Consensus of Hybrid Behavior for Graphical Coordination Games

Liqi Zhou, Jiamin Wang, Maojiao Ye, Bao-Lin Zhang, Yuanshi Zheng

***Abstract*—This brief is concerned with consensus of hybrid behavior for graphical coordination games. We take into account a multi-agent system with two agent-unions that have various preferences. Interactions among agents in the multi-agent system are modelled as a hybrid graphical coordination game, in which three kinds of payoff matrices are designed. By using potential game theory, necessary and sufficient conditions are established for achieving consensus of the hybrid graphical coordination game. Finally, several numerical simulations are provided to validate the effectiveness of our theoretical results.**

***Index Terms*—Graphical coordination game, hybrid behavior, consensus.**

I. Introduction

A lot of attention has been paid to games such as aggregative games [1], multi-player games [2]–[4], trust games [5], bimatrix games [6], graphical coordination games [7], etc. Among them, graphical coordination games are representative game models with numerous applications in biology, economics, and social sciences.

Graphical coordination games simulate the so-called strategic complements effects, that is, a choice an agent makes is more appealing for other agents [8]. They can be used to model social network characteristics like belief adoption and spread of new technologies. As a type of game with multiple pure Nash equilibria, the focus of studying such games is to solve an equilibrium selection problem. Log-linear learning algorithm [9], [10] is a distributed decision-making rule with the ability of optimal equilibrium selection, which corresponds to consensus of the agents [11]–[14]. Therefore, many authors have applied log-linear learning algorithm to solve consensus problem for graphical coordination games.

Based on the graphical coordination game and log-linear learning algorithm, consensus problem was considered in [15] for the graphical coordination game with adversaries. Collins et al. [16] looked at a multi-agent system (MAS) modelled as a graphical coordination game with log-linear learning algorithm. Moreover, they presented a threshold below which the graphical coordination game can guarantee consensus. In [17], the stochastically stable states were investigated for the log-linear learning algorithm, which ensured consensus of the graphical coordination game. By employing log-linear learning algorithm, the author [18] investigated consensus of the graphical coordination game under adversarial attacks.

It is worth mentioning that the previously mentioned works are concerned with graphical coordination games, in which all the agents have the same preference. However, two agent-unions with different preferences may exist in graphical coordination games. Therefore, we consider an MAS with two agent-unions that have different preferences in this brief. Note that hybrid generally refers to heterogeneity of nature or composition, such as different types of agents [19]. The interactions among agents are modelled as a hybrid graphical coordination game, in which three kinds of payoff matrices are designed. The log-linear learning algorithm is employed as the decision-making rule of agents. Difficulties arise from how to model and analyze the hybrid graphical coordination game between two agent-unions with diverse preferences. The main contributions of this brief are threefold. First, a novel class of hybrid graphical coordination game is developed to model the MAS composed of two agent-unions with different preferences. Second, necessary and sufficient conditions are established for consensus of the hybrid graphical coordination game developed in this brief. Third, we investigate how consensus is impacted by the number of agents in agent-unions.

The remainder of this brief is structured as follows. Some preliminaries are listed in Section II. Our problem is formulated in Section III. Main results are presented in IV. Several numerical examples are provided in Section V to illustrate the validity of our theoretical results. Conclusion is given in Section VI.

Notations: Throughout this brief, $\mathbb{R}$ denotes the set of real numbers. Null set is represented by $\emptyset$. For given sets A and B, $A\cup B$, $A\cap B$ and $A\times B$ indicate the set union, set intersection and cartesian product, respectively. $A \subseteq B$ ($A \subset B$) refers to that set A is the (proper) subset of set B. $|A|$ denotes the number of elements in set A. $P(A)$ is the set of all subsets of A.

II. Preliminaries

A. Graph theory

An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a vertex set $\mathcal{V} = \{1, 2, \cdots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$, where $(i, j) \in \mathcal{E}$

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if and only if $(j,i) \in \mathcal{E}$. Moreover, we assume that $(i,i) \notin \mathcal{E}$ for all $i \in \mathcal{V}$. The neighbor set of vertex i is denoted as $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j,i) \in \mathcal{E}\}$. A finite-ordered sequence of distinct edges $(i,k_1),(k_1,k_2),\cdots,(k_l,j)$ is called a path between two distinct vertices i and j. If there is a path between every two vertices, such an undirected graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ is connected.

