t_k) + hu_i(t_k), & i \in \mathcal{R},\end{cases} \tag{20}$$

where the sampling period $h = t_{k+1} - t_k > 0$ and $k \in \mathbb{N}$.

4.1. Case 1

Followers obtain their states and communicate with neighbors at the sampling time t_k. For the HMAS (20), the enclosing control protocol is designed as

$$\begin{cases} u_i(t) = \alpha \sum_{j\in\mathcal{F}} a_{ij}\big(x_j(t_k) - x_i(t_k)\big) + \sum_{j\in\mathcal{R}} a_{ij}\big(x_j(t_k) - x_i(t_k)\big), & i \in \mathcal{F}_1, \\ u_i(t) = \sum_{j\in\mathcal{F}} a_{ij}\big(x_j(t_k) - x_i(t_k)\big), & i \in \mathcal{F}_2, \\ u_i(t_k) = 0, & i \in \mathcal{R}, \end{cases} \tag{21}$$

where $t \in (t_k, t_{k+1}]$ and the parameter $\alpha < 0$.

Theorem 1. *Under Assumption 1, system (20) with protocol (21) can achieve enclosing control if the set $\mathcal{F}_1^{(s)}$ is nonempty for any $s \in \mathcal{R}$, and the parameter and the sampling period satisfy*

$$\begin{cases} 0 > \alpha > \max_{i\in\mathcal{F}_1} \dfrac{-\sum_{j\in\mathcal{R}} a_{ij}}{2\sum_{j\in\mathcal{F}} a_{ij}}, \\ 0 < h < \min_{i\in\mathcal{F}} \dfrac{2\mathrm{Re}(\lambda_i)}{|\lambda_i|^2}. \end{cases} \tag{22}$$

Proof. Using protocol (21) for HMAS (20), we obtain

$$\begin{cases} x_i(t) = x_i(t_k) + \Big[\alpha \sum_{j\in\mathcal{F}} a_{ij}\big(x_j(t_k) - x_i(t_k)\big) \\ \qquad + \sum_{j\in\mathcal{R}} a_{ij}\big(x_j(t_k) - x_i(t_k)\big) \Big] (t - t_k), & i \in \mathcal{F}_1, \\ x_i(t) = x_i(t_k) + \Big[\sum_{j\in\mathcal{F}} a_{ij}\big(x_j(t_k) - x_i(t_k)\big)\Big](t - t_k), & i \in \mathcal{F}_2, \\ x_i(t_{k+1}) = x_i(t_k), & i \in \mathcal{R}. \end{cases} \tag{23}$$

When $t = t_{k+1}$, since $0 > \alpha > \max_{i\in\mathcal{F}_1} \frac{-\sum_{j\in\mathcal{R}} a_{ij}}{2\sum_{j\in\mathcal{F}} a_{ij}}$ and $0 < h < \min_{i\in\mathcal{F}} \frac{2\mathrm{Re}(\lambda_i)}{|\lambda_i|^2}$, by the analysis of Lemma 4, we get

$$\lim_{t_k\to\infty} F_{\mathcal{F}\mathcal{F}} x_{\mathcal{F}}(t_k) + F_{\mathcal{F}\mathcal{R}} x_{\mathcal{R}}(t_k) = 0_{(n-m)\times 1}. \tag{24}$$

When $t \in (t_k, t_{k+1})$, we have

$$x_{\mathcal{F}}(t) - x_{\mathcal{F}}(t_k) = -(t - t_k)[F_{\mathcal{F}\mathcal{F}} x_{\mathcal{F}}(t_k) + F_{\mathcal{F}\mathcal{R}} x_{\mathcal{R}}(t_k)]. \tag{25}$$

Due to $h > t - t_k > 0$, we know that $t - t_k$ is bounded. Moreover, $t_k \to \infty$ as $t \to \infty$, then we have

$$\lim_{t\to\infty} \|x_{\mathcal{F}}(t) - x_{\mathcal{F}}(t_k)\| = 0. \tag{26}$$

