Automatica 157 (2023) 111259

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Monotonically convergent iterative learning control by time varying learning gain revisited[☆]

Jian Liu[a], Yuanshi Zheng[a,*], YangQuan Chen[b]

[a] Shaanxi Key Laboratory of Space Solar Power Station System, School of Mechano-electronic Engineering, Xidian University, Xi'an 710071, China
[b] School of Engineering, University of California at Merced, CA 95343, USA

ARTICLE INFO

Article history:
Received 7 November 2021
Received in revised form 16 November 2022
Accepted 21 July 2023
Available online xxxx

Keywords:
Iterative learning control
Monotonic convergence
Discrete-time system
Time-varying learning gain

ABSTRACT

This note considers the problem whether we can design a time-varying learning gain to ensure the monotone convergence of system output tracking errors (SOTEs) in the sense of unweighted $1/2/\infty$-norm for iterative learning control systems. Firstly, it points that the iterative learning control update law with the exponentially decaying learning factor considered in Moore et al. (2005) cannot ensure the monotonic convergence of SOTEs in the sense of unweighted 1-norm, 2-norm or ∞-norm. Secondly, it is strictly proven that there exists a time-varying learning gain to ensure the monotone convergence of SOTEs in the sense of unweighted ∞-norm and there is no time-varying learning gain to ensure the monotone convergence of SOTEs in the sense of unweighted 1-norm. Finally, this paper presents a sufficient condition under which there exists a time-varying learning gain to ensure the monotone convergence of SOTEs in the sense of unweighted 2-norm.

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1. Introduction

In iterative learning control (ILC), the main task is to utilize the available information of system dynamics and the historical input and output data to formulate an update law of system input sequence such that the system output sequence can track the given desired trajectory over the given finite time interval along the iteration axis (Ahn, Chen, & Moore, 2007; Arimoto, Kawamura, & Miyazaki, 1984; Chi, Huang, Hou, & Jin, 2018; Jin, 2018; Liu, Ruan, Zheng, Yi, & Wang, 2023; Shen & Li, 2019; Sun & Wang, 2002; Zhang & Meng, 2020). In practical application of ILC, the main concerns are the transient tracking performance and convergence rate of ILC process.

As reported in Lee and Bien (1997), although the classical proportional–derivative-type (PD-type) ILC update law can ensure the exponential convergence of the tracking error in the sense of unweighted 1, 2, ∞-norm for linear time-invariant (LTI) system, huge overshoot may appear before the tracking error converges to a desired level. Meanwhile, Lee and Bien (1997) pointed in simulations that the overshoot can be overcome by incorporating an exponentially decaying factor into the learning gain of PD-type ILC update law and gave an example to illustrate its effectiveness. However, Lee and Bien (1997) did not answer whether the ILC update law with exponentially decaying factor was able to eliminate the huge overshoot in theory. Clearly, the monotonic convergence technique is an effective method to eliminate the huge overshoot. Moore, Chen, and Bahl (2005) studied the monotone convergence of the Arimoto-type ILC update law with an exponentially decaying factor for LTI discrete-time system. It was claimed that the monotone convergence in the sense of unweighted 1/2-norm can be ensured via tuning the exponentially decaying factor. However, due to the defects of its analytical techniques, whether there is an exponentially decaying factor to ensure the monotone convergence in the sense of unweighted $1/2/\infty$-norm for discrete-time system is still unresolved. So far, it is still up in the air whether there is a time-varying learning gain in Arimoto-type ILC update law to ensure the monotone convergence in the sense of unweighted $1/2/\infty$-norm. This technique note attempts to answer the above questions.

The main contributions of this note are as follows.

- This note shows that for discrete-time system the Arimoto-type ILC update law with an exponentially decaying factor considered in Moore et al. (2005) cannot ensure the monotone convergence of SOTEs in the sense of unweighted $1/2/\infty$-norm.

☆ This work is funded by the National Natural Science Foundation of China (62273267), the Fundamental Research Funds for the Central Universities, China (ZYTS23021), the Nature Science Basic Research Plan in Shaanxi Province of China (2023-JC-YB-525 and 2023-JC-QN-0766) and the Natural Science Basic Research Program of Shaanxi, China (2022JC-46). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Raul Ordonez under the direction of Editor Miroslav Krstic.

* Corresponding author.
E-mail addresses: liujianzym@outlook.com (J. Liu), zhengyuanshi2005@163.com (Y. Zheng), ychen53@ucmerced.edu (Y. Chen).

https://doi.org/10.1016/j.automatica.2023.111259
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- It proves that there is no time-varying learning gain for Arimoto-type ILC update law to ensure the monotone convergence of SOTEs in the sense of 1-norm.
- A time-varying learning gain with an exponentially increasing factor is designed to guarantee the monotone convergence of SOTEs in the sense of unweighted ∞-norm.
- It gives a sufficient condition under which there is time-varying learning gain to guarantee the monotone convergence of SOTEs in the sense of unweighted 2-norm. Meanwhile, it reveals that the initial system input and the trial length may have a significant influence on both the transient tracking performance and the convergence rate of ILC process.

The rest is organized as follows. Section 2 gives the problem formation, and the convergence results about the ILC update law developed in Moore et al. (2005). In Section 3, a time-varying gain ILC update law that can ensure the monotone convergence of SOTEs is proposed. An illustrative example is given in Section 4. Section 5 concludes the paper.