B. Graphical coordination games

First, we introduce a two-agent coordination game [20]. Agent i in a two-agent coordination game selects a choice a_i in an action set $\Omega_i = \{x,y\}$, where $i = 1,2$. When agents adopt a joint action (a_1,a_2), their payoffs $u(a_1,a_2)$ depicted by the following payoff matrix with payoff gain $\alpha > 0$.

Agent 2 \ Agent 1	x	y
x	$1+\alpha,\ 1+\alpha$	0, 0
y	0, 0	1, 1

A graphical coordination game is played by a set of agents $V = \{1,2,\cdots,n\}$ over a connected undirected graph $\mathcal{G} = (V,\mathcal{E})$, in which each agent i plays the two-agent coordination game with agent j if $(i,j) \in \mathcal{E}$. The utility function of agent i is $U_i(a_i,a_{-i}) = \sum_{j\in\mathcal{N}_i} u(a_i,a_j)$ for a joint action $a = (a_i,a_{-i}) \in \Omega = \Omega_1 \times \Omega_2 \times \cdots \Omega_n$, where $a_{-i} = (a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_n)$ and $\Omega_i = \{x,y\}$ for all $i \in V$.

A Nash equilibrium is a joint action in which no agent can increase their utilities unilaterally [21]. Apparently, joint actions $\vec{x} = (x,x,\cdots,x)$ and $\vec{y} = (y,y,\cdots,y)$ are both Nash equilibria of the graphical coordination game. We refer to x as the preference of agents in $\mathcal{V}$ since payoff gain $\alpha > 0$. Then, which equilibrium will be evolved to depends on decision-making rules of agents.

C. Log-linear learning

A distributed stochastic algorithm known as log-linear learning is employed for making decisions at discrete-time instants $t = 0,1,2,\cdots$ in this brief. For arbitrarily determined initial joint action $a(0) \in \Omega$, under log-linear learning, a joint action at discrete-time $t+1$ is chosen in the following way [10]:

- One agent i is selected from $\mathcal{V}$ with uniform probability.
- The selected agent i updates its action to $z \in \Omega_i$ probabilistically according to

$$Pr[a_i(t+1) = z|a(t)] = \frac{\exp(\beta\cdot U_i(z,a_{-i}(t)))}{\exp(\beta\cdot U_i(x,a_{-i}(t))) + \exp(\beta\cdot U_i(y,a_{-i}(t)))},$$

  where $\beta > 0$ is a given rationality parameter.
- All other agents maintain their previous actions, i.e. $a_{-i}(t+1) = a_{-i}(t)$.

Then, one has a joint action $a(t+1) = (a_i(t+1),a_{-i}(t))$ at time $t+1$. As shown in [7], for any $\beta > 0$, the log-linear learning process induces an ergodic Markov chain over joint action space Ω, with a unique stationary distribution π_β. As $\beta \to \infty$, the limiting distribution $\pi = \lim_{\beta\to\infty} \pi_\beta$ exists and is unique.

Definition 1: [15] Under log-linear learning algorithm, a joint action $w \in \Omega$ is said to be strictly stochastically stable, if for any $\varepsilon > 0$, there exist $\mathcal{B} < \infty$ and $\tau < \infty$ such that

$$Pr[a(t) = w] > 1 - \varepsilon$$

for all $\beta > \mathcal{B}$, $t > \tau$, where $a(t)$ is the joint action at time t.

D. Potential games

Definition 2: [22] For a game with joint action set Ω, if there exists a potential function $\Phi : \Omega \to \mathbb{R}$, such that

$$U_i(a_i,a_{-i}) - U_i(a_i',a_{-i}) = \Phi(a_i,a_{-i}) - \Phi(a_i',a_{-i}) \qquad (1)$$

holds for all $i \in \mathcal{V}$, $a_i, a_i' \in \Omega_i$ and $a_i \neq a_i'$, then such a game is a potential game.