We can easily get $\lim_{t\to\infty} x_{\mathcal{F}}(t) = \lim_{t_k\to\infty} x_{\mathcal{F}}(t_k)$. From (23), we have $x_{\mathcal{R}}(t_k) = x_{\mathcal{R}}(0)$, then

$$\lim_{t\to\infty} F_{\mathcal{F}\mathcal{F}} x_{\mathcal{F}}(t) = \lim_{t_k\to\infty} F_{\mathcal{F}\mathcal{F}} x_{\mathcal{F}}(t_k) = -F_{\mathcal{F}\mathcal{R}} x_{\mathcal{R}}(0), \tag{27}$$

where $t \in (t_k, t_{k+1}]$. Since the set $\mathcal{F}_1^{(s)}$ is nonempty for any $s \in \mathcal{R}$, similar to the *Step 2* of Lemma 3, the HMAS (20) with protocol (21) achieves enclosing control. □

4.2. Case 2

Followers obtain their states in real time and communicate with neighbors at the sampling time t_k. For the HMAS (20), the enclosing control protocol is designed as

$$\begin{cases} u_i(t) = \alpha \sum_{j\in\mathcal{F}} a_{ij}\big(x_j(t_k) - x_i(t)\big) + \sum_{j\in\mathcal{R}} a_{ij}\big(x_j(t_k) - x_i(t)\big), & i \in \mathcal{F}_1, \\ u_i(t) = \sum_{j\in\mathcal{F}} a_{ij}\big(x_j(t_k) - x_i(t)\big), & i \in \mathcal{F}_2, \\ u_i(t_k) = 0, & i \in \mathcal{R}, \end{cases} \tag{28}$$

where $t \in (t_k, t_{k+1}]$ and the parameter $\alpha < 0$.

Theorem 2. *Under Assumption 1, system (20) with protocol (28) can achieve enclosing control if the set $\mathcal{F}_1^{(s)}$ is nonempty for any $s \in \mathcal{R}$, and the parameter satisfies*

$$0 > \alpha > \max_{i\in\mathcal{F}_1} \frac{-\sum_{j\in\mathcal{R}} a_{ij}}{2\sum_{j\in\mathcal{F}} a_{ij}}. \tag{29}$$

Proof. Using protocol (28) for HMAS (20), we obtain

$$\begin{cases} x_i(t) = x_i(t_k) + \dfrac{1 - e^{-d_i(t-t_k)}}{d_i} \Big[\alpha \sum_{j\in\mathcal{F}} a_{ij}\big(x_j(t_k) - x_i(t_k)\big) \\ \qquad + \sum_{j\in\mathcal{R}} a_{ij}\big(x_j(t_k) - x_i(t_k)\big) \Big], & i \in \mathcal{F}_1, \\ x_i(t) = x_i(t_k) + \dfrac{1 - e^{-d_i(t-t_k)}}{d_i}\Big[\sum_{j\in\mathcal{F}} a_{ij}\big(x_j(t_k) - x_i(t_k)\big)\Big], & i \in \mathcal{F}_2, \\ x_i(t_{k+1}) = x_i(t_k), & i \in \mathcal{R}, \end{cases} \tag{30}$$

where

$$d_i = (F_{\mathcal{F}\mathcal{F}})_{ii} = \begin{cases} \alpha \sum_{j\in\mathcal{F}} a_{ij} + \sum_{j\in\mathcal{R}} a_{ij}, & i \in \mathcal{F}_1, \\ \sum_{j\in\mathcal{F}} a_{ij}, & i \in \mathcal{F}_2. \end{cases}$$

Since $0 > \alpha > \max_{i\in\mathcal{F}_1} \frac{-\sum_{j\in\mathcal{R}} a_{ij}}{2\sum_{j\in\mathcal{F}} a_{ij}}$, we get $d_i = (F_{\mathcal{F}\mathcal{F}})_{ii} > 0$ for $i \in \mathcal{F}$. By the analysis of Lemma 3, the matrix $F_{\mathcal{F}\mathcal{F}}$ is reversible. When $t = t_{k+1}$, we have

