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Practical exponential stability of impulsive stochastic functional differential systems with distributed-delay dependent impulses

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ABSTRACT

This paper develops new practical stability criteria for impulsive stochastic functional differential systems with distributed-delay dependent impulses by using the Lyapunov–Razumikhin approach and some inequality techniques. In the given systems, the state variables on the impulses are concerned with a history time period, which is very appropriate for modelling some practical problems. Moreover, different from the existing practical stabilization results for the systems with unstable continuous stochastic dynamics and stabilizing impulsive effects, we take the systems with stable continuous stochastic dynamics and destabilizing impulsive effects into account. It shows that under the impulsive perturbations, the practical exponential stability of the stochastic functional differential systems can remain unchanged when the destabilizing distributed-delay dependent impulses satisfy some conditions on the frequency and amplitude of the impulses. In other words, it reveals that how to control the impulsive perturbations such that the corresponding stochastic functional differential systems still maintain practically exponentially stable. Finally, an example with its numerical simulation is offered to demonstrate the efficiency of the theoretical findings.

1. Introduction

The practical stability (PS), originally introduced in [1], has attracted great attention for many years due to its important roles in characterizing the stability of many dynamical systems, such as reaction–diffusion systems [2,3], impulsive systems [3–6], switched systems [7–10], chained systems [11], stochastic age-dependent capital systems [12], stochastic systems with uncertainties and disturbances [13]. The analysis of PS aims to investigate the stability of systems when the origin is not necessarily an equilibrium point (refer to [4,12]).

Needless to say that impulsive systems can be appropriate tools in mathematical modeling of many real world systems, which have crucial applications in a variety of areas, such as mechanical systems, complex networks, secure communication, population growth and biological systems [14–16]. On the other hand, delay effects are also frequently encountered in lots of practical systems [17–20]. Up to now, increasing attention has been paid to the stability analysis for the impulsive systems with delay effects (see e.g., [3,5,17,21–32]). For example, in [5], the authors discussed the practical exponential stability (PES) of impulsive stochastic functional differential systems with G-Brownian motion by

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employing stochastic analysis technique, Razumikhin-type theorem and vector G-Lyapunov function. In [21], the authors investigated input-to-state stability problem for impulsive and switching hybrid time-delay systems by using the method of multiple Lyapunov–Krasovskii functionals. In [22], the authors studied the robust stability, stabilization and H_∞-control for uncertain impulsive systems with time-delay by using Lyapunov–Razumikhin (LR) approach. However, for the PS problem of impulsive systems, one may observe that the state variables on impulses only considered the discrete delay effects in the literature. In fact, in some special fields, impulsive effects may be concerned with distributed delay when the jumps of systems states only rely on the accumulation of the system states over a history time period, such as population dynamics, financial markets, network connections [23,28,29,33]. Therefore, it is great meaningful to investigate the PS of the systems with distributed delay dependent impulses.

Moreover, to the best of our knowledge, an impulsive system can be viewed as a hybrid one, which generally comprises three classes of systems: (1) the system with unstable continuous stochastic dynamics and stabilizing impulses; (2) the system with stable continuous stochastic dynamics and destabilizing impulses; (3) the systems where both the continuous dynamics and the discrete dynamics are stable. Recently, numerous results have been reported for stability analysis of three classes of impulsive systems (see e.g., [4,5,21–32]). For example, [4,5,24,26–29,32] investigated the first class of impulsive systems. [8] discussed the second class of impulsive systems. [21,30,31] studied the first and second classes of impulsive systems. [23] considered the first and third classes of impulsive systems. [22] investigated the first and third classes of impulsive systems. However, it should be mentioned that only the first class of impulsive systems were considered in the analysis of PS in the literature (see e.g., [3,4]). In addition, for the systems with distributed-delay dependent impulses, the authors just investigated the exponential stability of the first and third classes of impulsive systems (see e.g., [23,28,29]). Therefore, in this paper, we will analyse the PS of the systems with distributed-delay dependent impulses, in the case that the continuous stochastic systems are practically exponentially stable (PES) and distributed-delay dependent impulses are destabilizing.

Inspired by the above discussion, by using the LR approach and some inequality techniques, we will investigate the PS of impulsive stochastic functional differential systems with destabilizing distributed-delay dependent impulses, and give some new sufficient conditions for the practical exponential stability in pth moment (PESpM) and almost sure practical exponential stability (ASPES). The contributions of this paper are summarized as follows: (1) The systems with distributed-delay dependent impulses are a type of more general systems. Up to now, the stability of this type of impulsive systems in the literature were only considered in [23,28,29], where the exponential stabilization of the corresponding first and third classes of impulsive systems were studied. In this paper, we not only further discuss the PESpM and ASPES for such systems, but also consider the corresponding second class of impulsive systems; (2) So far, there have been few results on the PS of the second class of impulsive systems, and our results can also be applied to the deterministic systems when we do not consider the stochastic effects. Moreover, the obtained practical stability results complement and generalize some existing results in the literature (see e.g., [4,5,23,28,29])

The remainder of this paper is organized as follows. In Section 2, we introduce the model, together with some basic assumptions and definitions. In Section 3, some sufficient conditions of PESpM and ASPEM are established for the studied system. In Section 4, we give an example with its numerical simulations. Finally, some conclusions are presented in Section 5.

Notations. The following notations are used throughout this paper. $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and $w(t) = (w_1(t), \ldots, w_m(t))^T$ be an m-dimensional standard Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. $\mathbb{N}$, $\mathbb{R}_+$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the set of positive integers, the set of nonnegative real numbers, the n-dimensional real space and $n \times m$-dimensional real matrix space, respectively. $\mathbb{E}$ stands for the mathematical expectation operator with regard to the given probability measure $\mathbb{P}$. If A is a vector or a matrix, its transpose is denoted by A^T. For $x \in \mathbb{R}^n$, $|x| = \sqrt{x^T x}$ denotes the Euclidian vector norm. Given $\tau > 0$, $\mathbb{PC}([-\tau, 0]; \mathbb{R}^n)$ denotes the family of piecewise continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$. For $\varphi \in \mathbb{PC}([-\tau, 0]; \mathbb{R}^n)$, the norm is defined as $\|\varphi\|_\tau = \sup_{s\in[-\tau,0]} |\varphi(s)|$. For $p > 0$ and $t \geq 0$, let $L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$ denote the family of all $\mathcal{F}_t$-measurable $\mathbb{PC}([-\tau, 0]; \mathbb{R}^n)$-valued random processes $\varphi = \{\varphi(-s) : -\tau \leq s \leq 0\}$ such that $\sup_{-\tau\leq s\leq 0} \mathbb{E}|\varphi(s)|^p < \infty$. Moreover, $\mathbb{PC}^b([-\tau, 0]; \mathbb{R}^n)$ denote the family of all bounded $\mathbb{PC}([-\tau, 0]; \mathbb{R}^n)$-valued functions.

2. Preliminaries

To begin, we introduce the following nonlinear stochastic system with distributed-delay dependent impulses

$$\begin{cases} \mathrm{d}y(t) = f(t, y_t)\,\mathrm{d}t + g(t, y_t)\,\mathrm{d}w(t), \ t \neq t_k, \ t \geqslant t_0, \\ \Delta y(t) = I_k\left(t, \int_{t-\gamma_k}^{t} y(s)\mathrm{d}s\right), \ t = t_k, \ k \in \mathbb{N}, \\ y_{t_0} = \phi, \ \text{i.e.,}\ y(t_0+s) = \phi(s), \ s \in [-\tau, 0] \end{cases} \tag{2.1}$$

with initial value $\phi \in \mathbb{PC}^b([-\tau, 0]; \mathbb{R}^n)$. $y(t) \in \mathbb{R}^n$ is right continuous at each $t = t_k$. $y_t = y(t+s) \in L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$. For all $k \in \mathbb{N}$, $\gamma_k > 0$ is the distributed delay satisfying $\gamma_k \leq \gamma \leq \tau$ with $\gamma = \sup_{k\in\mathbb{N}} \gamma_k$. $f : \mathbb{R}_+ \times L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$, $g : \mathbb{R}_+ \times L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^{n\times m}$, $I_k : \mathbb{R}_+ \times L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$, and $\Delta y(t_k) = y(t_k) - y(t_k^-)$ with the fixed moments of impulse times t_k satisfying $0 \leqslant t_0 < t_1 < \cdots < t_k \to \infty$ (as $k \to \infty$).

As a standard hypothesis, $f(t,\varphi)$, $g(t,\varphi)$ satisfy the Lipschitz condition. In addition, suppose that $f(t,0)\neq 0$, $g(t,0)\neq 0$ and $I_k(t,0)\neq 0$ for all $t\geqslant t_0$, $k\in\mathbb{N}$, then system (2.1) does not admit a trivial solution $y(t,0)\equiv 0$.

The following assumptions and definitions are presented to obtain the main results.

Assumption 2.1. For almost all $t\in[t_0,\infty)$, $f(t,\varphi)$ and $g(t,\varphi)$ are continuous, and there exist positive constants L_1 and L_2 such that $\|f(t,\varphi)\|\leq L_1\|\varphi\|_\tau$ and $\|g(t,\varphi)\|\leq L_2\|\varphi\|_\tau$ for any $(t,\varphi)\in\mathbb{R}_+\times L^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$.

Assumption 2.2. For all $t\in\mathbb{R}_+$, $x,y\in\mathbb{R}^n$, there exists a positive constant L_3 such that $\|I_k(t,x)-I_k(t,y)\|\leq L_3\|x-y\|$.

Remark 2.1. Assumption 2.1 implies that $f(t,\varphi)$ and $g(t,\varphi)$ satisfy the linear growth condition. Assumption 2.2 means that impulsive function satisfy Lipschitz condition and the linear growth condition. Therefore, there is a unique stochastic process $y(t,t_0,\phi)$ for system (2.1) by [4,23,29].

Assumption 2.3. For all $k\in\mathbb{N}$, there exist positive constants ε and h satisfying $\varepsilon=\inf_{k\in\mathbb{N}}\{t_k-t_{k-1}\}>0$ and $h=\sup_{k\in\mathbb{N}}\{t_k-t_{k-1}\}<\infty$, and $r\varepsilon<\gamma\leq(r+1)\varepsilon$, i.e., the impulse number is no more than r on each interval $[t_k-\gamma_k,t_k]$.