III. Problem Statement

Consider an MAS with n agents on a connected undirected graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$, in which the interactions among agents are modelled as a hybrid graphical coordination game $\mathbb{G}$. In this brief, we investigate consensus of such hybrid graphical coordination game $\mathbb{G}$ which is defined as follows:

- **Players** : There are two agent-unions $A = \{1,\cdots,n_A\}$ and $B = \{n_A+1,\cdots,n_A+n_B\}$, where $n_A, n_B > 0$ and $n_A+n_B = n$. The set of all players is denoted as $\mathcal{V} = A \cup B$.
- **Strategy** : Each agent i in $\mathcal{V}$ has initial action $a_i(0)$ and the same action set $\Omega_i = \{x,y\}$. They choose their actions according to log-linear learning algorithm at time t. Then, one has joint action $a(t) = (a_i(t),a_{-i}(t-1)) \in \Omega = \{x,y\}^n$, where i is the agent selected to update at time t.
- **Payoff** : Since there are two types of agents in the system, there may be three kinds of payoff matrices:
  - If both agents i and $j \in \mathcal{N}_i$ are in set A, their payoffs $u_{AA}(a_i,a_j)$ are depicted by the following payoff matrix with payoff gain $\alpha_1 > 0$.

Agent i \ Agent j	x	y
x	$1+\alpha_1,\ 1+\alpha_1$	0, 0
y	0, 0	1, 1

  - If both agents i and $j \in \mathcal{N}_i$ are in set B, their payoffs $u_{BB}(a_i,a_j)$ are depicted by the following payoff matrix with payoff gain $\alpha_2 > 0$.

Agent i \ Agent j	x	y
x	1, 1	0, 0
y	0, 0	$1+\alpha_2,\ 1+\alpha_2$

  - If agents i is in set A and $j \in \mathcal{N}_i$ is in set B (or i is in set B and $j \in \mathcal{N}_i$ is in set A), their payoffs $u_{AB}(a_i,a_j)$ are depicted by the following payoff matrix with payoff gains $\alpha_1, \alpha_2 > 0$.

Agent i \ Agent j	x	y
x	$1+\alpha_1/2,\ 1+\alpha_1/2$	0, 0
y	0, 0	$1+\alpha_2/2,\ 1+\alpha_2/2$

  The total utility of agent i in $\mathcal{V}$ is equal to the sum of the payoffs obtained by playing two-agent coordination games with its neighbors. Therefore, given a joint action $a = (a_i,a_{-i})$,the total utility of agent i is

$$U_i(a) = \begin{cases} \sum_{j\in\mathcal{N}_i\cap A} u_{AA}(a_i,a_j) + \sum_{j\in\mathcal{N}_i\cap B} u_{AB}(a_i,a_j), & i \in A \\ \sum_{j\in\mathcal{N}_i\cap B} u_{BB}(a_i,a_j) + \sum_{j\in\mathcal{N}_i\cap A} u_{AB}(a_i,a_j), & i \in B \end{cases}.$$

Remark 1: Because of mutual influence, when agents i in set A and $j \in \mathcal{N}_i$ in set B both select action x, their payoffs will both be half of the sum of payoffs $1+\alpha_1$ and 1. Therefore,

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the payoffs are designed as $1+\alpha_1/2$ when agents $i \in A$ and $j \in \mathcal{N}_i$ in set B both select action x. Similarly, the payoffs are designed as $1+\alpha_2/2$ when agents $i \in A$ and $j \in \mathcal{N}_i$ in set B both select action y.

Remark 2: The problem addressed in this brief extends the game of battle of sex from two to n agents, which is practical. For example, a group of men and women with different preferences would prefer to participate in the same activity together rather than separately participate in their preferred activities. However, given that everyone is involved in the same activity, both groups would prefer to participate in their preferred activities.

IV. Main Results

Consider a connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, in which interactions among agents in $\mathcal{V}$ are modelled as game $\mathbb{G}$ defined in Section III. In this section, we explore criteria for achieving consensus of game $\mathbb{G}$. The following is the definition of consensus of game $\mathbb{G}$.

Definition 3: Under log-linear learning algorithm, if strictly stochastically stable state of hybrid graphical coordination game $\mathbb{G}$ is either $\vec{x}$ or $\vec{y}$ (i.e, all the agents adopt action x or action y) for any initial joint action $a(0)$, then game $\mathbb{G}$ is said to achieve consensus.

Prior to stating the main results, we propose following useful lemmas.

Lemma 1: Hybrid graphical coordination game $\mathbb{G}$ is a potential game, and its potential function is $\Phi(a) = \frac{1}{2}\sum_{i\in\mathcal{V}} U_i(a)$.