$$\begin{cases} x_{\mathcal{F}}(t_{k+1}) = x_{\mathcal{F}}(t_k) - H\big(F_{\mathcal{F}\mathcal{F}} x_{\mathcal{F}}(t_k) + F_{\mathcal{F}\mathcal{R}} x_{\mathcal{R}}(t_k)\big), \\ x_{\mathcal{R}}(t_{k+1}) = x_{\mathcal{R}}(t_k), \end{cases} \tag{31}$$

where $H = \mathrm{diag}\{H_1, H_2, \dots, H_{n-m}\}$, $H_i = \frac{1-e^{-d_i h}}{d_i}$ for $i \in \mathcal{F}$. Let $\zeta_x(t_k) = x_{\mathcal{F}}(t_k) + F_{\mathcal{F}\mathcal{F}}^{-1} F_{\mathcal{F}\mathcal{R}} x_{\mathcal{R}}(t_k)$, then

$$\zeta_x(t_{k+1}) = Q\zeta_x(t_k), \tag{32}$$

where $Q = I_{n-m} - HF_{\mathcal{F}\mathcal{F}}$. Let q_i be the diagonal elements of the matrix Q, then we have $q_i = 1 - \frac{1-e^{-d_i h}}{d_i} d_i = e^{-d_i h}$, $0 < q_i < 1$, and $H_i > 0$ for $i \in \mathcal{F}$. By Lemma 2, we get

$$|z - q_i| \le \sum_{j\in\mathcal{F}, j\ne i} |-H_i (F_{\mathcal{F}\mathcal{F}})_{ij}| = \frac{1 - e^{-d_i h}}{d_i} \sum_{j\in\mathcal{F}, j\ne i} |-(F_{\mathcal{F}\mathcal{F}})_{ij}|, \quad i \in \mathcal{F}.$$

Due to $0 > \alpha > \max_{i\in\mathcal{F}_1} \frac{-\sum_{j\in\mathcal{R}} a_{ij}}{2\sum_{j\in\mathcal{F}} a_{ij}}$, from the *Step 1* of Lemma 3, we can obtain

$$|z - q_i| \le \frac{1 - e^{-d_i h}}{d_i} \sum_{j\in\mathcal{F}, j\ne i} (F_{\mathcal{F}\mathcal{F}})_{ij} < \frac{1 - e^{-d_i h}}{d_i} (F_{\mathcal{F}\mathcal{F}})_{ii} = 1 - e^{-d_i h}, \quad i \in \mathcal{F}_1,$$

$$|z - q_i| \le \frac{1 - e^{-d_i h}}{d_i} \sum_{j\in\mathcal{F}, j\ne i} -(F_{\mathcal{F}\mathcal{F}})_{ij} = \frac{1 - e^{-d_i h}}{d_i} (F_{\mathcal{F}\mathcal{F}})_{ii} = 1 - e^{-d_i h}, \quad i \in \mathcal{F}_2.$$

Therefore, we have $|z - q_i| \le 1 - e^{-d_i h} = 1 - q_i$ for $i \in \mathcal{F}$. Let μ_i be the eigenvalue of the matrix Q, by Lemma 2, we have $|\mu_i| \le 1$ with the equality holds if and only if $\mu_i = 1$. The characteristic equation of the matrix Q is given as

$$\begin{aligned} \det(\mu I_{n-m} - Q) &= \det\big(\mu I_{n-m} - (I_{n-m} - HF_{\mathcal{F}\mathcal{F}})\big) \\ &= \det\big((\mu - 1)I_{n-m} + HF_{\mathcal{F}\mathcal{F}}\big) \\ &= \prod_{i=1}^{n-m} (\mu_i - 1 + \mu_i') \\ &= 0, \end{aligned}$$

where μ_i' is the eigenvalue of matrix $HF_{\mathcal{F}\mathcal{F}}$. We assume that $\mu_i = 1$, then $\mu_i' = 0$ and $\det(HF_{\mathcal{F}\mathcal{F}}) = \det H \det F_{\mathcal{F}\mathcal{F}} = \prod_{i=1}^{n-m} H_i \det F_{\mathcal{F}\mathcal{F}} = 0$. Since $H_i = \frac{1-d^{-d_i h}}{d_i} > 0$ for $i \in \mathcal{F}$ and the matrix $F_{\mathcal{F}\mathcal{F}}$ is reversible, we have $\prod_{i=1}^{n-m} H_i \ne 0$ and $\det F_{\mathcal{F}\mathcal{F}} \ne 0$ that contradicts $\det(HF_{\mathcal{F}\mathcal{F}}) = 0$.