Definition 2.2 (*[4]*). System (2.1) is said to be practically exponentially stable in pth ($p>1$) moment (PESpM), if for all $\phi\in\mathbb{PC}^b([-\tau,0];\mathbb{R}^n)$, there are positive constants z, C and η such that the following inequality holds

$$\mathbb{E}\,|y(t,t_0,\phi)|^p\leq C\|\phi\|^p e^{-z(t-t_0)}+\eta,\ \ t\geq t_0. \tag{2.2}$$

Remark 2.3. Noting that in Definitions 2.2, if let $\eta=0$ and assume $y(t,0)\equiv 0$ for all $t\geq t_0$, we then obtain the classical definition of the exponential stability in pth ($p>1$) moment (ESpM).

Definition 2.4 (*[4]*). The ball $B_\eta:=\{y\in\mathbb{R}^n:|y|\leq\eta\}$, $\eta>0$, is said to be almost surely practically exponentially stable (ASPES) if it follows that

$$|y(t,t_0,\phi)|\leq C\|\phi\|^p e^{-z(t-t_0)}+\eta,\ \ \text{a.s.} \tag{2.3}$$

for any initial data $\phi\in\mathbb{PC}^b([-\tau,0];\mathbb{R}^n)$ and every $t\geq t_0$. Furthermore, if there exists $\eta>0$ such that B_η is ASPES, then system (2.1) is said to be ASPES.

Definition 2.5 (*[4]*). Let $\mathcal{C}^{1,2}([t_0-\tau,\infty)\times\mathbb{R}^n;\mathbb{R}_+)$ denote the family of all nonnegative functions $V(t,y)$ on $[t_0-\tau,\infty)\times\mathbb{R}^n$ that are continuously once differentiable in t and twice in y. For a function $V\in\mathcal{C}^{1,2}([t_0-\tau,\infty)\times\mathbb{R}^n;\mathbb{R}_+)$, we define the operator $\mathcal{L}V:[t_0,\infty)\times L^p_{\mathcal{F}_t}([-\tau,0],\mathbb{R}^n)\to\mathbb{R}$ for system (2.1) by

$$\mathcal{L}V(t,\varphi)=V_t(t,\varphi(0))+V_y(t,\varphi(0))f(t,\varphi)+\frac{1}{2}\,\mathrm{trace}\left[g^T(t,\varphi)V_{yy}(t,\varphi(0))g(t,\varphi)\right], \tag{2.4}$$

where $\varphi\in L^p_{\mathcal{F}_t}([-\tau,0],\mathbb{R}^n)$ and

$$V_t(t,y)=\frac{\partial V(t,y)}{\partial t},\ \ V_y(t,y)=\left(\frac{\partial V(t,y)}{\partial y_1},\ldots,\frac{\partial V(t,y)}{\partial y_n}\right),\ \ V_{yy}(t,y)=\left(\frac{\partial^2 V(t,y)}{\partial y_i\partial y_j}\right)_{n\times n}.$$

Remark 2.6. In the previous theorems given in [4,5], the continuous dynamics of the suggested systems may not be PES, and stabilizing impulses are considered such that the continuous stochastic dynamics are PES. In the forthcoming theorem, sufficient conditions are given to ensure that the continuous dynamics of system (2.1) can maintain PES, when the system is subjected to destabilizing impulses.

3. Main results

In this section, we will discuss the PESpM and the ASPES of system (2.1) by using the LR approach and some inequality techniques.

Theorem 3.1. *Let c_1, c_2, $\bar{c}$, z, α, C_p, q, W_1, W_2, $\kappa>1$, $p>1$ be positive constants and δ be nonnegative constants. If Assumptions 2.1–2.3 are satisfied, and there exist functions $V\in\mathcal{C}^{1,2}([t_0-\tau,\infty)\times\mathbb{R}^n;\mathbb{R}_+)$ and $c\in\mathbb{PC}([t_0-\tau,\infty);\mathbb{R}_+)$ such that for any $x,y\in\mathbb{R}^n$ and $\varphi\in L^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^n)$*

(**H1**) *$c_1|y|^p\leq V(t,y)\leq c_2|y|^p$ for all $t\geq t_0-\tau$;*

(**H2**) *$\mathbb{E}V(t,y+I_k(t,\gamma_k y))\leq\kappa\mathbb{E}V\left(t^-,y\right)+\rho_k e^{-z(t_k-t_0)}$ for all $t=t_k$, $k\in\mathbb{N}$, where $0<\rho_k\leq\rho$ with $\rho=\sup_{k\in\mathbb{N}}\rho_k<\infty$;*

(**H3**) *$\mathbb{E}\mathcal{L}V(t,\varphi)\leq -c(t)\mathbb{E}V(t,\varphi(0))+\alpha e^{-z(t-t_0)}$ for all $t\geqslant t_0$, $t\neq t_k$, $k\in\mathbb{N}$, if $\mathbb{E}V(t+s,\varphi(s))<q\mathbb{E}V(t,\varphi(0))+\delta$ for $s\in[-\tau,0]$;*

(**H4**) *$\mathbb{E}V(t,x+y)\le W_1\mathbb{E}V(t,x)+W_2\mathbb{E}V(t,y)$ for all $t=t_k$, $k\in\mathbb{N}$;*
(**H5**) *$1<W_1\kappa+W_2c_2\frac{3^{p-1}}{c_1}L_3^p\left(L_1^p\gamma^{2p}+L_2^p\gamma^{\frac{3p}{2}}C_p+L_3^p\gamma^{2p}r^p\right)e^{\bar{c}(\tau+\gamma)}<q<e^{\bar{c}\varepsilon}$,*
$\inf_{t\ge t_0}c(t)\ge\bar{c}$.

Then, system (2.1) under destabilizing distributed-delay dependent impulses is PESpM.

Proof. Given an initial data $\phi\in\mathbb{PC}^b([-\tau,0];\mathbb{R}^n)$, we write $y(t,t_0,\phi)=y(t)$ for simplicity. Let $d_1=W_1\kappa$, $d_2=W_2c_2\frac{3^{p-1}}{c_1}L_3^p\left(L_1^p\gamma^{2p}+L_2^p\gamma^{\frac{3p}{2}}C_p+L_3^p\gamma^{2p}r^p\right)$. By condition (**H5**), a sufficiently small $z>0$ can be chosen such that

$$1<e^{z(\tau+\gamma)}\left(d_1+d_2e^{\bar{c}(\tau+\gamma)}\right)<q<e^{(\bar{c}-z)\varepsilon},\ z<\bar{c}, \tag{3.1}$$

and there are $\bar{q}=qe^{-z(\tau+\gamma)}>1$ and $M>0$ such that $\bar{q}c_2<M$. Define $J(t,y(t))=e^{z(t-t_0)}V(t,y(t))$ for $t\ge t_0-\tau$. Then, from condition (**H1**), we see that

$$\mathbb{E}J(t,y(t))\le\mathbb{E}V(t,y(t))\le c_2\mathbb{E}\|\phi\|_\tau^p<\frac{1}{\bar{q}}M\mathbb{E}\parallel\phi|_\tau^p<M\mathbb{E}\|\phi\|_\tau^p+\varsigma e^{z(t-t_0)},\ \forall t\in[t_0-\tau,t_0], \tag{3.2}$$

where $\varsigma\ge\max\left\{\frac{\bar{q}\alpha h}{\bar{q}-1},\frac{\bar{q}\alpha}{\bar{c}-z}-\alpha h,\frac{\left(d_1+d_2e^{\bar{c}(\tau+\gamma)}\right)\alpha h}{\bar{q}-d_1-d_2e^{\bar{c}(\tau+\gamma)}}-\alpha h-\rho\right\}$.

We will show that, for all $t\in[t_{k-1},t_k)$, $k\in\mathbb{N}$,

$$\mathbb{E}J(t,y(t))\le M\mathbb{E}\|\phi\|_\tau^p+\varsigma e^{z(t-t_0)}. \tag{3.3}$$

In view of $z>0$, we can get that $\varsigma\le\varsigma e^{z(t-t_0)}$. This shows that if the following inequality holds, then (3.3) is also true

$$\mathbb{E}J(t,y(t))\le M\mathbb{E}\|\phi\|_\tau^p+\varsigma,\ \forall t\in[t_{k-1},t_k),\ k\in\mathbb{N}. \tag{3.4}$$

Next, we will further prove (3.4), and the following proof will be divided into several steps by induction.

Step 1: First, we will show that

$$\mathbb{E}J(t,y(t))\le M\mathbb{E}\|\phi\|_\tau^p+\varsigma,\ \forall t\in[t_0,t_1). \tag{3.5}$$

If (3.5) is not true, then there must hold that $\mathbb{E}J(t,y(t))>M\mathbb{E}\|\phi\|_\tau^p+\varsigma$ for some $t\in[t_0,t_1)$. Define $t^\triangle=\inf\left\{t\in[t_0,t_1):\mathbb{E}J(t,y(t))>M\mathbb{E}\|\phi\|_\tau^p+\varsigma\right\}$.

Combining the well-known Lebesgue dominated convergence theorem with condition (**H1**), we can get that $t\in[t_0,t_1)\mapsto\mathbb{E}V(t,y(t))$ is a continuous mapping. In addition, the property of continuous function shows that $\mathbb{E}J(t,y(t))=e^{z(t-t_0)}\mathbb{E}V(t,y(t))$ is also continuous over $t\in[t_0,t_1)$. Therefore, we have

$$\begin{aligned}\lim_{t\to t_0^-}\mathbb{E}J(t,y(t))&=\lim_{t\to t_0^+}\mathbb{E}J(t,y(t))\\&=\mathbb{E}J(t_0,y(t_0))\\&<M\mathbb{E}\|\phi\|^p+\varsigma,\end{aligned}$$

which implies that $t^\triangle\in(t_0,t_1)$. Then, there hold that

$$\mathbb{E}J\left(t^\triangle,y(t^\triangle)\right)=M\mathbb{E}\|\phi\|_\tau^p+\varsigma, \tag{3.6}$$

$$\mathbb{E}J(t,y(t))<M\mathbb{E}\|\phi\|_\tau^p+\varsigma,\ \forall t\in\left[t_0-\tau,t^\triangle\right). \tag{3.7}$$