Proof. Since $u_{AA}(a_i, a_j) = u_{AA}(a_j, a_i)$, $u_{AB}(a_i, a_j) = u_{AB}(a_j, a_i)$ and $u_{BB}(a_i, a_j) = u_{BB}(a_j, a_i)$, the fact $\Phi(a_i, a_{-i}) - \Phi(a_i', a_{-i}) = \frac{1}{2}\{[U_i(a_i, a_{-i}) - U_i(a_i', a_{-i})] + [U_i(a_{-i}, a_i) - U_i(a_{-i}, a_i')]\} = U_i(a_i, a_{-i}) - U_i(a_i', a_{-i})$ holds for all $i \in \mathcal{V}$, $a_i, a_i' \in \Omega_i$ and $a_i \neq a_i'$. Then, the proof is completed. □

Lemma 2: [15] Under the log-linear learning algorithm,a joint action $a \in \Omega$ is strictly stochastically stable if and only if it maximize the potential function Φ, i.e., $\Phi(a) > \Phi(a')$ holds for all $a, a' \in \Omega$ and $a \neq a'$.

Let $d(S, T) = |\{(i, j) \in \mathcal{E} : i \in S, j \in T\}|$ for any $S \subseteq \mathcal{V}$ and $T \subseteq \mathcal{V}$. For convenience, we define the following functions and sets: $V_1 = d(A, A) - d(T \cap A, T \cap A) + \frac{1}{2}[d(A, B) - d(T \cap A, T \cap B)]$, $V_2 = d((\mathcal{V}\backslash T) \cap B, (\mathcal{V}\backslash T) \cap B) + \frac{1}{2} d((\mathcal{V}\backslash T) \cap A, (\mathcal{V}\backslash T) \cap B)$, $V_3 = d(B, B) - d((\mathcal{V}\backslash T) \cap B, (\mathcal{V}\backslash T) \cap B) + \frac{1}{2}[d(A, B) - d((\mathcal{V}\backslash T) \cap A, (\mathcal{V}\backslash T) \cap B)]$, $V_4 = d(T \cap A, T \cap A) + \frac{1}{2} d(T \cap A, T \cap B)$, $\mathcal{T}_1 = \{T \in P(\mathcal{V}) : V_2 > 0, T \neq \mathcal{V}\}$, $\mathcal{T}_2 = \{T \in P(\mathcal{V}) : V_4 > 0, T \neq \emptyset\}$.

The following theorem establishes the necessary and sufficient conditions for consensus of game $\mathbb{G}$.

Theorem 1: For hybrid graphical coordination game $\mathbb{G}$ played over a connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$:

1) Joint action $\vec{x}$ is strictly stochastically stable if and only if $(\alpha_1, \alpha_2) \in D^{\vec{x}}$, where $D^{\vec{x}} = \bigcap_{T\in\mathcal{T}_1} \{(\alpha_1, \alpha_2) \mid 0 < \alpha_2 < \frac{V_1}{V_2}\alpha_1 + \frac{d(T, \mathcal{V}\backslash T)}{V_2}, \alpha_1 > 0\}$.
2) Joint action $\vec{y}$ is strictly stochastically stable if and only if $(\alpha_1, \alpha_2) \in D^{\vec{y}}$, where $D^{\vec{y}} = \bigcap_{T\in\mathcal{T}_2} \{(\alpha_1, \alpha_2) \mid 0 < \alpha_1 < \frac{V_3}{V_4}\alpha_2 + \frac{d(T, \mathcal{V}\backslash T)}{V_4}, \alpha_2 > 0\}$.

Proof. 1) (Necessity) Denote a joint action $a = (\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) \neq \vec{x}$ for $\forall T \subset \mathcal{V}$, in which $T = \{i : a_i = x\}$ and $\mathcal{V}\backslash T = \{i : a_i = y\}$. Since joint action $\vec{x}$ is strictly stochastically stable, it follows from Lemma 2 that $\Phi(\vec{x}) > \Phi(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T})$ for $\forall T \subset \mathcal{V}$.

In virtue of Lemma 1, one has

$$\Phi(\vec{x}) - \Phi(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) = d(T, \mathcal{V}\backslash T) + \alpha_1 V_1 - \alpha_2 V_2 > 0, \quad (2)$$

where $V_1 = d(A, A) - d(T \cap A, T \cap A) + \frac{1}{2}[d(A, B) - d(T \cap A, T \cap B)]$ and $V_2 = d((\mathcal{V}\backslash T) \cap B, (\mathcal{V}\backslash T) \cap B) + \frac{1}{2} d((\mathcal{V}\backslash T) \cap A, (\mathcal{V}\backslash T) \cap B)$. Clearly, $V_1, V_2 \geq 0$. Based on these preliminary observations, we enumerate the following cases:

Case 1: $V_1 = 0$ and $V_2 = 0$.