Therefore, $|\mu_i| < 1$ for $i \in \mathcal{F}$, which means that all μ_i are within the unit circle, and we get

$$\lim_{t_k \to \infty} \zeta_x(t_k) = \lim_{k \to \infty} Q^k \zeta_x(0) = 0_{(n-m)\times 1}. \tag{33}$$

From system (30), we can obtain $x_{\mathcal{R}}(t_k) = x_{\mathcal{R}}(0)$, then

$$\lim_{t_k \to \infty} F_{\mathcal{F}\mathcal{F}} x_{\mathcal{F}}(t_k) = -F_{\mathcal{F}\mathcal{R}} x_{\mathcal{R}}(0). \tag{34}$$

When $t \in (t_k, t_{k+1})$, we can rewrite (30) as

$$x_{\mathcal{F}}(t) - x_{\mathcal{F}}(t_k) = -H(t) F_{\mathcal{F}\mathcal{F}} \zeta_x(t_k), \tag{35}$$

where $H(t) = \operatorname{diag}\{H_1(t), H_2(t), \dots, H_{n-m}(t)\}$, $H_i(t) = \frac{1-e^{-d_i(t-t_k)}}{d_i}$ for $i \in \mathcal{F}$. Since $\dot{H}_i(t) = e^{-d_i(t-t_k)} > 0$, the function $H_i(t)$ increases monotonically over the interval (t_k, t_{k+1}). Therefore, we have $0 < H_i(t) < \frac{1-e^{-d_i h}}{d_i}$ for $i \in \mathcal{F}$, which means that all $H_i(t)$ are bounded. Moreover, $t_k \to \infty$ as $t \to \infty$, then

$$\lim_{t \to \infty} \|x_{\mathcal{F}}(t) - x_{\mathcal{F}}(t_k)\| = 0. \tag{36}$$

We can easily get $\lim_{t\to\infty} x_{\mathcal{F}}(t) = \lim_{t_k \to \infty} x_{\mathcal{F}}(t_k)$, then

$$\lim_{t \to \infty} F_{\mathcal{F}\mathcal{F}} x(t) = \lim_{t_k \to \infty} F_{\mathcal{F}\mathcal{F}} x_{\mathcal{F}}(t_k) = -F_{\mathcal{F}\mathcal{R}} x_{\mathcal{R}}(0), \tag{37}$$

where $t \in (t_k, t_{k+1}]$. Due to the set $\mathcal{F}_1^{(s)}$ is nonempty for any $s \in \mathcal{R}$, similar to the *Step 2* of Lemma 3, the HMAS (20) with protocol (28) achieves enclosing control. □

4.3. Case 3

Followers obtain their states and communicate with other followers in real time. The communications between followers and leaders occur at sampling time t_k. For the HMAS (20), the enclosing control protocol is designed as

$$\begin{cases} u_i(t) = \alpha \sum_{j\in\mathcal{F}} a_{ij}\big(x_j(t) - x_i(t)\big) + \sum_{j\in\mathcal{R}} a_{ij}\big(x_j(t_k) - x_i(t)\big), & i \in \mathcal{F}_1, \\ u_i(t) = \sum_{j\in\mathcal{F}} a_{ij}\big(x_j(t) - x_i(t)\big), & i \in \mathcal{F}_2, \\ u_i(t_k) = 0, & i \in \mathcal{R}, \end{cases} \tag{38}$$

where $t \in (t_k, t_{k+1}]$ and the parameter $\alpha < 0$.