2. Is there time-varying learning gain to ensure monotone convergence?

Consider the following discrete-time, linear time-invariant (LTI) system that has been considered in Moore et al. (2005):

$$y_k(t) = \sum_{i=0}^{t-1} h_{t-i} u_k(i), \tag{1}$$

where t in $\mathscr{S}^+ = \{1, 2, \ldots, T\}$ is the discrete time, k in $\mathbb{N}^* = \{1, 2, \ldots\}$ labels the trial number, $y_k(t)(t \in \mathscr{S}^+)$ and $u_k(t)(t \in \mathscr{S}^- = \{0, 1, \ldots, T-1\})$ are the system output and input at the k trial, $h_i(i \in \mathscr{S}^+)$ are the Markov parameters and T is the trial length.

The problem we care about is whether there is a time-varying learning gain γ_{t+1} to ensure the monotone convergence of SOTEs in the sense of 1-norm, 2-norm or ∞-norm when the ILC update law (2) is applied to system (1).

$$u_{k+1}(t) = u_k(t) + \gamma_{t+1}[y_d(t+1) - y_k(t+1)], \tag{2}$$

where $y_d(t)$ is the given desired trajectory over the finite time interval $\mathscr{S}^+$.

Remark 1. In practical applications of ILC, non-monotone convergence ILC algorithms can be accepted, but huge overshoot should be avoided. In this paper, we use the technique of monotone convergence to completely eliminate overshoot.

The following definitions are useful.

Definition 1. For system (1) with a given ILC update law, the SOTEs are said to be monotonically convergent in the sense of 1-norm if for any given initial system input $u_1(t)(t \in \mathscr{S}^-)$, $\lim_{k\to\infty} y_k(t) = y_d(t)$ and

$$\sum_{t=1}^{T} |y_d(t) - y_{k+1}(t)| \le \sum_{t=1}^{T} |y_d(t) - y_k(t)|$$

for all $k \in \mathbb{N}^*$.

Definition 2. For system (1) with a given ILC update law, the SOTEs are said to be monotonically convergent in the sense of ∞-norm if for any given $u_1(t)(t \in \mathscr{S}^-)$, $\lim_{k\to\infty} y_k(t) = y_d(t)$ and

$$\max_{1\le t\le T} |y_d(t) - y_{k+1}(t)| \le \max_{1\le t\le T} |y_d(t) - y_k(t)|$$

for all $k \in \mathbb{N}^*$.

Definition 3. For system (1) with a given ILC update law, the SOTEs are said to be monotonically convergent in the sense of 2-norm if for any given $u_1(t)(t \in \mathscr{S}^-)$, $\lim_{k\to\infty} y_k(t) = y_d(t)$ and

$$\sqrt{\sum_{t=1}^{T} |y_d(t) - y_{k+1}(t)|^2} \le \sqrt{\sum_{t=1}^{T} |y_d(t) - y_k(t)|^2}$$

for all $k \in \mathbb{N}^*$.

Remark 2. As pointed in Lee and Bien (1997) and Moore et al. (2005), when applying the non-monotone convergence ILC algorithms to the repetitive control system, we may observe the huge overshoot in the sense of 1-norm, 2-norm and ∞-norm, which is unacceptable in practical applications, and may hinder the application of non-monotone ILC algorithms. How to eliminate the huge overshoot in iterative learning process is a challenging problem in ILC community. The monotone convergence ILC algorithms can completely eliminate the overshoot phenomenon, which are more popular in practical applications than those non-monotone convergence ILC algorithms. Moreover, monotone convergence is more in line with the expectation of iterative learning, i.e., we can get better tracking precision from trial to trial.

Remark 3. It is well known that ILC is developed for repetitive control systems with unknown dynamics information. Therefore, we can only use the rough information of system dynamics to design ILC update law. Furthermore, the influence of system initial input on the tracking performance of ILC system is very complex. To the best of our knowledge, up to now, there is no good way to quantify and analyze the impact of initial input on the tracking performance of ILC system. Therefore, in a certain sense, it is necessary to require that the monotone convergence is independent of the initial input signals.

When the time-varying learning gain γ_t takes the form of $\gamma_t = \eta e^{-\alpha t}$, the ILC update law (2) becomes (3) that has been considered in Moore et al. (2005).

$$u_{k+1}(t) = u_k(t) + \eta e^{-\alpha t}[y_d(t+1) - y_k(t+1)], \tag{3}$$

where η is a constant and $\alpha > 0$ is an exponentially decaying factor of the time-varying learning gain.

Remark 4. When the ILC update law (3) is applied to system (1), it is easy to verify that the output sequence $y_k(t)$ of ILC system (1)–(3) is convergent to the given desired output $y_d(t)$ if and only $|1 - \eta h_1| < 1$. The convergence condition $|1 - \eta h_1| < 1$ tells that the parameter η must have the same sign as h_1. The main function of η is to determine the direction of ILC and the main function of α is to improve the transient tracking performance of the output sequence of ILC process by tuning the learning step size at each time instant. Therefore, in ILC update law (3), the parameter η characterizes the control direction and the positive constant α characterizes the learning step size.

Firstly, we show that for system (1) with ILC update law (3), the monotone convergence of SOTEs in the sense of 1-norm cannot be ensured only by tuning the parameters α and η if $|h_1| < \sum_{j=2}^{T} |h_j|$.

Theorem 1. *For system (1) with ILC update law (3), there are no α and η to ensure the monotone convergence of $\sum_{t=1}^{T} |y_d(t) - y_k(t)|$ if $|h_1| < \sum_{j=2}^{T} |h_j|$.*

Proof. Let

$$Y_k = [y_k(1), \ldots, y_k(T)]^\top,$$

$Y_d = [y_d(1), \dots, y_d(T)]^\top,$
$U_k = [u_k(0), \dots, u_k(T-1)]^\top,$
$\Gamma = \eta \mathrm{diag}\{1, e^{-\alpha}, e^{-2\alpha}, \dots, e^{-\alpha(T-1)}\},$
$\sigma_i = 1 - \eta h_1 e^{-\alpha(i-1)} (i = 1, \dots, T),$

$$H = \begin{bmatrix} h_1 & & & \\ h_2 & h_1 & & \\ \vdots & \ddots & \ddots & \\ h_T & \cdots & h_2 & h_1 \end{bmatrix}.$$

Then, system (1) becomes

$$Y_k = HU_k \tag{4}$$

and ILC update law (3) can be written as

$$U_{k+1} = U_k + \Gamma E_k, \tag{5}$$

where $E_k = Y_d - Y_k$.