Under this circumstance, $A \subset T$, which implies that $T \neq \emptyset$. Then, it is easy to verify that $\Phi(\vec{x}) - \Phi(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) = d(T, \mathcal{V}\backslash T) > 0$ is always true as long as $\alpha_1, \alpha_2 > 0$.

Case 2: $V_1 = 0$ and $V_2 > 0$.

Similar to Case 1, one has $T \neq \emptyset$ in this case, which leads to $d(T, \mathcal{V}\backslash T) > 0$. Then, (2) becomes $\Phi(\vec{x}) - \Phi(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) = d(T, \mathcal{V}\backslash T) - \alpha_2 V_2 > 0$. Hence, $\alpha_1 > 0$ and $0 < \alpha_2 < \frac{d(T, \mathcal{V}\backslash T)}{V_2}$.

Case 3: $V_1 > 0$ and $V_2 = 0$.

For this configuration, (2) can be rewritten as $\Phi(\vec{x}) - \Phi(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) = d(T, \mathcal{V}\backslash T) + \alpha_1 V_1 > 0$, which always holds as long as $\alpha_1, \alpha_2 > 0$ since $d(T, \mathcal{V}\backslash T) \geq 0$ and $V_1 > 0$.

Case 4: $V_1 > 0$ and $V_2 > 0$.

In this situation, it can be deduced from (2) that $0 < \alpha_1$ and $0 < \alpha_2 < \frac{V_1}{V_2}\alpha_1 + \frac{d(T, \mathcal{V}\backslash T)}{V_2}$, where $d(T, \mathcal{V}\backslash T) \geq 0$.

As a consequence, combining Cases 1-4 gives rise to that joint action $\vec{x}$ is strictly stochastically stable only if $(\alpha_1, \alpha_2) \in D^{\vec{x}}$, where $D^{\vec{x}} = \bigcap_{T\in\mathcal{T}_1} \{(\alpha_1, \alpha_2) \mid 0 < \alpha_2 < \frac{V_1}{V_2}\alpha_1 + \frac{d(T, \mathcal{V}\backslash T)}{V_2}, \alpha_1 > 0\}$. This completes the proof of necessity.

(Sufficiency) When $V_2 = 0$ and $V_1 \geq 0$, $\Phi(\vec{x}) > \Phi(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T})$ holds for $\forall \alpha_1, \alpha_2 > 0$ according to the analysis of Cases 1 and 3. When $V_2 > 0$ and $V_1 \geq 0$, a simple manipulation yields that $d(T, \mathcal{V}\backslash T) + \alpha_1 V_1 - \alpha_2 V_2 > 0$. Therefore, $\Phi(\vec{x}) - \Phi(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) > 0$. By Lemma 2, we can conclude that joint action $\vec{x}$ is strictly stochastically stable when $(\alpha_1, \alpha_2) \in D^{\vec{x}}$. Then, the proof of sufficiency is completed.

2) The proof of conclusion 2) is similar to that of 1). For saving space, we omit it. □

Remark 3: According to Definition 3, Theorem 1 provides criteria for achieving consensus of game $\mathbb{G}$. Moreover, necessary and sufficient conditions are presented for agents in game $\mathbb{G}$ to converge to the specified consensus actions.

Remark 4: The strictly stochastic stable state of game $\mathbb{G}$ will never be $\vec{x}$ or $\vec{y}$ if $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is not connected. Therefore, every theorem and lemmas in this brief require the topology of the MAS to be connected.

Remark 5: As shown in the proof of 1) in Theorem 1, Cases 1 and 3 imply that agents in game $\mathbb{G}$ can all converge to action x for any set A and payoff gains $\alpha_1, \alpha_2 > 0$ under specific topologies. Otherwise, if joint action $\vec{x}$ is required to be strictly stochastically stable, it is necessary to constrain the payoff gain α_2 according to Cases 2 and 4 of Theorem 1. In other words, it is possible to achieve consensus for game $\mathbb{G}$ either by limiting payoff gains or modifying network topology.