Theorem 3. *Under Assumption 1, system (20) with protocol (38) can achieve enclosing control if the set $\mathcal{F}_1^{(s)}$ is nonempty for any $s \in \mathcal{R}$, and the parameter satisfies*

$$0 > \alpha > \max_{i\in\mathcal{F}_1} \frac{-\sum_{j\in\mathcal{R}} a_{ij}}{2\sum_{j\in\mathcal{F}} a_{ij}}. \tag{39}$$

Proof. Using protocol (38) for HMAS (20), we obtain

$$\begin{cases} \dot{x}_i(t) = \alpha \sum_{j\in\mathcal{F}} a_{ij}\big(x_j(t) - x_i(t)\big) + \sum_{j\in\mathcal{R}} a_{ij}\big(x_j(t_k) - x_i(t)\big), & i \in \mathcal{F}_1, \\ \dot{x}_i(t) = \sum_{j\in\mathcal{F}} a_{ij}\big(x_j(t) - x_i(t)\big), & i \in \mathcal{F}_2, \\ x_i(t_{k+1}) = x_i(t_k), & i \in \mathcal{R}. \end{cases} \tag{40}$$

According to (40), we have $x_{\mathcal{R}}(t_k) = x_{\mathcal{R}}(0)$, then

$$\dot{x}_{\mathcal{F}}(t) = -F_{\mathcal{F}\mathcal{F}} x_{\mathcal{F}}(t) - F_{\mathcal{F}\mathcal{R}} x_{\mathcal{R}}(t_k) = -F_{\mathcal{F}\mathcal{F}} x_{\mathcal{F}}(t) - F_{\mathcal{F}\mathcal{R}} x_{\mathcal{R}}(0). \tag{41}$$

Let $\delta_x(t) = F_{\mathcal{F}\mathcal{F}} x_{\mathcal{F}}(t) + F_{\mathcal{F}\mathcal{R}} x_{\mathcal{R}}(0)$, then

$$\dot{\delta}_x(t) = -F_{\mathcal{F}\mathcal{F}} \delta_x(t). \tag{42}$$

Since $0 > \alpha > \max_{i\in\mathcal{F}_1} \frac{-\sum_{j\in\mathcal{R}} a_{ij}}{2\sum_{j\in\mathcal{F}} a_{ij}}$, from the *Step 1* of Lemma 3, we can obtain that $\operatorname{Re}(\lambda_i) > 0$ for $i \in \mathcal{F}$. Therefore, we get

$$\lim_{t\to\infty} \delta_x(t) = \lim_{t\to\infty} e^{-F_{\mathcal{F}\mathcal{F}} t} \delta_x(0) = 0_{(n-m)\times 1}. \tag{43}$$

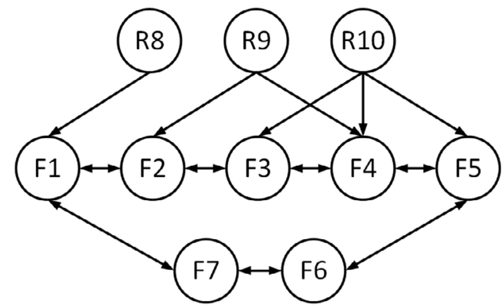


Fig. 1. A directed graph.

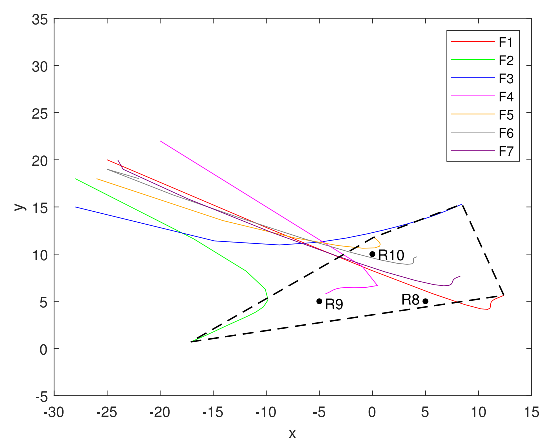


Fig. 2. State trajectories of agents in Example 1.