Using (4) and (5) gives rise to

$$E_{k+1} = \Xi E_k, \tag{6}$$

where

$$\Xi = \begin{bmatrix} \sigma_1 & & & & \\ -\eta h_2 & \sigma_2 & & & \\ -\eta h_3 & -\eta h_2 e^{-\alpha} & \sigma_3 & & \\ \vdots & \vdots & \vdots & \ddots & \\ -\eta h_T & -\eta h_{T-1} e^{-\alpha} & -\eta h_{T-2} e^{-2\alpha} & \cdots & \sigma_T \end{bmatrix}. \tag{7}$$

As shown in Moore et al. (2005), the system outputs $y_k(t)$ of ILC (1)–(3) are convergent to $y_d(t)$ over the finite time interval $\mathscr{S}^+$ if and only if $|1 - \eta h_1| < 1$. Therefore, to complete the proof, what we need to do is to show that under the conditions $|1 - \eta h_1| < 1$ and $|h_1| < \sum_{j=2}^T |h_j|$ there is no α to ensure the monotone convergence of SOTEs in the sense of 1-norm.

Using $|1 - \eta h_1| < 1$ yields that $0 < \eta h_1 < 2$.

In the case $0 < \eta h_1 \le 1$, using $|h_1| < \sum_{j=2}^T |h_j|$ yields

$$|1 - \eta h_1| + |\eta| \Big(\sum_{i=2}^T |h_i| \Big) > 1 - \eta h_1 + |\eta||h_1| = 1. \tag{8}$$

In the case $1 < \eta h_1 < 2$, using $|h_1| < \sum_{j=2}^T |h_j|$ gives

$$|1 - \eta h_1| + |\eta| \Big(\sum_{i=2}^T |h_i| \Big) > \eta h_1 > 1. \tag{9}$$

By $|1 - \eta h_1| < 1$, we get $h_1 \neq 0$, which implies that H is nonsingular. Thus, there exists a unique initial system input $u_1(t) (t \in \mathscr{S}^-)$ such that $y_1(1) = y_d(1) - 1$ and $y_1(t) = y_d(t)$ for $t = 2, \dots, T$. Then, using (8) and (9) together with a simple computation yields that for any given α, $\|E_1\|_1 = 1$ and $\|E_2\|_1 = |1 - \eta h_1| + |\eta| \Big(\sum_{i=2}^T |h_i| \Big) > 1$.

Therefore, there exists an initial system input such that for any given α, $\|E_2\|_1 > \|E_1\|_1$. This completes the proof. □

Secondly, we illustrate that the monotone convergence of SOTEs in the sense of ∞-norm cannot be guaranteed only by adjusting α and η in (2) if $|h_1| < \sum_{i=2}^T |h_i|$.

Theorem 2. *For system (1) with ILC update law (3), there are no α and η to ensure the monotone convergence of $\max_{1 \le t \le T} |y_d(t) - y_k(t)|$ if $|h_1| < \sum_{i=2}^T |h_i|$.*

Proof. What we need to do is to illustrate that under the conditions $|1 - \eta h_1| < 1$ and $|h_1| < \sum_{i=2}^T |h_i|$, there is no $\alpha > 0$ to ensure the monotone convergence of SOTEs in the sense of ∞-norm.

Using $|1 - \eta h_1| < 1$ together with $\alpha > 0$ yields $|1 - \eta h_1 e^{-(T-1)\alpha}| < 1$, which implies $0 < \eta h_1 e^{-(T-1)\alpha} < 2$.

In the case $0 < \eta h_1 e^{-(T-1)\alpha} \le 1$, using $|h_1| < \sum_{i=2}^T |h_i|$ and $\alpha > 0$ gives rise to

$$|\eta| \Big(\sum_{i=2}^T |h_{T-i+2}| e^{-(i-2)\alpha} \Big) + |1 - \eta h_1 e^{-(T-1)\alpha}| > 1 \tag{10}$$

for all $\alpha > 0$.

In the case $1 < \eta h_1 e^{-(T-1)\alpha} < 2$, using $|h_1| < \sum_{i=2}^T |h_i|$ and $\alpha > 0$ yields

$$|\eta| \Big(\sum_{i=2}^T |h_{T-i+2}| e^{-(i-2)\alpha} \Big) + |1 - \eta h_1 e^{-(T-1)\alpha}| > 1 \tag{11}$$

for all $\alpha > 0$.