Clearly, there exists $t_\triangledown\in[t_0,t^\triangle)$ satisfying $t_\triangledown=\sup\{t\in[t_0,t^\triangle]:\mathbb{E}J(t,y(t))\le\frac{1}{\bar{q}}M\mathbb{E}\|\phi\|_\tau^p+\frac{\varsigma}{\bar{q}}\}$ such that

$$\mathbb{E}J(t_\triangledown,y(t_\triangledown))=\frac{1}{\bar{q}}M\mathbb{E}\|\phi\|_\tau^p+\frac{\varsigma}{\bar{q}}, \tag{3.8}$$

$$\mathbb{E}J(t,y(t))>\frac{1}{\bar{q}}M\mathbb{E}\|\phi\|_\tau^p+\frac{\varsigma}{\bar{q}},\ \forall t\in\left(t_\triangledown,t^\triangle\right]. \tag{3.9}$$

Consequently, for all $t\in\left[t_\triangledown,t^\triangle\right]$, we get from (3.6)–(3.9) that

$$\mathbb{E}J(t+s,y(t+s))\le M\mathbb{E}\|\phi\|_\tau^p+\varsigma\le\bar{q}\mathbb{E}J(t,y(t)),\ s\in[-\tau,0],$$

which leads to

$$\begin{aligned}\mathbb{E}V(t+s,y(t+s))&=e^{-z(t+s-t_0)}\mathbb{E}J(t+s,y(t+s))\\&\le\bar{q}e^{-z(t+s-t_0)}\mathbb{E}J(t,y(t))\\&\le\bar{q}e^{z\tau}\mathbb{E}V(t,y(t))\\&<q\mathbb{E}V(t,y(t))+\delta,\ s\in[-\tau,0].\end{aligned} \tag{3.10}$$

This together with condition (**H3**) shows that

$$\begin{aligned}\mathbb{E}\mathcal{L}J(t,y_t) &= e^{z(t-t_0)}(z\mathbb{E}V(t,y(t))+\mathbb{E}\mathcal{L}V(t,y_t))\\ &\le (z-c(t))\mathbb{E}J(t,y(t))+\alpha,\ \ \forall t\in\left[t_\nabla,t^\triangle\right].\end{aligned}\tag{3.11}$$

Then, using (3.11) and the Itô formula on $[t_\nabla, t^\triangle]$, we further have

$$\begin{aligned}\mathbb{E}J\left(t^\triangle,y(t^\triangle)\right) &= \mathbb{E}J\left(t_\nabla,y(t_\nabla)\right)+\int_{t_\nabla}^{t^\triangle}\mathbb{E}\mathcal{L}J(s,y_s)\mathrm{d}s\\ &\le \mathbb{E}J\left(t_\nabla,y(t_\nabla)\right)+\int_{t_\nabla}^{t^\triangle}[(z-c(s))\mathbb{E}J(s,y(s))+\alpha]\mathrm{d}s\\ &=\mathbb{E}J\left(t_\nabla,y(t_\nabla)\right)+\int_{t_\nabla}^{t^\triangle}(z-c(s))\mathbb{E}J(s,y(s))\mathrm{d}s+\alpha\left(t^\triangle-t_\nabla\right),\end{aligned}$$

and thus,

$$\mathbb{E}J\left(t^\triangle,y(t^\triangle)\right)\le \mathbb{E}J\left(t_\nabla,y(t_\nabla)\right)+\int_{t_\nabla}^{t^\triangle}(z-c(s))\mathbb{E}J(s,y(s))\mathrm{d}s+\alpha h.\tag{3.12}$$

Thus, from (3.1), (3.6), (3.8) and the Gronwall inequality, we infer that

$$\begin{aligned}\mathbb{E}J\left(t^\triangle,y(t^\triangle)\right) &\le \left(\mathbb{E}J\left(t_\nabla,y(t_\nabla)\right)+\alpha h\right)e^{\int_{t_\nabla}^{t^\triangle}(z-c(s))\mathrm{d}s}\\ &\le \left(\mathbb{E}J\left(t_\nabla,y(t_\nabla)\right)+\alpha h\right)e^{(z-\bar{c})\left(t^\triangle-t_\nabla\right)}\\ &\le \frac{1}{\bar{q}}M\mathbb{E}\|\phi\|_\tau^p+\frac{1}{\bar{q}}\varsigma+\alpha h\\ &< M\mathbb{E}\|\phi\|_\tau^p+\frac{1}{\bar{q}}\varsigma+\alpha h\\ &\le M\mathbb{E}\|\phi\|_\tau^p+\varsigma\\ &= \mathbb{E}J\left(t^\triangle,y(t^\triangle)\right),\end{aligned}$$

which is a contradiction. It is easy to see that (3.5) is true and (3.4) must hold for $k=1$. Therefore, (3.3) is satisfied for $k=1$.

Step 2: Assume that for $k=1,2,\ldots,l$ ($l\in\mathbb{N}$, $l\ge 1$),

$$\mathbb{E}J(t,y(t))\le M\mathbb{E}\|\phi\|_\tau^p+\varsigma,\ \ \forall t\in[t_{k-1},t_k).\tag{3.13}$$

Next, we turn to show that (3.4) holds as well for $k=l+1$, i.e.,

$$\mathbb{E}J(t,y(t))\le M\mathbb{E}\|\phi\|_\tau^p+\varsigma,\ \ \forall t\in[t_l,t_{l+1}).\tag{3.14}$$

Suppose (3.14) is not true and further define $t^\triangle=\inf\{t\in[t_l,t_{l+1}):\mathbb{E}J(t,y(t))>M\mathbb{E}\|\phi\|_\tau^p+\ell\}$, where $\ell=\varrho+\alpha h$ with $\varrho=\varsigma+\alpha h+\rho$. Then, for $t\in[t_l-\gamma_l,t_l)$, integrating system (2.1) on both sides from t to t_l^- follows that

$$y\left(t_l^-\right)-y(t)=\int_t^{t_l}f\left(s,y_s\right)\mathrm{d}s+\int_t^{t_l}g\left(s,y_s\right)\mathrm{d}w_s+\sum_{i=1}^{r(t)}I_{l-i}\left(t_{l-i},\int_{t_{l-i}-\gamma_{l-i}}^{t_{l-i}}y(s)\mathrm{d}s\right),\tag{3.15}$$

where $r(t)$ stands for the number of impulses on the interval $\left[t,t_l^-\right)$. Now, let us further integrate both sides of (3.15) over $[t_l-\gamma_l,t_l]$,

$$\begin{aligned}&\left\|\gamma_l y\left(t_l^-\right)-\int_{t_l-\gamma_l}^{t_l}y(s)\mathrm{d}s\right\|\\ &=\Bigg\|\int_{t_l-\gamma_l}^{t_l}\int_t^{t_l}f\left(s,y_s\right)\mathrm{d}s\,\mathrm{d}t+\int_{t_l-\gamma_l}^{t_l}\int_t^{t_l}g\left(s,y_s\right)\mathrm{d}w_s\,\mathrm{d}t\\ &\quad+\int_{t_l-\gamma_l}^{t_l}\left(\sum_{i=1}^{r(t)}I_{l-i}\left(t_{l-i},\int_{t_{l-i}-\gamma_{l-i}}^{t_{l-i}}y(s)\mathrm{d}s\right)\right)\mathrm{d}t\Bigg\|\\ &\le\left\|\int_{t_l-\gamma_l}^{t_l}\int_t^{t_l}f\left(s,y_s\right)\mathrm{d}s\,\mathrm{d}t\right\|+\left\|\int_{t_l-\gamma_l}^{t_l}\int_t^{t_l}g\left(s,y_s\right)\mathrm{d}w_s\,\mathrm{d}t\right\|\\ &\quad+\left\|\int_{t_l-\gamma_l}^{t_l}\left(\sum_{i=1}^{r(t)}I_{l-i}\left(t_{l-i},\int_{t_{l-i}-\gamma_{l-i}}^{t_{l-i}}y(s)\mathrm{d}s\right)\right)\mathrm{d}t\right\|.\end{aligned}\tag{3.16}$$

Define $\Delta I_l := I_l\left(t_l, \int_{t_l-\gamma_l}^{t_l} y(s)\mathrm{d}s\right) - I_l\left(t_l, \gamma_l y\left(t_l^-\right)\right)$. By Assumption 2.2, we have

$$\mathbb{E}\left\|\Delta I_l\right\|^p \le L_3^p \mathbb{E}\left\|\gamma_l y\left(t_l^-\right) - \int_{t_l-\gamma_l}^{t_l} y(s)\mathrm{d}s\right\|^p. \tag{3.17}$$

This together with C_r-inequality yields that

$$\begin{aligned}
&\mathbb{E}\left\|\gamma_l y\left(t_l^-\right) - \int_{t_l-\gamma_l}^{t_l} y(s)\mathrm{d}s\right\|^p \\
&\le \mathbb{E}\Big(\left\|\int_{t_l-\gamma_l}^{t_l}\int_t^{t_l} f\left(s, y_s\right)\mathrm{d}s\,\mathrm{d}t\right\| + \left\|\int_{t_l-\gamma_l}^{t_l}\int_t^{t_l} g\left(s, y_s\right)\mathrm{d}w_s\,\mathrm{d}t\right\| \\
&\quad + \left\|\int_{t_l-\gamma_l}^{t_l}\left(\sum_{i=1}^{r(t)} I_{l-i}\left(t_{l-i}, \int_{t_{l-i}-\gamma_{l-i}}^{t_{l-i}} y(s)\mathrm{d}s\right)\right)\mathrm{d}t\right\| \Big)^p \\
&\le 3^{p-1}\mathbb{E}\left\|\int_{t_l-\gamma_l}^{t_l}\int_t^{t_l} f\left(s, y_s\right)\mathrm{d}s\,\mathrm{d}t\right\|^p + 3^{p-1}\mathbb{E}\left\|\int_{t_l-\gamma_l}^{t_l}\int_t^{t_l} g\left(s, y_s\right)\mathrm{d}w_s\,\mathrm{d}t\right\|^p \\
&\quad + 3^{p-1}\mathbb{E}\left\|\int_{t_l-\gamma_l}^{t_l}\left(\sum_{i=1}^{r(t)} I_{l-i}\left(t_{l-i}, \int_{t_{l-i}-\gamma_{l-i}}^{t_{l-i}} y(s)\mathrm{d}s\right)\right)\mathrm{d}t\right\|^p.
\end{aligned} \tag{3.18}$$