Thus, we have

$$\lim_{t\to\infty} F_{\mathcal{F}\mathcal{F}} x_{\mathcal{F}}(t) = -F_{\mathcal{F}\mathcal{R}} x_{\mathcal{R}}(0), \tag{44}$$

where $t \in (t_k, t_{k+1}]$. Since the set $\mathcal{F}_1^{(s)}$ is nonempty for any $s \in \mathcal{R}$, similar to the *Step 2* of Lemma 3, the HMAS (20) with protocol (38) achieves enclosing control. □

5. Followers with DT dynamics

Consider the HMASs consisting of CT dynamic leaders and DT dynamic followers. The dynamics of agents are given as

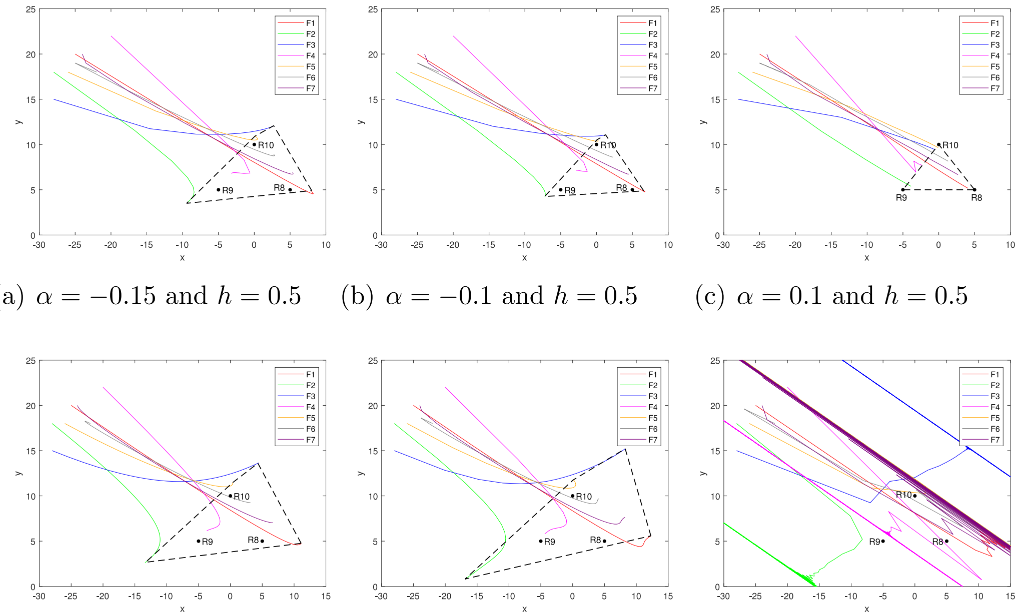
$$\begin{cases} x_i(t_{k+1}) = x_i(t_k) + h u_i(t_k), & i \in \mathcal{F}, \\ \dot{x}(t) = u_i(t), & i \in \mathcal{R}, \end{cases} \tag{45}$$

where the sampling period $h = t_{k+1} - t_k > 0$, $k \in \mathbb{N}$. In this case, followers can only obtain their states and communicate with neighbors at the sampling time t_k. For the HMAS (45), the enclosing control protocol is designed as

$$\begin{cases} u_i(t_k) = \alpha \sum_{j\in\mathcal{F}} a_{ij}\big(x_j(t_k) - x_i(t_k)\big) + \sum_{j\in\mathcal{R}} a_{ij}\big(x_j(t_k) - x_i(t_k)\big), & i \in \mathcal{F}_1, \\ u_i(t_k) = \sum_{j\in\mathcal{F}} a_{ij}\big(x_j(t_k) - x_i(t_k)\big), & i \in \mathcal{F}_2, \\ u_i(t) = 0, & i \in \mathcal{R}, \end{cases} \tag{46}$$

where $t \in (t_k, t_{k+1}]$ and the parameter $\alpha < 0$.

Theorem 4. *Under Assumption 1, system (45) with protocol (46) can achieve enclosing control if the set $\mathcal{F}_1^{(s)}$ is nonempty for any $s \in \mathcal{R}$, and the*



(a) $\alpha = -0.15$ and $h = 0.5$ (b) $\alpha = -0.1$ and $h = 0.5$ (c) $\alpha = 0.1$ and $h = 0.5$

(d) $\alpha = -0.22$ and $h = 0.1$ (e) $\alpha = -0.22$ and $h = 0.3$ (f) $\alpha = -0.22$ and $h = 0.8$

Fig. 3. State trajectories of agents in Example 2 with different α and h.