It follows from $|1 - \eta h_1| < 1$ that $h_1 \neq 0$, which implies H is nonsingular. Therefore, there exists a unique $u_1(t) (t \in \mathscr{S}^-)$ such that

$$y_1(t) = y_d(t) + \mathrm{sgn}(\eta h_{T-t+1}) \text{ for } 1 \le t \le T-1$$

and $y_1(T) = y_d(T) - \mathrm{sgn}(1 - \eta h_1 e^{-(T-1)\alpha})$, where $\mathrm{sgn}(\cdot)$ is defined as

$$\mathrm{sgn}(z) = \begin{cases} 1 \text{ if } z \ge 0, \\ -1 \text{ if } z < 0. \end{cases}$$

Then, using (10) and (11) gives that for any given α, $\|E_1\|_\infty = 1$ and

$$\|E_2\|_\infty \ge |\eta| \Big(\sum_{i=2}^T |h_{T-i+2}| e^{-(i-2)\alpha} \Big) + |1 - \eta h_1 e^{-(T-1)\alpha}| > 1.$$

Therefore, for any given α, there exists an initial system input such that $\|E_2\|_\infty > \|E_1\|_\infty$. This completes the proof. □

Remark 5. In the case $\alpha = 0$, (3) becomes the Arimoto-type ILC update law

$$u_{k+1}(t) = u_k(t) + \eta[y_d(t+1) - y_k(t+1)]. \tag{12}$$

It is easy to prove that under the conditions $0 < \eta h_1 \le 1$ and $|h_1| \ge \sum_{i=2}^T |h_i|$, the SOTEs of system (1) with update law (12) are monotonically convergent in the sense of 1-norm and ∞-norm.

Remark 6. Let

$$\Phi = \begin{bmatrix} 1 - \eta h_1 & & & & \\ -\eta h_2 & 1 - \eta h_1 & & & \\ -\eta h_3 & -\eta h_2 & 1 - \eta h_1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ -\eta h_T & -\eta h_{T-1} & -\eta h_{T-2} & \cdots & 1 - \eta h_1 \end{bmatrix}.$$

It is easy to derive that the output sequence of system (1) with update law (12) satisfies

$$E_{k+1} = \Phi E_k.$$

Clearly, the conditions $0 < \eta h_1 \le 1$ and $\sum_{i=2}^T |h_i| \le |h_1|$ imply $\|\Phi\|_1 \le 1$ and $\|\Phi\|_\infty \le 1$. Therefore, using the fact $\|\Phi\|_2^2 \le \|\Phi\|_\infty \|\Phi\|_1$ yields that the SOTEs of system (1) with update law (12) in the sense of 2-norm are monotonically convergent if $0 < \eta h_1 \le 1$ and $\sum_{i=2}^T |h_i| \le |h_1|$.

Next, we will give as example to show that under the conditions $|1 - \eta h_1| < 1$ and $\sum_{i=2}^T |h_i| > |h_1|$, there may be no α

to ensure the monotone convergence of the SOTEs of system (1) with ILC update law (3) in the sense of 2-norm.

Example 1. Let the trial length T being 2, $h_1 = 1$, $h_2 = 2$ and $\gamma = 1.5$. It is obvious that $\rho_1 = |1-\gamma h_1| = 0.5 < 1$, $\Xi_1 = \begin{bmatrix} 1-\gamma h_1 & 0 \\ -\gamma h_2 & 1-\gamma h_1 e^{-\alpha} \end{bmatrix}$. Since

$$\Xi_1^\top \Xi_1 = \begin{bmatrix} 9.25 & -3(1-1.5e^{-\alpha}) \\ -3(1-1.5e^{-\alpha}) & (1-1.5e^{-\alpha})^2 \end{bmatrix},$$

it follows from Gerŝgorin disk theorem that for all $\alpha > 0$, the maximum eigenvalue of $\Xi_1^\top \Xi_1$ is greater than 6.25. This means that for all $\alpha > 0$, $\|\Xi_1\|_2 > 2.5$. Therefore, there does not exist an α to ensure the monotone convergence of SOTEs in the sense of 2-norm.

Remark 7. We have proven that under the condition $|1-\eta h_1| < 1$, only by tuning the exponentially decaying factor α, the monotone convergence of the SOTEs of ILC system (1) with (3) in the sense of 1-norm, ∞-norm or 2-norm cannot be ensured. It should be pointed that although the ILC update law (3) cannot guarantee the monotone convergence of the SOTEs, it can guarantee the monotone convergence of the system input tracking errors. This is the reason why the ILC update law (3) can be used to improve the transient tracking performance of the SOTEs in simulations.

Remark 8. In Moore et al. (2005), the authors claimed that for ILC system (1) with (3), there is α to ensure the monotone convergence of SOTEs in the sense of 1-norm/2-norm. Obviously, this is inconsistent with the results obtained in the paper. The main reason is that there exists a technical defect in the proof of Theorem 4 in Moore et al. (2005), which is used to derive the monotone convergence of the SOTEs in the sense of 1-norm/2-norm. It is easy to check that for all $k \in \mathbb{N}^*$ and $\alpha > 0$, the output signals $y_k(1)$ and $y_k(2)$ of ILC system (1)–(3) satisfy

$$|\bar{e}_{k+1}(1)| \le \rho_1 |\bar{e}_k(1)|, \tag{13}$$

$$|\bar{e}_{k+1}(2)| \le \rho_2 |\bar{e}_k(2)| + e^{-\alpha}|\eta h_2||\bar{e}_k(1)|, \tag{14}$$

where $\rho_1 = |1-\eta h_1|$, $\rho_2 = |1-e^{-\alpha}\eta h_1|$, $\bar{e}_k(1) = y_d(1) - y_k(1)$ and $\bar{e}_k(2) = e^{-\alpha}[y_d(2) - y_k(2)]$. Using (13) together with $\rho_1 < 1$ yields that

$$|\bar{e}_{k+1}(1)| \le |\bar{e}_k(1)| \text{ for all } k \in \mathbb{N}^* \text{ and} \alpha > 0. \tag{15}$$

Then, Moore et al. (2005) claimed that due to the boundedness of $|\bar{e}_k(1)|$ and $|\eta h_2|$, it follows from (14) and $\rho_2 < 1$ that there exists a large enough α such that

$$|\bar{e}_{k+1}(2)| \le |\bar{e}_k(2)| \text{ for all } k \in \mathbb{N}^*. \tag{16}$$

It follows from the property of the limit that if $\bar{e}_k(2)$ are independent of α, then (16) holds. Unfortunately, $\bar{e}_k(2)$ highly depends on α. Therefore, by (14) together with the boundedness of $|\bar{e}_k(1)|$ and $|\eta h_2|$, we cannot get the inequalities presented in (16) for arbitrarily given initial system input $u_1(\cdot)$ if $h_2 \neq 0$. This means that Theorem 4 in Moore et al. (2005) is incorrect. Therefore, it cannot be used to derive the monotone convergence of the SOTEs in the sense of 1-norm/2-norm.