Theorem 1 indicates that whether the strictly stochastic stable state is required to be $\vec{x}$ or $\vec{y}$, payoff gains α_1, α_2

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are mutually conditioned. Then, scenario $\alpha_1 = \alpha_2 = \alpha$ is considered in the following lemma. Moreover, we define sets $\mathcal{T}_3 = \{T \in P(\mathcal{V}) : V_2 - V_1 > 0, T \neq \mathcal{V}\}$ and $\mathcal{T}_4 = \{T \in P(\mathcal{V}) : V_4 - V_3 > 0, T \neq \emptyset\}$.

Lemma 3: For hybrid graphical coordination game $\mathbb{G}$ with $\alpha_1 = \alpha_2 = \alpha > 0$ played over a connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$:

1) Joint action $\vec{x}$ is strictly stochastically stable if and only if $d(B,B) < d(A,A)$ and $\alpha \in \bigcap_{T\in\mathcal{T}_3} \{0 < \alpha < \frac{d(T,\mathcal{V}\backslash T)}{V_2-V_1}\}$.
2) Joint action $\vec{y}$ is strictly stochastically stable if and only if $d(A,A) < d(B,B)$ and $\alpha \in \bigcap_{T\in\mathcal{T}_4} \{0 < \alpha < \frac{d(T,\mathcal{V}\backslash T)}{V_4-V_3}\}$.

Proof. 1) (Necessity) Similar to the proof of Theorem 1, it can be concluded that

$$\Phi(\vec{x}) - \Phi(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) = d(T, \mathcal{V}\backslash T) + \alpha(V_1 - V_2) > 0 \quad (3)$$

for any joint action $a = (\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) \neq \vec{x}$, in which $T \subset \mathcal{V}$ and $T = \{i : a_i = x\}$ and $\mathcal{V}\backslash T = \{i : a_i = y\}$.

When $d(T, \mathcal{V}\backslash T) = 0$, one has $(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) = \vec{y}$ since $(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) \neq \vec{x}$. It follows from (3) that $\Phi(\vec{x}) - \Phi(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) = \alpha(V_1 - V_2) = \alpha(d(A,A) - d(B,B)) > 0$. Therefore, one has $d(B,B) < d(A,A)$.

When $d(T, \mathcal{V}\backslash T) > 0$, we contemplate the following two cases:

Case 1: $V_1 \geq V_2$.

For this configuration, $\Phi(\vec{x}) - \Phi(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) = d(T, \mathcal{V}\backslash T) + \alpha(V_1 - V_2) > 0$ always holds. Then, one has $\alpha > 0$.

Case 2: $V_1 < V_2$.

In this case, $\Phi(\vec{x}) - \Phi(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) = d(T, \mathcal{V}\backslash T) + \alpha(V_1 - V_2) > 0$ provides $0 < \alpha < \frac{d(T,\mathcal{V}\backslash T)}{V_2-V_1}$.

The above analysis yields that joint action $\vec{x}$ is strictly stochastically stable only if $d(B,B) < d(A,A)$ and $\alpha \in \bigcap_{T\in\mathcal{T}_3} \{0 < \alpha < \frac{d(T,\mathcal{V}\backslash T)}{V_2-V_1}\}$. This completes the proof of necessity.

(Sufficiency) When $V_2 - V_1 \leq 0$ and $d(B,B) < d(A,A)$, $\Phi(\vec{x}) > \Phi(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T})$ holds for $\forall \alpha > 0$. When $V_2 - V_1 > 0$ and $d(B,B) < d(A,A)$, a simple manipulation yields that $d(T, \mathcal{V}\backslash T) + \alpha(V_1 - V_2) > 0$. Therefore, $\Phi(\vec{x}) - \Phi(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) > 0$.

In the light of Lemma 2, we can conclude that joint action $\vec{x}$ is strictly stochastically stable as long as $d(B,B) < d(A,A)$ and $\alpha \in \bigcap_{T\in\mathcal{T}_3} \{0 < \alpha < \frac{d(T,\mathcal{V}\backslash T)}{V_2-V_1}\}$. Then, the proof of sufficiency is completed.

2) The derivation is analogous to the evidence in 1), which is omitted here. □

Remark 6: By comparison, the result of lemma 3 is the intersection of region $D^{\vec{x}}$ (or $D^{\vec{y}}$) in Theorem 1 and line $\alpha_1 = \alpha_2$, which is in accordance with our intuition. However, unlike Theorem 1, Lemma 3 limits the relationship between $d(A,A)$ and $d(B,B)$. This is due to that when α_1 and α_2 are equal, the potential function evaluated at $\vec{x}$ cannot be strictly larger or smaller than the potential function evaluated at $\vec{y}$ by limiting payoff gains, requiring topology to be further constrained.