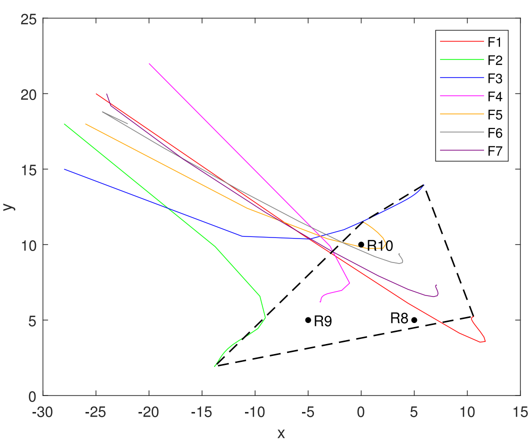


Fig. 4. State trajectories of agents in Example 3.

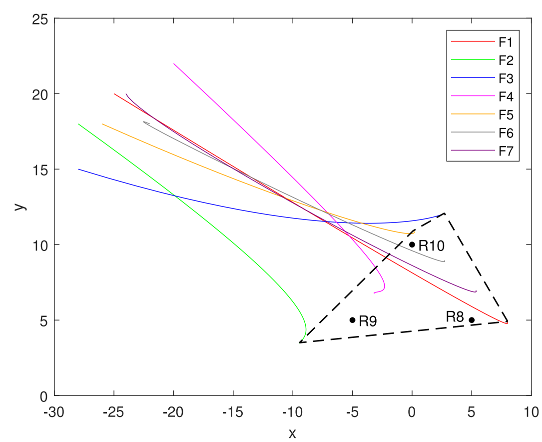


Fig. 5. State trajectories of agents in Example 4.

parameter and the sampling period satisfy

$$\begin{cases} 0 > \alpha > \max\limits_{i \in \mathcal{F}_1} \dfrac{-\sum_{j \in \mathcal{R}} a_{ij}}{2\sum_{j \in \mathcal{F}} a_{ij}}, \\ 0 < h < \min\limits_{i \in \mathcal{F}} \dfrac{2\mathrm{Re}(\lambda_i)}{|\lambda_i|^2}. \end{cases} \tag{47}$$

Proof. Using protocol (46) for HMAS (45), we obtain $x_{\mathcal{R}}(t) = x_{\mathcal{R}}(t_k) = x_{\mathcal{R}}(0)$. Similar to Lemma 4, we can obtain

$$\lim_{t_k \to \infty} F_{\mathcal{F}\mathcal{F}} x_{\mathcal{F}}(t_k) = \lim_{t \to \infty} -F_{\mathcal{F}\mathcal{R}} x_{\mathcal{R}}(t) = -F_{\mathcal{F}\mathcal{R}} x_{\mathcal{R}}(0). \tag{48}$$

Due to the set $\mathcal{F}_1^{(s)}$ is nonempty for any $s \in \mathcal{R}$, similar to the *Step 2* of Lemma 3, the HMAS (45) with protocol (46) achieves enclosing control. □

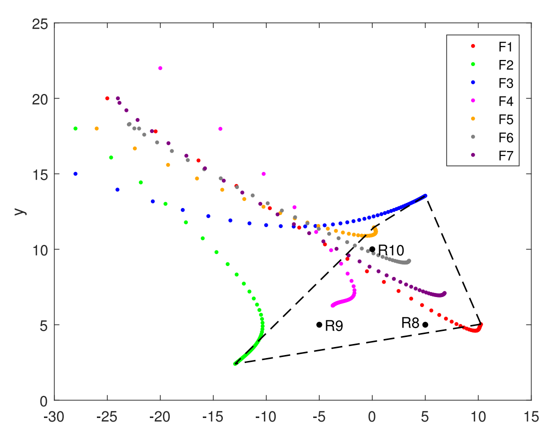


Fig. 6. State trajectories of agents in Example 5.