3. There does exist time-varying learning gain to ensure monotone convergence

In Section 2, we have proven that the time-varying gain ILC update law (3) developed in Moore et al. (2005) cannot ensure the monotone convergence of the SOTEs in the sense of unweighted 1-norm, 2-norm and ∞-norm, respectively. In this section, we further explore whether there exists the time-varying gain ILC update law taking the form of (2) to ensure the monotone convergence of the SOTEs in the sense of unweighted 1-norm, 2-norm or ∞-norm.

The following theorem tells that there exists a time-varying learning gain γ_t to ensure the monotone convergence of SOTEs in the sense of ∞-norm.

Theorem 3. *Assume that the time-varying learning gain in (2) takes the form of $\gamma_t = \eta e^{-\alpha(T-t)} (t \in \mathscr{S}^+)$, where α is a positive constant. If $0 < \eta h_1 \le 1$, then there is an α to ensure the monotone convergence of the SOTEs in the sense of ∞-norm.*

Proof. By the definitions of U_k, Y_k, Y_d and E_k, ILC update law (2) can be written as

$$U_{k+1} = U_k + \Gamma_1 E_k, \tag{17}$$

where $\Gamma_1 = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_T\}$.

By (4) and (17), we get

$$E_{k+1} = \Omega E_k, \tag{18}$$

where

$$\Omega = \begin{bmatrix} 1-\gamma_1 h_1 & & & & \\ -\gamma_1 h_2 & 1-\gamma_2 h_1 & & & \\ -\gamma_1 h_3 & -\gamma_2 h_2 & 1-\gamma_3 h_1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ -\gamma_1 h_T & -\gamma_2 h_{T-1} & -\gamma_3 h_{T-2} & \cdots & 1-\gamma_T h_1 \end{bmatrix}. \tag{19}$$

Let $\phi_1 = 1 - \gamma_1 h_1$ and

$$\phi_i = \sum_{j=1}^{i-1} |\eta h_{i-j+1}| e^{-\alpha(T-j)} + 1 - \eta h_1 e^{-\alpha(T-i)} \tag{20}$$

for $i = 2, \ldots, T$.

It follows from the condition $0 < \eta h_1 \le 1$ that $h_1 \neq 0$. Using the fact

$$\lim_{\alpha \to +\infty} \sum_{j=2}^{T} |h_j| e^{-\alpha(j-1)} = 0 \tag{21}$$

yields that there is an α such that

$$\sum_{j=2}^{T} |h_j| e^{-\alpha(j-1)} < |h_1|. \tag{22}$$

Using (20) and (22) together with the condition $0 < \eta h_1 \le 1$ yields that there is an α such that

$$0 < \phi_i < 1 \text{ for all } i \in \mathscr{S}^+. \tag{23}$$

Therefore, there is an α such that

$$\|\Omega\|_\infty = \max_{1 \le i \le T} \phi_i < 1, \tag{24}$$

which implies that $\|E_k\|_\infty$ are monotonically convergent along the k axis. This completes the proof. □

The following theorem shows that there does not exist a time-varying learning gain $\gamma_t (t \in \mathscr{S}^+)$ to ensure the monotone convergence of SOTEs in the sense of 1-norm if $\sum_{j=2}^{T} |h_j| > |h_1|$.

Theorem 4. *Assume that $\sum_{j=2}^{T} |h_j| > |h_1|$. Then, for system (1) with ILC update law (2), there does not exist a learning gain $\gamma_t (t \in \mathscr{S}^+)$ to ensure the monotone convergence of the SOTEs in the sense of 1-norm.*

Proof. Under the condition $\max_{1\le j\le T}|1-\gamma_j h_j|\ge 1$, the SOTEs of ILC system (1)–(2) do not converge to zero if the initial system input is not the desired one. Therefore, assume that $\max_{1\le j\le T}|1-\gamma_j h_j|<1$, which implies $|1-\gamma_1 h_1|<1$.

In the case $0<\gamma_1 h_1\le 1$, it follows from $\sum_{j=2}^{T}|h_j|>|h_1|$ that

$$1-\gamma_1 h_1+|\gamma_1 h_2|+\cdots+|\gamma_1 h_T|>1. \tag{25}$$

Thus,

$$\|\Phi\|_1>1, \tag{26}$$

which implies that $\|E_k\|_1$ diverges.

In the case $1<\gamma_1 h_1<2$, it follows from $\sum_{j=2}^{T}|h_j|>|h_1|$ that

$$\gamma_1 h_1-1+|\gamma_1 h_2|+\cdots+|\gamma_1 h_T|>1. \tag{27}$$

Therefore, $\|\Phi\|_1>1$, which implies that $\|E_k\|_1$ diverges.