Then, using the Hölder inequality, Burkholder–Davis–Gundy inequality and Assumption 2.1, we arrive at

$$\begin{aligned}
&\mathbb{E}\left\|\int_{t_l-\gamma_l}^{t_l}\int_t^{t_l} f\left(s, y_s\right)\mathrm{d}s\,\mathrm{d}t\right\|^p \\
&\le \mathbb{E}\left(\int_{t_l-\gamma_l}^{t_l}\int_t^{t_l} \left\|f\left(s, y_s\right)\right\|\mathrm{d}s\,\mathrm{d}t\right)^p \\
&\le \mathbb{E}\left(\int_{t_l-\gamma_l}^{t_l}\int_t^{t_l} L_1\left\|y_s\right\|_\tau \mathrm{d}s\,\mathrm{d}t\right)^p \\
&\le L_1^p \gamma_l^p \mathbb{E}\left(\int_{t_l-\gamma_l}^{t_l} \left\|y_s\right\|_\tau ds\right)^p \\
&\le L_1^p \gamma_l^p \mathbb{E}\left(\int_{t_l-\gamma_l}^{t_l} \sup_{\xi\in[-\tau,0]} \left\|y(s+\xi)\right\|\mathrm{d}s\right)^p \\
&\le L_1^p \gamma_l^{2p} \mathbb{E}\left(\sup_{\xi\in[-\tau-\gamma_l,0]} \left\|y\left(t_l^- + \xi\right)\right\|\right)^p \\
&\le L_1^p \gamma^{2p} \sup_{\xi\in[-\tau-\gamma,0]} \mathbb{E}\left\|y\left(t_l^- + \xi\right)\right\|^p
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
&\mathbb{E}\left\|\int_{t_l-\gamma_l}^{t_l}\int_t^{t_l} g\left(s, y_s\right)\mathrm{d}w_s\,\mathrm{d}t\right\|^p \\
&\le \gamma_l^p \mathbb{E}\left(\sup_{t\in[t_l-\gamma_l,t_l]} \left\|\int_t^{t_l} g\left(s, y_s\right)\mathrm{d}w_s\right\|^p\right) \\
&\le \gamma_l^p C_p \mathbb{E}\left(\int_{t_l-\gamma_l}^{t_l} \left\|g\left(s, y_s\right)\right\|^2 \mathrm{d}s\right)^{\frac{p}{2}} \\
&\le \gamma_l^p \gamma_l^{\frac{p}{2}-1} C_p \mathbb{E}\left(\int_{t_l-\gamma_l}^{t_l} \left\|g\left(s, y_s\right)\right\|^p \mathrm{d}s\right) \\
&\le L_2^p \gamma_l^{\frac{3p}{2}-1} C_p \mathbb{E}\left(\int_{t_l-\gamma_l}^{t_l} \left\|y_s\right\|^p \mathrm{d}s\right) \\
&\le L_2^p \gamma_l^{\frac{3p}{2}-1} C_p \mathbb{E}\left(\int_{t_l-\gamma_l}^{t_l} \left(\sup_{\xi\in[-\tau,0]} \left\|y\left(s+\xi\right)\right\|\right)^p \mathrm{d}s\right) \\
&\le L_2^p \gamma_l^{\frac{3p}{2}} C_p \sup_{\xi\in[-\tau-\gamma_l,0]} \mathbb{E}\left\|y\left(t_l^- + \xi\right)\right\|^p \\
&\le L_2^p \gamma^{\frac{3p}{2}} C_p \sup_{\xi\in[-\tau-\gamma,0]} \mathbb{E}\left\|y\left(t_l^- + \xi\right)\right\|^p,
\end{aligned} \tag{3.20}$$

where $C_p > 0$ is a positive constant given by the Burkholder–Davis–Gundy inequality, and only depends on p. Meanwhile, applying again Hölder inequality and Assumptions 2.2, 2.3, we arrive at

$$\begin{aligned}
&\mathbb{E}\left\|\int_{t_l-\gamma_l}^{t_l}\left(\sum_{i=1}^{r(t)} I_{l-i}\left(t_{l-i},\int_{t_{l-i}-\gamma_{l-i}}^{t_{l-i}} y(s)\mathrm{d}s\right)\right)\mathrm{d}t\right\|^p\\
&\le \mathbb{E}\left(\int_{t_l-\gamma_l}^{t_l}\left(\sum_{i=1}^{r(t)}\left\|I_{l-i}\left(t_{l-i},\int_{t_{l-i}-\gamma_{l-i}}^{t_{l-i}} y(s)\mathrm{d}s\right)\right\|\right)\mathrm{d}t\right)^p\\
&\le \mathbb{E}\left(\int_{t_l-\gamma_l}^{t_l}\left(\sum_{i=1}^{r(t)} L_3\left\|\int_{t_{l-i}-\gamma_{l-i}}^{t_{l-i}} y(s)\mathrm{d}s\right\|\right)\mathrm{d}t\right)^p\\
&\le L_3^p\gamma_l^p\mathbb{E}\left(\sum_{i=1}^{r}\int_{t_{l-i}-\gamma_{l-i}}^{t_{l-i}}\|y(s)\|\mathrm{d}s\right)^p\\
&\le L_3^p\gamma_l^p\mathbb{E}\left(\sum_{i=1}^{r}\gamma_{l-i}\sup_{\xi\in[-\gamma_{l-i},0]}\|y\left(t_{l-i}+\xi\right)\|\right)^p\\
&\le L_3^p\gamma_l^p\gamma^p r^p\sup_{\xi\in[-2\gamma,0]}\mathbb{E}\left\|y\left(t_l^-+\xi\right)\right\|^p\\
&\le L_3^p\gamma^{2p} r^p\sup_{\xi\in[-\tau-\gamma,0]}\mathbb{E}\left\|y\left(t_l^-+\xi\right)\right\|^p.
\end{aligned}\tag{3.21}$$

Thus, it follows from (3.18)–(3.21) that

$$\begin{aligned}
&\mathbb{E}\left\|\gamma_l y\left(t_l^-\right)-\int_{t_l-\gamma_l}^{t_l} y(s)\mathrm{d}s\right\|^p\\
&\le 3^{p-1}\left(L_1^p\gamma^{2p}+L_2^p\gamma^{\frac{3p}{2}}C_p+L_3^p\gamma^{2p}r^p\right)\sup_{\xi\in[-\tau-\gamma,0]}\mathbb{E}\left\|y\left(t_l^-+\xi\right)\right\|^p.
\end{aligned}\tag{3.22}$$

According to (3.22) and condition (**H1**), we can derive that

$$\begin{aligned}
\mathbb{E}\left\|\varDelta I_l\right\|^p &\le L_3^p\mathbb{E}\left\|\gamma_l y\left(t_l^-\right)-\int_{t_l-\gamma_l}^{t_l} y(s)\mathrm{d}s\right\|^p\\
&\le 3^{p-1}L_3^p\left(L_1^p\gamma^{2p}+L_2^p\gamma^{\frac{3p}{2}}C_p+L_3^p\gamma^{2p}r^p\right)\sup_{\xi\in[-\tau-\gamma,0]}\mathbb{E}\left\|y\left(t_l^-+\xi\right)\right\|^p\\
&\le \frac{3^{p-1}}{c_1}L_3^p\left(L_1^p\gamma^{2p}+L_2^p\gamma^{\frac{3p}{2}}C_p+L_3^p\gamma^{2p}r^p\right)\sup_{\xi\in[-\tau-\gamma,0]}\mathbb{E}V\left(t_l^-+\xi\right).
\end{aligned}\tag{3.23}$$

This together with conditions (**H1**), (**H2**) and (**H4**) further gives that

$$\begin{aligned}
\mathbb{E}V\left(t_l,y(t_l)\right) &= \mathbb{E}V\left(t_l,y\left(t_l^-\right)+I_l\left(t_l,\int_{t_l-\gamma_l}^{t_l} y(s)\mathrm{d}s\right)\right)\\
&= \mathbb{E}V\left(t_l,y\left(t_l^-\right)+I_l\left(t_l,\gamma_l y\left(t_l^-\right)\right)+\varDelta I_l\right)\\
&\le W_1\mathbb{E}V\left(t_l,y\left(t_l^-\right)+I_l\left(t_l,\gamma_l y\left(t_l^-\right)\right)\right)+W_2\mathbb{E}V\left(t_l,\varDelta I_l\right)\\
&\le W_1\kappa\mathbb{E}V\left(t_l^-,y\left(t_l^-\right)\right)+W_2c_2\mathbb{E}\left\|\varDelta I_l\right\|^p+\rho_l e^{-z(t_l-t_0)}.
\end{aligned}\tag{3.24}$$

By using (3.23) and (3.24), we have

$$\begin{aligned}
&\mathbb{E}J\left(t_l,y(t_l)\right)\\
&\le e^{z(t_l-t_0)}\mathbb{E}V\left(t_l,y(t_l)\right)\\
&\le W_1\kappa\mathbb{E}J\left(t_l^-,y\left(t_l^-\right)\right)+W_2c_2e^{z(t_l-t_0)}\mathbb{E}\left\|\varDelta I_l\right\|^p+\rho_l\\
&\le W_1\kappa\mathbb{E}J\left(t_l^-,y\left(t_l^-\right)\right)+W_2c_2e^{z(t_l-t_0)}\Big[\frac{3^{p-1}}{c_1}L_3^p\left(L_1^p\gamma^{2p}+L_2^p\gamma^{\frac{3p}{2}}C_p+L_3^p\gamma^{2p}r^p\right)\\
&\sup_{\xi\in[-\tau-\gamma,0]}\mathbb{E}V\left(t_l^-+\xi\right)\Big]+\rho_l\\
&\le W_1\kappa\mathbb{E}J\left(t_l^-,y\left(t_l^-\right)\right)+W_2c_2\Big[\frac{3^{p-1}}{c_1}L_3^p\left(L_1^p\gamma^{2p}+L_2^p\gamma^{\frac{3p}{2}}C_p+L_3^p\gamma^{2p}r^p\right)e^{z(\tau+\gamma)}\\
&\sup_{\xi\in[-\tau-\gamma,0]}\mathbb{E}J\left(t_l^-+\xi\right)\Big]+\rho_l.
\end{aligned}\tag{3.25}$$

In what follows, we first claim that

$$\mathbb{E}J\left(t_l^-, y(t_l^-)\right) \leq \frac{1}{q} M\mathbb{E}\|\phi\|_\tau^p + \frac{1}{q}(\varsigma + \alpha h). \tag{3.26}$$

Arguing by contradiction once more, we assume that (3.26) is not true, and then, two possible cases will be examined.