6. Simulations

We consider the enclosing control of HMASs on a two-dimensional plane. The communication network $\mathcal{G}$ is given in Fig. 1, where followers and leaders are labeled as F1-F7 and R8-R10, respectively. Then we have $\mathcal{F}_1$ = {F1, F2, F3, F4, F5}, $\mathcal{F}_2$ = {F6, F7}, $\mathcal{F}_1^{(8)}$ = F1, $\mathcal{F}_1^{(9)}$ = F2, and $\mathcal{F}_1^{(10)}$ = {F3, F5}. Suppose the weight of each edge is 1, then $0 > \alpha > \max_{i\in\mathcal{F}_1} \frac{-\sum_{j\in\mathcal{R}} a_{ij}}{2\sum_{j\in\mathcal{F}} a_{ij}} = -0.25$. In Examples 1, 2, 3, and 4, we consider the HMAS (20), and the HMAS (45) is considered in Example 5. Moreover, the same initial states are adopted for agents in all examples.

Example 1. Let $\alpha = -0.22$, we can obtain $0 < h < \min_{i\in\mathcal{F}} \frac{2\mathrm{Re}(\lambda_i)}{|\lambda_i|^2} \approx 0.68847$. Thus, we choose $h = 0.5$. Fig. 2 displays the state trajectories of agents with protocol (21). Followers dynamically span the convex hull enclosing all leaders. The result is the same as Theorem 1.

Example 2. To reflect the influence of different parameters on the simulation results, Fig. 3 gives the state trajectories of agents with protocol (21) under different α and h. Comparing with Example 1, we can see that the closer parameter α approaches 0, the final positions of followers will be closer to the leaders, which means the smaller convex hull spanned by followers. When $\alpha = 0.1 > 0$, the followers will actively enter and stay in the convex hull formed by the leaders. In this case, containment control is achieved instead of enclosing control. Moreover, the smaller the sampling period h, the smoother the state trajectories of the agents. When $h = 0.8 > \min_{i\in\mathcal{F}} \frac{2\mathrm{Re}(\lambda_i)}{|\lambda_i|^2}$, the system is divergent and cannot achieve enclosing control.

Example 3. Let $\alpha = -0.2$ and $h = 0.8$. Fig. 4 depicts the state trajectories of agents with protocol (28). All leaders are enclosed in the convex hull dynamically spanned by followers, which verifies the correctness of Theorem 2.

Example 4. Let $\alpha = -0.15$ and $h = 1.2$. Fig. 5 displays the state trajectories of agents with protocol (38). Followers dynamically span the convex hull enclosing all leaders. The result is the same as Theorem 3.

Example 5. Let $\alpha = -0.2$, we can get $0 < h < \min_{i\in\mathcal{F}} \frac{2\mathrm{Re}(\lambda_i)}{|\lambda_i|^2} \approx 0.68667$. Therefore, we choose $h = 0.15$. Fig. 6 depicts the state trajectories of agents with protocol (46). All leaders are enclosed in the convex hull dynamically spanned by followers, which verifies the correctness of Theorem 4.

7. Conclusion

The enclosing control of HMASs is investigated in this paper. Followers are divided into two groups based on whether they get information directly from leaders, and some criteria are obtained for CT and DT MASs to achieve enclosing control. Then, we propose four effective distributed protocols for two kinds of HMASs. The first kind of HMAS consists of DT dynamic leaders and CT dynamic followers, and the second one contains CT dynamic leaders and DT dynamic followers. Finally, we prove that the enclosing control of HMASs can be achieved and give the corresponding sufficient conditions. Future work will consider the enclosing control of high-order and heterogeneous HMASs.

CRediT authorship contribution statement

Yapeng Jia: Conceptualization, Methodology, Visualization, Writing – original draft. **Qi Zhao:** Data curation, Software, Validation, Formal analysis, Writing – original draft. **Dong Zhang:** Writing – review & editing, Supervision. **Yuanshi Zheng:** Investigation, Supervision, Writing – review & editing, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China (62273267); in part by the Natural Science Basic Research Program of Shaanxi, China (2022JC-46).

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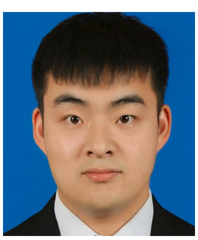

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