The above proves that under the condition $\sum_{j=2}^{T}|h_j|>|h_1|$, there does not exist a time-varying learning gain to ensure the monotone convergence of SOTEs in the sense of 1-norm. This completes the proof. □

Theorem 5. *Assume that $\sum_{j=2}^{T-1}|h_j|<2|h_1|$. Then, for system (1) with update law (2) in which $\gamma_t=\eta e^{-\alpha(T-t)}(t\in\mathscr{S}^+)$, there do exist η and α to ensure the monotone convergence of the SOTEs in the sense of 2-norm.*

Proof. Since $\sum_{j=2}^{T-1}|h_j|<2|h_1|$, it follows that $h_1\ne 0$. Therefore, there exists η satisfying $\eta h_1=1$.

The proof is completed by showing that under the condition $\sum_{j=2}^{T-1}|h_j|<2|h_1|$, there exists a positive α to ensure that the maximum eigenvalue of $\Omega^\top\Omega$ is less than 1, i.e., $\|\Omega\|_2<1$, where Ω is defined by (19).

Let $\Phi=[\varphi_{i,j}]=\Omega^\top\Omega$,

$$\psi_1=1-\left(2-\sum_{j=2}^{T-1}\left|\frac{h_j}{h_1}\right|\right)e^{-\alpha(T-1)}+\frac{Th^2e^{-\alpha T}}{1-e^{-\alpha}}, \tag{28}$$

$$\psi_{T-1}=1-2e^{-\alpha}+\frac{(2h^2+h)e^{-2\alpha}}{1-e^{-\alpha}}, \tag{29}$$

$$\psi_i=1-\left(2-\sum_{j=2}^{T-i}\left|\frac{h_j}{h_1}\right|\right)e^{-\alpha(T-i)}+\frac{(Th^2+h)e^{-\alpha(T-i+1)}}{1-e^{-\alpha}}\ \text{for } i=2,\ldots,T-2, \tag{30}$$

where $h=\max_{1\le j\le T}\left|\frac{h_j}{h_1}\right|$.

It is easy to check that $\sum_{j=1}^{T}|\varphi_{T,j}|=0$ and

$$\sum_{j=1}^{T}|\varphi_{i,j}|\le\psi_i \text{ for } i=1,2,\ldots,T-1. \tag{31}$$

Since $\sum_{j=2}^{T-1}|h_j|<2|h_1|$, it follows from (28)–(31) that there exists an α such that

$$\sum_{j=1}^{T}|\varphi_{i,j}|\le\psi_i<1 \text{ for } i=1,2,\ldots,T. \tag{32}$$

Using (32) together with Gerŝgorin disk theorem yields that there exists an α such that the maximum eigenvalue of $\Omega^\top\Omega$ is less than 1. This completes the proof. □

Remark 9. It follows from the monotone convergence conditions that the trial length may have a significant influence on the monotone convergence of ILC process. As the trial length increases, the SOTEs in the sense of 1-norm/2-norm may change from monotone convergence to non-monotone convergence.

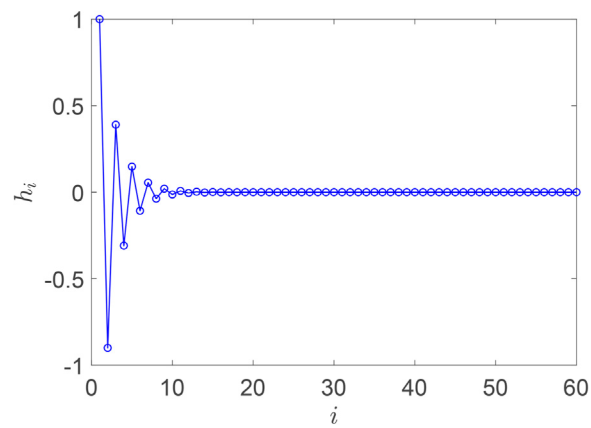

Fig. 1. The profile of Markov parameters $h_1,h_2,\ldots,h_{60}$.

4. Illustrative examples

In this section, we use the stable oscillatory system to show our findings, whose z-transfer function is given in (33). In simulations, all the initial conditions of system (33) are set to 0. The profile of the involved Markov parameters are displayed in Fig. 1.

$$H_1(z)=\frac{z-0.8}{(z-0.5)(z+0.6)}. \tag{33}$$

Let the desired trajectory be defined as

$$y_d(t)=\begin{cases}\frac{2t}{T},\ t\in\{1,2,\ldots,\frac{T}{2}\},\\ \frac{2(T-t)}{T},\ t\in\{\frac{T}{2}+1,\frac{T}{2}+2,\ldots,T\},\end{cases} \tag{34}$$

which is a triangle with a maximum height 1.

4.1. Tracking performances of system (33) with (3)

In this subsection, we aim to show that under the same settings as those in Moore et al. (2005), the ILC update law (3) cannot ensure the monotone convergence of the SOTEs in the sense of 1-norm and 2-norm, respectively.

As done in Moore et al. (2005), let $T=60$, $\eta=0.9$ and $\alpha=0.025$. Since $h_1=1$, it follows that $|1-\eta h_1|=0.1$, which means that the convergence condition of system (33) with update law (3) is satisfied. Due to $\sum_{i=2}^{60}|h_i|=2$, it follows from Theorem 1 that there is no α to ensure the monotone convergence of $\|E_k\|_1=\sum_{t=1}^{60}|y_d(t)-y_k(t)|$ for any given initial system input U_1.

Fig. 2 shows that the SOTEs in the sense of 2-norm with the initial system input U_1 being set to 0 converge monotonically. However, when the initial input $U_1=H^{-1}(Y_d-V_1)$ is applied to ILC system (33)–(3), the monotone convergence of the SOTEs in the sense of 2-norm does not hold any more. Here, V_1 satisfying $\|V_1\|_2=1$ is the eigenvector corresponding to the eigenvalue 1.799 of the matrix $\Xi^\top\Xi$, in which Ξ is defined by (7).