Case 1: For all $t \in [t_{l-1}, t_l)$, $\mathbb{E}J\left(t, y(t)\right) > \frac{1}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{1}{\bar{q}}(\varsigma + \alpha h)$. Combining this with (3.13), one can see that, for $t \in [t_{l-1}, t_l)$,

$$\mathbb{E}J(t+s, y(t+s)) \leq M\mathbb{E}\|\phi\|_\tau^p + \varsigma < \bar{q}\mathbb{E}J(t, y(t)) - \alpha h < \bar{q}\mathbb{E}J(t, y(t)),\ s \in [-\tau, 0]. \tag{3.27}$$

Similar to (3.11), one can check that $\mathbb{E}\mathcal{L}J(t, y_t) \leq (z - c(t))\mathbb{E}J(t, y(t)) + \alpha$ for all $t \in \left[t_{l-1}, t_l^-\right]$. Then, applying the Itô formula on $\left[t_{l-1}, t_l^-\right]$, we know that

$$\begin{aligned}
\mathbb{E}J\left(t_l^-, y(t_l^-)\right) &= \mathbb{E}J\left(t_{l-1}, y(t_{l-1})\right) + \int_{t_{l-1}}^{t_l^-} \mathbb{E}\mathcal{L}J(s, y_s)\mathrm{d}s \\
&\leq \mathbb{E}J\left(t_{l-1}, y(t_{l-1})\right) + \int_{t_{l-1}}^{t_l^-} (z - c(s))\mathbb{E}J(s, y(s))\mathrm{d}s + \alpha(t_l^- - t_{l-1}).
\end{aligned} \tag{3.28}$$

By (3.1), (3.13), the Gronwall inequality and the property of definite integral, we further know that

$$\begin{aligned}
\mathbb{E}J\left(t_l^-, y(t_l^-)\right) &\leq \left[\mathbb{E}J\left(t_{l-1}, y(t_{l-1})\right) + \alpha\left(t_l^- - t_{l-1}\right)\right] e^{\int_{t_{l-1}}^{t_l^-}(z - c(s))\mathrm{d}s} \\
&\leq M\mathbb{E}\|\phi\|_\tau^p e^{(z-\bar{c})\left(t_l - t_{l-1}\right)} + e^{(z-\bar{c})\left(t_l - t_{l-1}\right)}(\varsigma + \alpha h) \\
&\leq \frac{1}{q} M\mathbb{E}\|\phi\|_\tau^p + \frac{1}{q}(\varsigma + \alpha h) \\
&< \frac{1}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{1}{\bar{q}}(\varsigma + \alpha h) \\
&< \mathbb{E}J\left(t_l^-, y(t_l^-)\right),
\end{aligned} \tag{3.29}$$

which is a contradiction. Clearly, (3.26) is true in Case 1.

Case 2: There exists $t \in [t_{l-1}, t_l)$ such that $\mathbb{E}J\left(t, y(t)\right) > \frac{1}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{1}{\bar{q}}(\varsigma + \alpha h)$. In this case, define $\bar{t} = \sup\{t \in [t_{l-1}, t_l) : \mathbb{E}J(t, y(t)) \leq \frac{1}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{1}{\bar{q}}(\varsigma + \alpha h)\}$. Then,

$$\mathbb{E}J(\bar{t}, y(\bar{t})) = \frac{1}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{1}{\bar{q}}\left(\varsigma + \alpha h\right), \tag{3.30}$$

$$\mathbb{E}J(t, y(t)) > \frac{1}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{1}{\bar{q}}\left(\varsigma + \alpha h\right),\ \forall t \in \left(\bar{t}, t_l\right). \tag{3.31}$$

Similar to the argument in Case 1, from (3.13) and (3.31), we know that (3.27) also holds for $t \in \left[\bar{t}, t_l\right)$ in Case 2. Then, according to the proof processes of (3.10) and (3.11), it can be checked that $\mathbb{E}\mathcal{L}J(t, y_t) \leq (z - c(t))\mathbb{E}J(t, y(t)) + \alpha$ for all $t \in \left[\bar{t}, t_l^-\right]$. Obviously, by using the Itô formula once more, one can see that $\mathbb{E}J\left(t_l^-, y(t_l^-)\right) \leq \mathbb{E}J\left(\bar{t}, y(\bar{t})\right) + \int_{\bar{t}}^{t_l^-}(z - c(s))\mathbb{E}J(s, y(s))\mathrm{d}s + \alpha(t_l^- - \bar{t})$. Now, combining this with the Gronwall inequality, the property of definite integral and (3.30), we have

$$\begin{aligned}
\mathbb{E}J\left(t_l^-, y(t_l^-)\right) &\leq [\mathbb{E}J(\bar{t}, y(\bar{t})) + \alpha(t_l^- - \bar{t})]e^{\int_{\bar{t}}^{t_l^-}(z - c(s))\mathrm{d}s} \\
&\leq [\frac{1}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{1}{\bar{q}}(\varsigma + \alpha h) + \alpha\left(t_l^- - \bar{t}\right)]e^{(z-\bar{c})(t_l - \bar{t})} \\
&\leq \frac{1}{q} M\mathbb{E}\|\phi\|_\tau^p + \frac{1}{\bar{q}} e^{(z-\bar{c})(t_l - \bar{t})}(\varsigma + \alpha h) + e^{(z-\bar{c})(t_l - \bar{t})}\alpha\left(t_l^- - \bar{t}\right) \\
&\leq \frac{1}{q} M\mathbb{E}\|\phi\|_\tau^p + \frac{1}{\bar{q}} e^{(z-\bar{c})(t_l - \bar{t})}(\varsigma + \alpha h) + e^{(z-\bar{c})(t_l - \bar{t})}\alpha\left(t_l - \bar{t}\right).
\end{aligned}$$

Together with the fact that $e^{(z-\bar{c})(t_l - \bar{t})} < 1$, we can find that $\frac{1}{\bar{q}} e^{(z-\bar{c})(t_l - \bar{t})}(\varsigma + \alpha h) + e^{(z-\bar{c})(t_l - \bar{t})}\alpha\left(t_l - \bar{t}\right) \leq \frac{1}{q}(\varsigma + \alpha h)$ is allowed to be true, which can be rewritten as

$$\varsigma + \alpha h \geq \frac{\bar{q} e^{(z-\bar{c})(t_l - \bar{t})}\alpha\left(t_l - \bar{t}\right)}{1 - e^{(z-\bar{c})(t_l - \bar{t})}}.$$

On the other hand, it is noted that if we consider all possible $t \in [t_{l-1}, t_l)$ satisfying $\mathbb{E}J(t, y(t)) > \frac{1}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{1}{\bar{q}}(\varsigma + \alpha h)$, then $\bar{t}$ will become a continuous random variable on $[t_{l-1}, t_l)$. Thus, we can define the following function:

$$\Psi(\bar{t}) = \frac{\bar{q} e^{(z-\bar{c})(t_l-\bar{t})} \alpha \left(t_l - \bar{t}\right)}{1 - e^{(z-\bar{c})(t_l-\bar{t})}}, \;\; \bar{t} \in [t_{l-1}, t_l) .$$

Then, to prove (3.26), we only need to verify that $\varsigma + \alpha h \in [\min \Psi(\bar{t}), b]$ is true, where b is a arbitrary constant satisfying $b < \infty$ (i.e., for all $t \in [t_{l-1}, t_l)$ satisfying $\mathbb{E}J(t, y(t)) > \frac{1}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{1}{\bar{q}}(\varsigma + \alpha h)$, there must be at least one $\bar{t} \in [t_{l-1}, t_l)$ such that $\varsigma + \alpha h \geq (\frac{\bar{q} e^{(z-\bar{c})(t_l-\bar{t})} \alpha (t_l - \bar{t})}{1 - e^{(z-\bar{c})(t_l-\bar{t})}})$. To simplify computation, let $u = (\bar{c} - z)(t_l - \bar{t})$, then

$$\Psi(t_l - \frac{u}{\bar{c} - z}) = \frac{\bar{q}\alpha u}{(z - \bar{c})(1 - e^u)}, \;\; u \in (0, (\bar{c} - z)(t_l - t_{l-1})] ,$$

and

$$\frac{\mathrm{d}\Psi(t_l - \frac{u}{\bar{c}-z})}{\mathrm{d}u} = -\bar{q}\alpha \frac{e^u - 1 - ue^u}{(e^u - 1)^2}.$$

Obviously, $\frac{\mathrm{d}\Psi(t_l - \frac{u}{\bar{c}-z})}{\mathrm{d}u} > 0$ for all $u \in (0, (\bar{c} - z)(t_l - t_{l-1})]$. Thus, $\Psi(t_l - \frac{u}{\bar{c}-z})$ is a strictly monotone increasing function on $(0, (\bar{c} - z)(t_l - t_{l-1})]$. Meanwhile, it can be checked that $\lim_{u\to 0} \Psi(t_l - \frac{u}{\bar{c}-z}) = \frac{\bar{q}\alpha}{\bar{c}-z}$. Now, we can conclude that if $\varsigma \geq \frac{\bar{q}\alpha}{\bar{c}-z} - \alpha h$, then there must exist some $t \in [t_{l-1}, t_l)$ such that $\varsigma + \alpha h \geq (\frac{\bar{q} e^{(z-\bar{c})(t_l-\bar{t})} \alpha (t_l - \bar{t})}{1 - e^{(z-\bar{c})(t_l-\bar{t})}})$. Then, we easily get that

$$\begin{aligned} \mathbb{E}J\left(t_l^-, y(t_l^-)\right) &\leq \frac{1}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{1}{\bar{q}}(\varsigma + \alpha h) \\ &< \mathbb{E}J\left(t_l^-, y(t_l^-)\right), \end{aligned}$$

which is a contradiction. Thus, we can conclude that (3.26) also holds in Case 2. By simple induction, (3.26) must be true for all possible cases.

Similar to (3.26), we can further claim that

$$\mathbb{E}J\left(t_l^- + \xi, y(t_l^- + \xi)\right) \leq \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}}(\varsigma + \alpha h), \;\; \xi \in [-\tau - \gamma, 0]. \tag{3.32}$$

If (3.32) is not true, then there must exist $\Lambda \in [-\tau - \gamma, 0]$ such that $\mathbb{E}J\left(t_l^- + \Lambda, y(t_l^- + \Lambda)\right) > \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}}(\varsigma + \alpha h)$. To avoid loss of generality, we assume that $t_l + \Lambda \in (t_{m-1}, t_m]$, $m \in \mathbb{N}$, $m \leq l$. In the following, there also exist two cases to be considered.