Remark 7: Apparently, $V_2 - V_1 = V_4 - V_3$ and $\mathcal{T}_3 = \mathcal{T}_4$ hold, which mean that $\bigcap_{T\in\mathcal{T}_3} \{0 < \alpha < \frac{d(T,\mathcal{V}\backslash T)}{V_2-V_1}\} = \bigcap_{T\in\mathcal{T}_4} \{0 < \alpha < \frac{d(T,\mathcal{V}\backslash T)}{V_4-V_3}\}$. In other words, whether $\vec{x}$ or $\vec{y}$ is required to be strictly stochastically stable, the range of α is the same. The reason for the different notation here is to ensure that the logic is complete from a mathematical symbolic standpoint. As illustrated by Lemma 3, when $\alpha_1 = \alpha_2 = \alpha > 0$, preferred action for more closely connected union is more likely to be strictly stochastic stable for the entire system.

Theorem 1 and Lemma 3 investigate the effect of payoff gains on consensus of game $\mathbb{G}$. In the following, we consider impact of number of agents in different agent-unions (i.e., n_A and n_B) on consensus of game $\mathbb{G}$, when payoff gains α_1 and α_2 are fixed.

Lemma 4: For hybrid graphical coordination game $\mathbb{G}$ with fixed $\alpha_1, \alpha_2 > 0$ played over a ring graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$:

1) Joint action $\vec{x}$ is strictly stochastically stable if and only if $n_A > \max_{T\in P(\mathcal{V})\backslash\mathcal{V}} \{V_4 + \frac{\alpha_2}{\alpha_1} V_2 - \frac{d(T,\mathcal{V}\backslash T)}{\alpha_1}\}$.
2) Joint action $\vec{y}$ is strictly stochastically stable if and only if $n_B > \max_{T\in P(\mathcal{V})\backslash\emptyset} \{V_2 + \frac{\alpha_1}{\alpha_2} V_4 - \frac{d(T,\mathcal{V}\backslash T)}{\alpha_2}\}$.

Proof. 1) (Necessity) Denote a joint action $a = (\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) \neq \vec{x}$ for $\forall T \subset \mathcal{V}$, in which $T = \{i : a_i = x\}$.

On a ring graph, A can be decomposed into p $(1 \leq p \leq n_A)$ components A_i, $i = 1, ..., p$, each of which is directly connected internally (or a separate part), and any two components are not directly connected. Then, one has $A = A_1 \cup A_2 \cdots \cup A_p$, $1 \leq p \leq n_A$, $\sum_{i=1}^{p} |A_i| = n_A$ and $A_i \cap A_j = \emptyset$ for $\forall i \neq j$. Likewise, we decompose B into p components and get $B = B_1 \cup B_2 \cdots \cup B_p$, $1 \leq p \leq n_B$, $\sum_{i=1}^{p} |B_i| = n_B$ and $B_i \cap B_j = \emptyset$ for $\forall i \neq j$.

A simple manipulation yields that $d(A,A) = n_A - p$, $d(B,B) = n_B - p$ and $d(A,B) = 2p$. Then, one has $V_1 = n_A - V_4$. According to Lemma 2, when joint action $\vec{x}$ is strictly stochastically stable, $\Phi(\vec{x}) - \Phi(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T}) = d(T, \mathcal{V}\backslash T) + \alpha_1(n_A - V_4) - \alpha_2 V_2 > 0$ holds for $\forall T \subset \mathcal{V}$. Therefore, $n_A > V_4 + \frac{\alpha_2}{\alpha_1} V_2 - \frac{d(T,\mathcal{V}\backslash T)}{\alpha_1}$ holds for $\forall T \subset \mathcal{V}$.

This completes the proof of necessity.

(Sufficiency) $n_A > \max_{T\in P(\mathcal{V})\backslash\mathcal{V}} \{V_4 + \frac{\alpha_2}{\alpha_1} V_2 - \frac{d(T,\mathcal{V}\backslash T)}{\alpha_1}\}$ implies that $\Phi(\vec{x}) > \Phi(\vec{x}_T, \vec{y}_{\mathcal{V}\backslash T})$ holds for $\forall T \subset \mathcal{V}$. By Lemma 2, one has that joint action $\vec{x}$ is strictly stochastically stable. Then, the proof of sufficiency is completed.