Fig. 3 shows that the SOTEs in the sense of 1-norm with the initial system input U_1 being set to 0 are monotonically convergent. And when applying

$$U_1=H^{-1}(Y_d-V_1)(V_1=[1,0,\ldots,0]^\top)$$

to ILC system (33)–(3), we get $E_1=V_1$ and

$$E_2=[1-\eta h_1,-\eta h_2,\ldots,-\eta h_T]^\top.$$

Then, $\|E_1\|_1=1$ and $\|E_2\|_1=1.9$. This means that the ILC update law (3) with α being setting to 0.025 cannot ensure the monotone convergence of the SOTEs in the sense of 1-norm.

The above facts imply that if $\|\Xi\|_1(\|\Xi\|_2)$ is not less than or equal to 1, the monotone convergence of the SOTEs in the sense of 1-norm (2-norm) does depend on the initial system input.

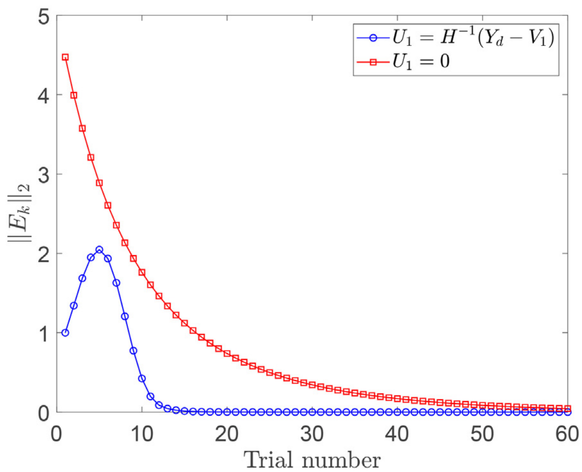

Fig. 2. SOTEs of system (33) with update law (3) in the sense of 2-norm with different U_1.

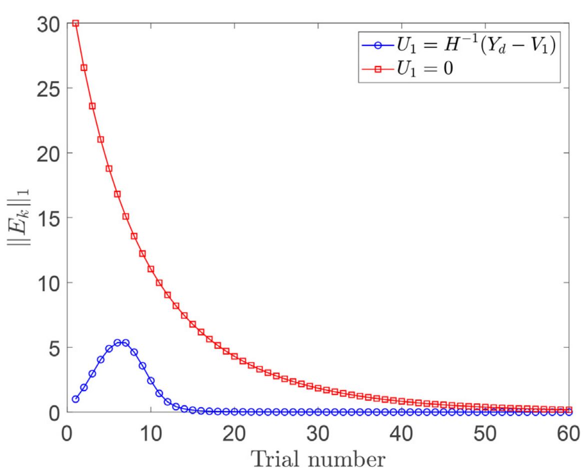

Fig. 3. SOTEs of system (33) with update law (3) in the sense of 1-norm with different U_1.

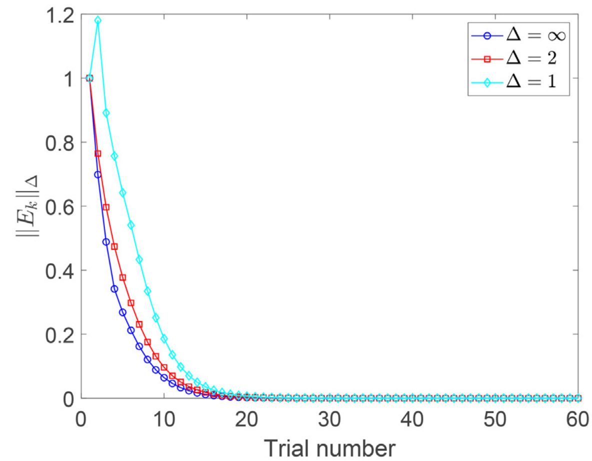

Fig. 4. SOTEs of ILC system (33)–(2) with $\gamma_t = e^{-0.4(20-t)}$ in the sense of ∞, 2 and 1-norm, respectively.

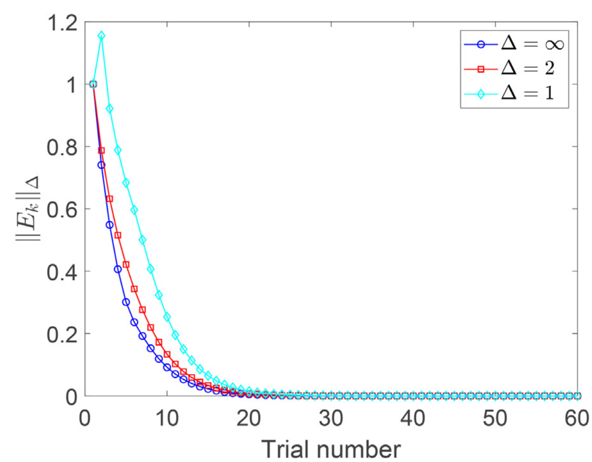

Fig. 5. SOTEs of ILC system (33)–(2) with $\gamma_t = e^{-0.45(20-t)}$ in the sense of ∞, 2 and 1-norm, respectively.

4.2. Tracking performances of system (33) with update law (2) in which $\gamma_t = \eta e^{-\alpha(T-t)}$

In this subsection, we aim to show the ILC update law (2) with the time-varying gain γ_t being set to $\eta e^{-\alpha(T-t)}$ can ensure the monotone convergence of the SOTEs in the sense of ∞-norm and can ensure the monotone convergence of the SOTEs in the sense of 2-norm when $\sum_{j=2}^{T-1} |h_j| < 2|h_1|$.