Case 1: $\mathbb{E}J(t, y(t)) > \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}}(\varsigma + \alpha h)$ over $t \in [t_{m-1}, t_l + \Lambda)$. Together this with (3.13), for all $t \in [t_{m-1}, t_l + \Lambda)$, we can obtain that

$$\begin{aligned} \mathbb{E}J(t + \xi, y(t + \xi)) &\leq M\mathbb{E}\|\phi\|_\tau^p + \varsigma \\ &< e^{(\bar{c}-z)(\tau+\gamma)} M\mathbb{E}\|\phi\|_\tau^p + \varsigma \\ &< \bar{q}\mathbb{E}J(t, y(t)) - \alpha h \\ &< \bar{q}\mathbb{E}J(t, y(t)), \;\; \xi \in [-\tau - \gamma, 0]. \end{aligned} \tag{3.33}$$

By the similar proof processes of (3.11) and (3.12), one can check that $\mathbb{E}\mathcal{L}J(t, y_t) \leq (z - c(t))\mathbb{E}J(t, y(t)) + \alpha$ for all $t \in \left[t_{m-1}, t_l^- + \Lambda\right]$, and then $\mathbb{E}J\left(t_l^- + \Lambda, y(t_l^- + \Lambda)\right) \leq \mathbb{E}J\left(t_{m-1}, y(t_{m-1})\right) + \int_{t_{m-1}}^{t_l^- + \Lambda} (z - c(s))\mathbb{E}J(s, y(s))$ $\mathrm{d}s + \alpha(t_l^- + \Lambda - t_{m-1})$. Combining this with the proof of (3.29), we consequently have

$$\begin{aligned} \mathbb{E}J\left(t_l^- + \Lambda, y(t_l^- + \Lambda)\right) &\leq \left[\mathbb{E}J\left(t_{m-1}, y(t_{m-1})\right) + \alpha\left(t_l^- + \Lambda - t_{m-1}\right)\right] e^{\int_{t_{m-1}}^{t_l^- + \Lambda} (z - c(s))\mathrm{d}s} \\ &\leq \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{q^{l-m+1}} M\mathbb{E}\|\phi\|_\tau^p + \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{q^{l-m+1}}(\varsigma + \alpha h) \\ &< \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}}(\varsigma + \alpha h) \\ &< \mathbb{E}J\left(t_l^- + \Lambda, y(t_l^- + \Lambda)\right), \end{aligned}$$

which is a contradiction. Thus, (3.32) holds in this case.

Case 2: There exists $t \in [t_{m-1}, t_l + \Lambda)$ such that $\mathbb{E}J(t, y(t)) > \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}}(\varsigma + \alpha h)$. Then, there must exist $\underline{t} \in [t_{m-1}, t_l + \Lambda)$ satisfying $\underline{t} = \sup\{t \in [t_{m-1}, t_l + \Lambda) : \mathbb{E}J(t, y(t)) \le \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}}(\varsigma+\alpha h)\}$ such that

$$\mathbb{E}J(\underline{t}, y(\underline{t})) = \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}}(\varsigma + \alpha h), \tag{3.34}$$

$$\mathbb{E}J(t, y(t)) > \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}}(\varsigma + \alpha h), \ \forall t \in (\underline{t}, t_l + \Lambda). \tag{3.35}$$

The rest is also similar to Case 2 in the proof of (3.26), it can be verified that

$\mathbb{E}J\left(t_l^- + \Lambda, y(t_l^- + \Lambda)\right) \le \mathbb{E}J\left(\underline{t}, y(\underline{t})\right) + \int_{\underline{t}}^{t_l^-+\Lambda}(z - c(s))\mathbb{E}J(s, y(s))\mathrm{d}s + \alpha(t_l^- + \Lambda - \underline{t})$ for $t \in \left[\underline{t}, t_l^- + \Lambda\right]$, and then, there exists $\varsigma \ge \frac{\bar{q}e^{(z-\bar{c})(\tau+\gamma)}\alpha}{\bar{c}-z} - \alpha h$ such that

$$\begin{aligned}\mathbb{E}J\left(t_l^- + \Lambda, y(t_l^- + \Lambda)\right) &\le \left[\mathbb{E}J\left(\underline{t}, y(\underline{t})\right) + \alpha\left(t_l^- + \Lambda - \underline{t}\right)\right] e^{\int_{\underline{t}}^{t_l^-+\Lambda}(z-c(s))\mathrm{d}s} \\ &\le \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}}(\varsigma + \alpha h) \\ &< \mathbb{E}J\left(t_l^- + \Lambda, y(t_l^- + \Lambda)\right), \ \Lambda \in [-\tau - \gamma, 0].\end{aligned}$$

Thus, (3.32) also holds in this case. Obviously, (3.32) must be true for all possible cases.

Consequently, by (3.1), (3.25), (3.26) and (3.32), one can see that

$$\begin{aligned}&\mathbb{E}J\left(t_l, y(t_l)\right) \\ &\le d_1\left[\frac{1}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{1}{\bar{q}}(\varsigma + \alpha h)\right] + d_2 e^{z(\tau+\gamma)}\left[\frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{e^{(\bar{c}-z)(\tau+\gamma)}}{\bar{q}}(\varsigma + \alpha h)\right] + \rho_l \\ &\le \frac{d_1 + d_2 e^{\bar{c}(\tau+\gamma)}}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \frac{d_1 + d_2 e^{\bar{c}(\tau+\gamma)}}{\bar{q}}(\varsigma + \alpha h) + \rho_l \\ &< M\mathbb{E}\|\phi\|_\tau^p + \varsigma + \alpha h + \rho,\end{aligned}$$

and thus,

$$\mathbb{E}J\left(t_l, y(t_l)\right) < M\mathbb{E}\|\phi\|_\tau^p + \varrho. \tag{3.36}$$

In view of (3.36), the definition of $t^\triangle$ in step 2 and the continuity of $\mathbb{E}J(t, y(t))$, $t \in [t_l, t_{l+1})$, we derive that $t^\triangle \in (t_l, t_{l+1})$. Meanwhile, it is easy to find that

$$\mathbb{E}J\left(t^\triangle, y(t^\triangle)\right) = M\mathbb{E}\|\phi\|_\tau^p + \ell, \tag{3.37}$$

$$\mathbb{E}J(t, y(t)) < M\mathbb{E}\|\phi\|_\tau^p + \ell, \ \forall t \in \left[t_l, t^\triangle\right). \tag{3.38}$$

Obviously, there must exist $t_\nabla \in \left[t_l, t^\triangle\right)$ satisfying $t_\nabla = \sup\left\{t \in [t_0 - \tau, t^\triangle] : \mathbb{E}J(t, y(t)) \le \frac{d_1+d_2e^{\bar{c}(\tau+\gamma)}}{\bar{q}} M \mathbb{E}\|\phi\|_\tau^p + \varrho\right\}$ such that

$$\mathbb{E}J\left(t_\nabla, y(t_\nabla)\right) = \frac{d_1 + d_2 e^{\bar{c}(\tau+\gamma)}}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \varrho, \tag{3.39}$$

$$\mathbb{E}J(t, y(t)) > \frac{d_1 + d_2 e^{\bar{c}(\tau+\gamma)}}{\bar{q}} M\mathbb{E}\|\phi\|_\tau^p + \varrho, \ \forall t \in \left(t_\nabla, t^\triangle\right]. \tag{3.40}$$

Let us now fix any $t \in \left[t_\nabla, t^\triangle\right]$, and suppose that $t+s \ge t_l$ for all $s \in [-\tau, 0]$. For all $s \in [-\tau, 0]$, (3.37)–(3.40) show that

$$\begin{aligned}\mathbb{E}J(t + s, y(t + s)) &\le M\mathbb{E}\|\phi\|_\tau^p + \ell \\ &\le \frac{\bar{q}}{d_1 + d_2 e^{\bar{c}(\tau+\gamma)}}\mathbb{E}J(t, y(t)) - \frac{\bar{q}\varrho}{d_1 + d_2 e^{\bar{c}(\tau+\gamma)}} + \ell \\ &\le \bar{q}\mathbb{E}J(t, y(t)) - \frac{\bar{q}\varrho}{d_1 + d_2 e^{\bar{c}(\tau+\gamma)}} + \ell \\ &< \bar{q}\mathbb{E}J(t, y(t)),\end{aligned}$$

which implies that $\mathbb{E}V(t + s, y(t + s)) < q\mathbb{E}V(t, y(t)) + \delta$ for $t \in \left[t_\nabla, t^\triangle\right]$, $s \in [-\tau, 0]$. Then, using the same arguments in (3.11) and (3.12), for $t \in \left[t_\nabla, t^\triangle\right]$,

$$\mathbb{E}J\left(t^\triangle, y(t^\triangle)\right) \le \mathbb{E}J\left(t_\nabla, y(t_\nabla)\right) + \int_{t_\nabla}^{t^\triangle}(z - c(s))\mathbb{E}J(s, y(s))\mathrm{d}s + \alpha h.$$

Consequently, employing (3.1), (3.37), (3.39) and the Gronwall inequality, we arrive at

$$\begin{aligned}\mathbb{E}J\left(t^{\triangle}, y(t^{\triangle})\right) &\le \left(\mathbb{E}J\left(t_{\nabla}, y(t_{\nabla})\right)+\alpha h\right) e^{\int_{t_{\nabla}}^{t^{\triangle}}(z-c(s))\mathrm{d}s} \\ &\le \left(\mathbb{E}J\left(t_{\nabla}, y(t_{\nabla})\right)+\alpha h\right) e^{(z-\bar{c})\left(t^{\triangle}-t_{\nabla}\right)} \\ &< \frac{d_1+d_2 e^{\bar{c}(\tau+\gamma)}}{\bar{q}} M\mathbb{E}\|\phi\|_{\tau}^{p}+(\varrho+\alpha h) \\ &< M\mathbb{E}\|\phi\|_{\tau}^{p}+(\varrho+\alpha h),\end{aligned}$$

and thus,

$$\mathbb{E}J\left(t^{\triangle}, y(t^{\triangle})\right) < M\mathbb{E}\|\phi\|_{\tau}^{p}+\ell, \tag{3.41}$$

which is a contradiction. Obviously, (3.14) must be true and (3.4) holds for $k=l+1$. Thus, (3.3) holds as well for $k=l+1$.

By the method of induction, (3.3) must hold for any $k \in \mathbb{N}$.