2) The derivation is similar to the proof in 1), which is omitted here. □

Remark 8: Lemma 4 indicates that the number of agents in different agent-unions has an impact on consensus of game $\mathbb{G}$. Consider a scenario that payoff gains α_1 and α_2 are fixed. If the strictly stochastically stable of the MAS is required to be the preferred action of an agent-union, the number of agents in this union should be sufficiently large.

V. Simulations

In the following, numerical simulations for our theoretical results are presented.

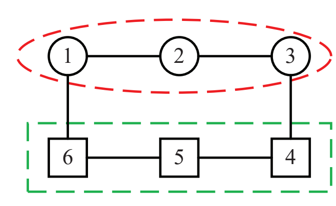

Fig. 1. A graph consists of two agent-unions which are denoted by $A = \{1, 2, 3\}$ (in round) and $B = \{4, 5, 6\}$ (in square), respectively.

We consider an MAS composed of 6 agents. The graph $\mathcal{G}$ is depicted in Fig. 1, where nodes 1, 2 and 3 belong to union A (in circle), and nodes 4, 5 and 6 belong to union B (in square). Model interactions among agents $1, 2, \cdots, 6$ as hybrid graphical coordination game $\mathbb{G}$. Then, we set $a(0) = (x, x, y, y, x, x)$, and intend to verify strictly stochastic stable state of game $\mathbb{G}$ for different payoff gains $\alpha_1, \alpha_2 > 0$.

$D^{\vec{x}}$ and $D^{\vec{y}}$ mentioned in Theorem 1 are depicted in Fig. 2. The purple region corresponds to set $D^{\vec{x}}$, and the yellow region corresponds to set $D^{\vec{y}}$.

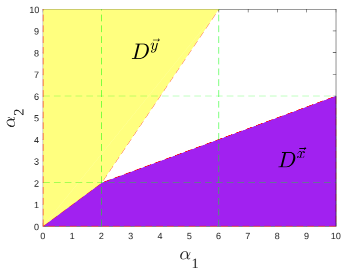

Fig. 2. Diagrams of sets $D^{\vec{x}}$ and $D^{\vec{y}}$.

For different gains $(\alpha_1, \alpha_2) = (0.5, 9)$, $(2, 6)$, $(6, 3)$ and $(9, 0.5)$, trends of the proportion of action y in stationary distribution of game $\mathbb{G}$ as increasing of rationality β are respectively shown in the Fig. 3.

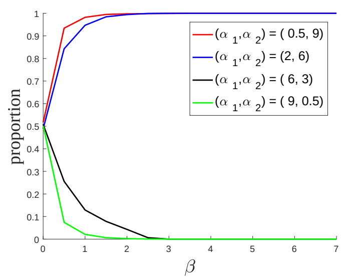

Fig. 3. The proportion of action y in stationary distribution.

$(\alpha_1, \alpha_2) = (0.5, 9)$ and $(2, 6)$ are both in the yellow region of Fig. 2, which imply that $(0.5, 9)$ and $(2, 6)$ are in set $D^{\vec{y}}$. Then, Fig. 3 shows that the proportion of action y converges to 1 in the stationary distribution of game $\mathbb{G}$ under these two pairs of payoff parameters, which is consistent with 1) of Theorem 1. $(\alpha_1, \alpha_2) = (6, 3)$ and $(9, 0.5)$ are in the purple region of Fig. 2, which indicate that $(6, 3)$ and $(9, 0.5)$ are in set $D^{\vec{x}}$. Fig. 3 depicts that in the stationary distribution of game $\mathbb{G}$ under these two pairs of payoff parameters, the proportion of action y converges to 0, which is in conformity to 2) of Theorem 1.

VI. Conclusion

In this brief, we considered consensus of hybrid graphical coordination game $\mathbb{G}$, in which agents are divided into two unions according to various payoff matrices. The log-linear learning algorithm was employed as the decision-making rule. By using potential game theory, we established necessary and sufficient conditions for achieving consensus of the hybrid graphical coordination game. Furthermore, we investigated how the number of agents in these two unions affects the consensus. Future work will concentrate on resilient consensus of hybrid graphical coordination games with malicious agents.

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