Let $T = 20$. Since $\sum_{i=2}^{19} |h_i| = 1.9998$ and $h_1 = 1$, it follows that $\sum_{i=2}^{19} |h_i| < 2|h_1|$, which means that there exist η and α to ensure the monotone convergence of the SOTEs in the sense of 2-norm. Due to $|h_1| < \sum_{i=2}^{20} |h_i|$, it follows from Theorem 4 that there do not exist η and α to guarantee the monotone convergence of the SOTEs in the sense of 1-norm.

In simulations, we consider the following two cases.

Case 1: $\eta = 1$ and $\alpha = 0.4$.

A direct computation yields $\|\Omega\|_2 = 0.999651$ and $\|\Omega\|_\infty = 0.999787$, which imply $\eta = 1$ and $\alpha = 0.4$ can ensure the monotone convergence of the SOTEs in the sense of 2-norm and ∞-norm, respectively. The initial system input is set as $U_1 = H^{-1}(Y_d - V_1)$, in which the 17th component of V_1 is 1 and others are set to 0. It is easy to check that $\|E_2\|_1 = \|\Omega\|_1 = \|\Omega V_1\|_1 = 1.180415$ and $\|E_1\|_1 = \|V_1\|_1 = 1$. Therefore, $\eta = 1$ and $\alpha = 0.4$ cannot ensure the monotone convergence of the SOTEs in the sense of 1-norm. Fig. 4 shows the SOTEs in the sense of ∞, 2 and 1-norm, respectively. Clearly, $\|E_k\|_2$ and $\|E_k\|_\infty$ are monotonically convergent and $\|E_k\|_1$ are not.

Case 2: $\eta = 1$ and $\alpha = 0.45$.

It is easy to check that $\|\Omega\|_2 = 0.999860$ and $\|\Omega\|_\infty = 0.999873$. Thus, $\eta = 1$ and $\alpha = 0.45$ can ensure the monotone convergence of the SOTEs in the sense of 2-norm and ∞-norm, respectively. The initial system input is set as $U_1 = H^{-1}(Y_d - V_1)$, in which the 17th component of V_1 is 1 and others are set to 0. It is easy to check that $\|E_2\|_1 = \|\Omega\|_1 = \|\Omega V_1\|_1 = 1.155285$ and $\|E_1\|_1 = \|V_1\|_1 = 1$. Therefore, $\eta = 1$ and $\alpha = 0.45$ cannot ensure the monotone convergence of the SOTEs in the sense of 1-norm. Fig. 5 shows the SOTEs in the sense of ∞, 2 and 1-norm, respectively. Clearly, $\|E_k\|_2$ and $\|E_k\|_\infty$ are monotonically convergent and $\|E_k\|_1$ are not.

Remark 10. It should be pointed that when the upper and lower bounds of the Markov parameters h_i and the sign of h_1 are known, we can give a conservative design method about η and α in the time-varying gain $\gamma_t = \eta e^{-\alpha(T-t)}$ to ensure the monotone convergence of the SOTEs. However, large α may significantly reduce the convergence rate of ILC process, which may be unacceptable in practices. Note that the transient tracking performance of ILC system (1)–(2) with $\gamma_t = \eta e^{-\alpha(T-t)}$ is continuously dependent on the parameter α. As a rule of thumb, a relatively large α is sufficient to eliminate the unacceptable huge overshoot of ILC process.

5. Conclusion

The major discoveries in the note are as follows. The ILC update law with an exponentially decaying learning factor developed in Moore et al. (2005) cannot ensure that the SOTEs in the sense of unweighted 1, 2 and ∞-norm are monotonically convergent. There exists a time-varying learning gain to ensure the monotone convergence of SOTEs in the sense of unweighted ∞-norm. There does not exist a time-varying learning gain that makes the SOTEs in the sense of unweighted 1-norm converge monotonically if $|h_1| < \sum_{j=2}^{T} |h_j|$. Under the condition $\sum_{j=2}^{T-1} |h_j| < 2|h_1|$, there exists a time-varying learning gain to ensure the monotone convergence of SOTEs in the sense of unweighted 2-norm. And the initial system input and the trial length may have a significant influence on the tracking performance of ILC process.

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Jian Liu received his B.S. in Mathematics from Fuyang Normal University, China, in 2007, his M.S. in Mathematics from Dong Hua University, China, in 2010 and his Ph.D in Mathematics from Xi'an Jiaotong University, China, in 2017. Since 2017, he has been with Xidian University, China. His research interests include iterative learning control, networked control systems, and reinforcement learning.

Yuanshi Zheng was born in Jiangshan, China. He received the bachelor's and master's degrees in applied mathematics from Ningxia University, Yinchuan, China, in 2006 and 2009, respectively, and the Ph.D. degree from Xidian University, Xi'an, China, in 2012. He is currently a Professor with Xidian University. His research interests include coordination of multiagent systems, game theory, and collective intelligence.

YangQuan Chen earned his Ph.D. from Nanyang Technological University, Singapore, in 1998. He had been a faculty of Electrical Engineering at Utah State University (USU) from 2000–12. He joined the School of Engineering, University of California, Merced (UCM) in summer 2012 teaching "Mechatronics", "Engineering Service Learning" and "Unmanned Aerial Systems" for undergraduates; "Fractional Order Mechanics", "Nonlinear Controls" and "Advanced Controls: Optimality and Robustness" for graduates. His research interests include mechatronics for sustainability, cognitive process control, small multi-UAV based cooperative multi-spectral "personal remote sensing", applied fractional calculus in controls, modeling and complex signal processing; distributed measurement and control of distributed parameter systems with mobile actuator and sensor networks. He is listed in Highly Cited Researchers by Clarivate Analytics from 2018 to 2021. He received Research of the Year awards from USU (12) and UCM (20).