Step 3: Finally, by condition (**H1**) and (3.3), we derive that

$$\begin{aligned}\mathbb{E}\|y(t)\|^{p} &\le \frac{\mathbb{E}V(t)}{c_1}=\frac{\mathbb{E}J(t)e^{-z(t-t_0)}}{c_1} \\ &\le \frac{M}{c_1}\mathbb{E}\|\phi\|_{\tau}^{p} e^{-z(t-t_0)}+\frac{\varsigma}{c_1}.\end{aligned} \tag{3.42}$$

Therefore, system (2.1) is PESpM with $\eta=\frac{\varsigma}{c_1}$. $\square$

Remark 3.2. The continuous-time stochastic dynamics of system (2.1) are PES when the condition (H1) and (H3) are satisfied, which is a straightforward extension of the results given in [5].

Remark 3.3. In Theorem 3.1, the condition $\kappa > 1$ is assumed, which implies that the distributed-delay dependent impulses are destabilizing. Moreover, $c(t) > 0$ shows that the continuous dynamics of system (2.1) are PES for almost all $t \ge t_0$. Therefore, the system studied in Theorem 3.1 can be viewed as the second class of impulsive system, which is described in the Introduction.

Remark 3.4. A lower bound of ε is given explicitly in condition (**H5**), which implies that the destabilizing impulses should not happen so frequently. Meanwhile, combining Assumption 2.3 with condition (**H5**), one can see that an upper bound of h is also required, which implies that the destabilizing impulses should also be frequent in some degree. Therefore, the impulsive intervals must be bounded, and meet some necessary conditions. Moreover, from conditions (**H3**) and (**H5**), one may find that the amplitude of the impulses must be appropriately associate with the decline rate of $\mathbb{E}V$, distributed delay γ_k and time delay τ. Theorem 3.1 tells us what extent we can reduce the restriction on the distributed-delay dependent impulses such that system (2.1) still maintain the PESpM, in the case when the impulses potentially destroy the PES of the given system.

Remark 3.5. The derivative of the Razumikhin function $c(t)$, which is a time-varying function, is allowed to take values on $\mathbb{R}_+$ in our results. This is more general than [28], where the derivative is given by a fixed constant c. In fact, the constant c frequently does not exist in lots of systems, especially the systems with time-varying coefficients (refer to [31]).

Remark 3.6. In [4,5], the PES of the systems with unstable continuous stochastic dynamics and stabilizing impulsive effects is investigated. In this paper, we discussed the PES of the systems with stable continuous stochastic dynamics and destabilizing impulsive effects. To the best of our knowledge to date, so far little is known concerned with this case, and the aim of Theorem 3.1 is to close the gap. Moreover, as far as the addressed stability is concerned, the stability concept in this paper is also more general than [8,23,24,28,30,32]

Corollary 3.7. *If the required conditions in Theorem 3.1 are all fulfilled, except that condition (H5) is replaced by* $1 < W_1\kappa + W_2 c_2 \frac{2^{p-1}}{c_1} L_3^p \left(L_1^p \gamma^{2p} + L_3^p \gamma^{2p} r^p\right) e^{\bar{c}(\tau+\gamma)} < q < e^{\bar{c}\varepsilon}$, $\inf_{t\ge t_0} c(t) \ge \bar{c}$, *we then derive that the deterministic impulsive system with distributed-delay dependent impulses is PESpM. Moreover, if we further assume* $y(t,0) \equiv 0$ *for all* $t \ge t_0$ *and* $\alpha=\rho=\delta=0$, *then this system is ESpM.*

We now give some sufficient conditions for the ASPES of system (2.1).

Theorem 3.8. *Assume the required conditions of Theorem 3.1 with* $p \ge 2$ *are all satisfied, and there exist positive constants* L, μ *and* $H \ge 1$ *such that for all* $(t,\varphi) \in [t_0,\infty) \times L_{\mathcal{F}_t}^{p}([-\tau,0];\mathbb{R}^n)$,

$$\mathbb{E}\left(|f(t,\varphi)|^p + |g(t,\varphi)|^p\right) \le L \sup_{s\in[-\tau,0]} \mathbb{E}|\varphi(s)|^p + \mu. \tag{3.43}$$

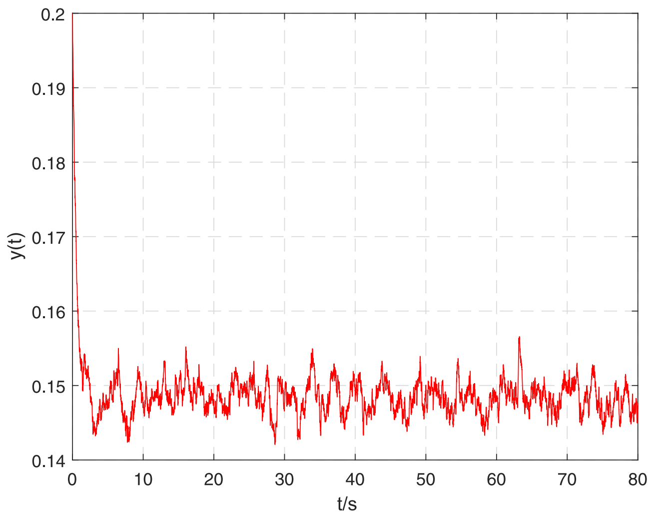

Fig. 1. Trajectory of y to system (4.1) without impulses.

Then, (3.42) *implies that*

$$|y(t)| \leq C\|\phi\|_\tau^p e^{-z(t-t_0)} + H^{\frac{1}{p}}, \ \forall t \geq t_0, \ a.s. \tag{3.44}$$

In other words, for system (2.1), *the PESpM implies the ASPES.*

Proof. The detailed proof of Theorem 3.8 is omitted since it can be completed by using the similar proof process of Theorem 3.3 in [4] for impulsive stochastic functional differential system. □

Remark 3.9. Noting that the system given in [4] is the first class of impulsive system, which is described in the Introduction. Therefore, in [4], impulses are utilized to stabilize the unstable continuous stochastic dynamics of the given system in the PES sense, which is different from our results.

Remark 3.10. In this paper, Theorems 3.1 and 3.8 can be interpreted as two kinds of PS results because the system in the absence of distributed-delay dependent impulses is PESpM, but the system with destabilizing ones is still PESpM if we impose some basic constraints on the ones. Also, if the system is already PESpM before the distributed-delay dependent impulses are considered, the obtained results give sufficient conditions for the system with destabilizing ones to maintain PESpM. The above discussion shows that the stable systems may bear some controlled impulsive perturbations, which implies the robustness.

4. An example

In this section, an example will be provided to verify the efficiency of the proposed results.

Example 4.1. Let us consider the following system with distributed-delay dependent impulses:

$$\begin{cases} \mathrm{d}y(t) = (-2y(t) + 0.5\,|\cos(\tanh(y(t-\tau) + 0.5\pi))|)\,\mathrm{d}t + \\ \qquad\qquad 0.5y(t-\tau)\mathrm{d}w(t), t \neq t_k, \ t \geq 0, \\ \Delta y\,(t_k) = 0.09 \int_{t_k-\gamma_k}^{t_k} y(s)\mathrm{d}s, \quad k \in \mathbb{N}, \\ y_{t_0} = 0.2, \end{cases} \tag{4.1}$$

where the system data are given by $0.55 \leq t_k - t_{k-1} \leq 1$, $\gamma_k = \gamma = 0.6$ and $\tau = 0.7$. Then, $f(t, y_t) = -2y(t) + 0.5|\cos(\tanh(y(t-0.7) + 0.5\pi))|$ and $g(t, y_t) = 0.5y(t-0.7)$. The impulses can be regarded as perturbations of the continuous stochastic dynamics of system (4.1). Fig. 1 shows that system (4.1) is PES when there is no impulsive effect.

In what follows, we will apply Theorem 3.1 to derive a PESpM result of system (4.1). Here, we can choose $p = 2$ and $V(t, y) = |y|^2$, then $C_p = 4$ (see [34]) and

$$\mathbb{E}|f(t, y_t)|^2 + \mathbb{E}|g(t, y_t)|^2 \leq 8\left(\mathbb{E}|y(t)|^2 + \mathbb{E}|y(t-0.7)|^2\right) + 0.1.$$

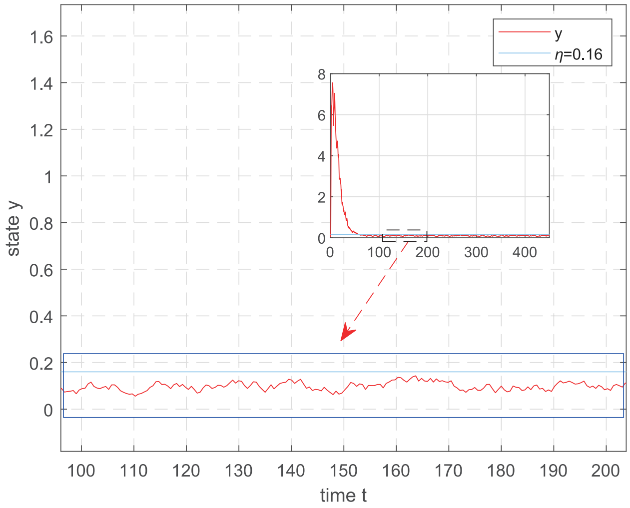

Fig. 2. Trajectory of y to system (4.1) under destabilizing distributed-delay dependent impulses..

Therefore, (3.43) is satisfied for system (4.1) with $L = 8$ and $\mu = 0.1$. By a simple computation, the constants in Theorem 3.1 can be chosen as $c_1 = c_2 = 1$, $q = 4.4$, $\delta = 0$, $r = 1$, $L_1 = 2.5$, $L_2 = 0.5$ and $L_3 = 0.1$. Obviously, we can see that

$$\begin{aligned}\mathbb{E}\mathcal{L}V(t, y_t) &\le -4\mathbb{E}|y(t)|^2 + \mathbb{E}|y(t)y(t-0.7)| + 0.25\mathbb{E}|y(t-0.7)|^2\\ &\le -2.75\mathbb{E}V(t,y) + 0.005e^{-z(t_k-t_0)},\end{aligned}$$

and

$$\mathbb{E}V(t, x+y) \le 2\mathbb{E}V(t,x) + 2\mathbb{E}V(t,y).$$

Thus, $c(t) \equiv \bar{c} = 2.75$, $\alpha = 0.005$, $W_1 = 2$ and $W_2 = 2$. As mentioned in [28], condition (**H2**) can be satisfied with $k = (1+0.09\gamma)^2 = 1.110916$ and $\rho_k \equiv \rho = 0.002$. Moreover, one can see that (**H5**) also holds with $d_1 = W_1\kappa = 2.221832$ and $d_2 = W_2 c_2 \frac{3^{p-1}}{c_1} L_3^p \left(L_1^p \gamma^{2p} + L_2^p \gamma^{\frac{3p}{2}} C_p + L_3^p \gamma^{2p} r^p\right) = 0.05$. Now, let $z = 0.05$, we have

$$\varsigma \ge \max\left\{\frac{\bar{q}\alpha h}{\bar{q}-1}, \frac{\bar{q}\alpha}{\bar{c}-z} - \alpha h, \frac{\left(d_1 + d_2 e^{\bar{c}(\tau+\gamma)}\right)\alpha h}{\bar{q} - d_1 - d_2 e^{\bar{c}(\tau+\gamma)}} - \alpha h - \rho\right\} = 0.16,$$

and thus, we can set $\varsigma = 0.16$.

Therefore, from the abovementioned discussion, the required conditions in Theorem 3.1 are all fulfilled. Hence, system (4.1) is PES in mean square and ASPES with $\eta = \frac{\varsigma}{c_1} = 0.16$. It is clearly demonstrated from Fig. 1 and Fig. 2 that the PS property can be preserved irrespective of destabilizing distributed-delay dependent impulses.

Remark 4.2. [4,28,29] considered the systems with unstable continuous-time dynamics and stable discrete-time dynamics, which is different from this paper. For system (4.1), the condition (iii) of Theorem 3.1 in [4] satisfies

$$\begin{aligned}\mathbb{E}\mathcal{L}V(t, y_t) &\le -4\mathbb{E}|y(t)|^2 + \mathbb{E}|y(t)y(t-0.7)| + 0.25\mathbb{E}|y(t-0.7)|^2\\ &\le -2.75\mathbb{E}V(t,y) + 0.005.\end{aligned}$$

One can get $c = -2.75$, which is a contradiction with the fact that c is a positive constant in [4]. On the other hand, if we do not consider the stochastic effect in system (4.1), then the condition (ii) of Theorem in [28] satisfies

$$\begin{aligned}\mathcal{L}V(t, y_t) &\le -4|y(t)|^2 + |y(t)y(t-0.7)|\\ &\le -3V(t,y).\end{aligned}$$

One can see $c = -3$, which is also a contradiction with the fact that c is a positive constant in [28]. Moreover, in [29], it is easy to verify that $0 < \upsilon < 1$. However, for system (4.1), there holds that $\upsilon = (1+0.09\gamma)^2 = 1.110916$. Therefore, system (4.1) does not satisfy the sufficient conditions presented in [4,28,29]. Together with the argument in Example 4.1, the PES result of system (4.1) is further verified.

5. Conclusions

In this paper, we have investigated the PS problem of stochastic functional differential system subject to destabilizing distributed-delay dependent impulses, and obtained some Razumikhin-type conditions for the PESpM and the ASPES of the suggested system. It has been shown that, the PES of the system subject to impulsive perturbations can be guaranteed when we impose some conditions on the destabilizing impulses. Finally, an example has been given to illustrate the effectiveness of the stability results. In the future, it would be interesting to consider the synchronization (or practical synchronization) problem of the nonlinear impulsive stochastic systems with delay effects (see e.g., [35–37]).

CRediT authorship contribution statement

Weijun Ma: Conceptualization, Methodology, Writing – original draft, Software, Supervision. **Bo Yang:** Conceptualization, Methodology, Writing – original draft. **Yuanshi Zheng:** Conceptualization, Methodology, Writing – reviewing, Visualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] J.L. Salle, S. Lefschetz, Stability By Liapunov's Direct Method with Applications, Academic Press, New York, 1961.
[2] Y. Li, Y. Ren, Stability for stochastic reaction–diffusion systems driven by G-Brownian motion, Internat. J. Control 95 (7) (2022) 1913–1921.
[3] Q. Yao, P. Lin, L. Wang, Y. Wang, Practical exponential stability of impulsive stochastic reaction–diffusion systems with delays, IEEE Trans. Cybern. 52 (5) (2022) 2687–2697.
[4] T. Caraballo, M.A. Hammami, L. Mchiri, Practical exponential stability of impulsive stochastic functional differential equations, Systems Control Lett. 109 (2017) 43–48.
[5] D. Zhu, J. Yang, X. Liu, Practical stability of impulsive stochastic delayed systems driven by G-Brownian motion, J. Franklin Inst. 359 (8) (2022) 3749–3767.
[6] M. Dlala, M.A. Hammami, Uniform exponential practical stability of impulsive perturbed systems, J. Dyn. Control Syst. 13 (3) (2007) 373–386.
[7] A. Mendoza-Torres, I. Cervantes, H. Cabrera-Ibarra, Practical stability of switched uncertain nonlinear systems using state-dependent switching laws, Nonlinear Anal. Hybrid Syst. 18 (2015) 72–84.
[8] X. Xu, Y. Li, H. Zhang, W. Chen, Q. Zheng, Practical stability for switched affine systems via time-dependent switching function, Int. J. Robust Nonlinear Control 31 (18) (2021) 9731–9744.
[9] L.N. Egidio, G.S. Deaecto, Novel practical stability conditions for discrete-time switched affine systems, IEEE Trans. Automat. Control 64 (11) (2019) 4705–4710.
[10] M. Liang, Y. Tian, Practical stability of switched homogeneous positive nonlinear systems: Max-separable Lyapunov function method, Appl. Math. Comput. 428 (2022) 127190.
[11] P. Morin, C. Samson, Practical and asymptotic stabilization of chained systems by the transverse function control approach, SIAM J. Control Optim. 43 (1) (2004) 32–57.
[12] W. Ma, X. Luo, Q. Zhu, Practical exponential stability of stochastic age-dependent capital system with Lévy noise, Systems Control Lett. 144 (2020) 104759.
[13] W. Ma, Z. Yao, B. Yang, Practical stability of continuous-time stochastic nonlinear system via event-triggered feedback control, J. Franklin Inst. 360 (3) (2023) 1733–1751.
[14] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[15] J.L. Mancilla-Aguilar, H. Haimovich, P. Feketa, Uniform stability of nonlinear time-varying impulsive systems with eventually uniformly bounded impulse frequency, Nonlinear Anal. Hybrid Syst. 38 (2020) 100933.
[16] W.H. Chen, X. Deng, W. Zheng, Sliding-mode control for linear uncertain systems with impulse effects via switching gains, IEEE Trans. Automat. Control 67 (4) (2022) 2044–2051.
[17] K. Zhang, E. Braverman, Time-delay systems with delayed impulses: A unified criterion on asymptotic stability, Automatica 125 (2021) 109470.
[18] X. Zong, T. Li, G. Yin, L.Y. Wang, J.F. Zhang, Stochastic consentability of linear systems with time delays and multiplicative noises, IEEE Trans. Automat. Control 63 (4) (2018) 1059–1074.
[19] Z. Zhang, S. Chen, Y. Zheng, Fully distributed scaled consensus tracking of high-order multiagent systems with time delays and disturbances, IEEE Trans. Ind. Inform. 18 (1) (2022) 305–314.
[20] X. Zong, T. Li, J.F. Zhang, Consensus conditions of continuous-time multi-agent systems with time-delays and measurement noises, Automatica 99 (2019) 412–419.

[21] J. Liu, X. Liu, W. Xie, Input-to-state stability of impulsive and switching hybrid systems with time-delay, Automatica 47 (5) (2011) 899–908.
[22] W. Chen, W. Zheng, Robust stability and H_∞-control of uncertain impulsive systems with time-delay, Automatica 45 (1) (2009) 109–117.
[23] J. Li, Q. Zhu, Stability of neutral stochastic delayed systems with switching and distributed-delay dependent impulses, Nonlinear Anal. Hybrid Syst. 47 (2023) 101279.
[24] W. Hu, Q. Zhu, H.R. Karimi, Some improved Razumikhin stability criteria for impulsive stochastic delay differential systems, IEEE Trans. Automat. Control 64 (12) (2019) 5207–5213.
[25] G. Li, Y. Zhang, Y. Guan, W. Li, Stability analysis of multi-point boundary conditions for fractional differential equation with non-instantaneous integral impulse, Math. Biosci. Eng. 20 (4) (2023) 7020–7041.
[26] M. Xia, L. Liu, J. Fang, Y. Zhang, Stability analysis for a class of stochastic differential equations with impulses, Mathematics 11 (2023) 1541.
[27] X. Fu, Q. Zhu, Stability of nonlinear impulsive stochastic systems with Markovian switching under generalized average dwell time condition, Sci. China Inf. Sci. 61 (2018) 112211.
[28] X. Liu, K. Zhang, Stabilization of nonlinear time-delay systems: Distributed-delay dependent impulsive control, Systems Control Lett. 120 (2018) 17–22.
[29] W. Hu, Q. Zhu, Stability criteria for impulsive stochastic functional differential systems with distributed-delay dependent impulsive effects, IEEE Trans. Syst. Man, Cybern. Syst. 51 (3) (2021) 2027–2032.
[30] P. Cheng, F. Deng, F. Yao, Exponential stability analysis of impulsive stochastic functional differential systems with delayed impulses, Commun. Nonlinear Sci. Numer. Simul. 19 (6) (2014) 2104–2114.
[31] W. Hu, Q. Zhu, H.R. Karimi, On the pth moment integral input-to-state stability and input-to-state stability criteria for impulsive stochastic functional differential equations, Int. J. Robust Nonlinear Control 29 (16) (2019) 5609–5620.
[32] P. Cheng, F. Deng, Global exponential stability of impulsive stochastic functional differential systems, Statist. Probab. Lett. 80 (2010) 1854–1862.
[33] H.S. Hurd, J.B. Kaneene, J.W. Lloyd, A stochastic distributed-delay model of disease processes in dynamic populations, Prev. Vet. Med. 16 (1) (1993) 21–29.
[34] X. Mao, Stochastic Differential Equations and their Applications, Second ed., Horwood Publisher, Chichester, 2008.
[35] R. Rao, Z. Lin, X. Ai, J. Wu, Synchronization of epidemic systems with Neumann boundary value under delayed impulse, Mathematics 10 (2022) 2064.
[36] Y. Tang, L. Zhou, J. Tang, Y. Rao, H. Fan, J. Zhu, Hybrid impulsive pinning control for mean square synchronization of uncertain multi-link complex networks with stochastic characteristics and hybrid delays, Mathematics 11 (2023) 1697.
[37] S. Dong, X. Liu, S. Zhong, K. Shi, H. Zhu, Practical synchronization of neural networks with delayed impulses and external disturbance via hybrid control, Neural Netw. 157 (2023) 54–64.