# Cluster Synchronization of a Nonlinear Network With Fixed and Switching Topologies 

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#### Abstract

This article mainly addresses the synchronization problem of leaderless and leader-follower clusters in directed topologically coupled nonlinear systems. The relationship between nodes within each cluster is cooperative, and nodes belonging to different clusters may compete with each other. For each case of leaderless and leader-following, we consider both fixed and switching topologies. The vector field at each node satisfies the one-sided Lipschitz condition. For situations where there is no leader and leader to follow, we do not need to use the in-degree balanced condition that is often required in most existing literature. Under the leaderless framework, the cluster synchronization (CS) problem is transformed into a stability problem through variable transformation. Under the leader-following framework, by equipping each cluster with a virtual leader, we design a new class of leader-following protocols that can be used to achieve CS of coupled nonlinear systems. Finally, two numerical examples are provided to illustrate the validity of the obtained results.


Index Terms-Cluster synchronization (CS), leader-following, leaderless, nonlinear systems, switching topologies.

## I. Introduction

IN RECENT years, the synchronization (consensus) problem of coupled nonlinear systems with competitive interactions has drawn significant attention of scholars from many fields due to its important role in many areas, such as social networks [1], [2], [3], [4], [5], competing species and companies [6], [7], [8]. The competition and cooperation are often abstracted as a signed graph, and the problem becomes the synchronization problem of the coupled nonlinear systems on the signed graph. When all agents can be divided into two hostile camps and the signed graph is structurally balanced, the bipartite and modulus synchronization are often studied in literatures [1], [2], [9], [10], [11], [12], [13], [14], [15], [16], [17].

In order to describe the actual system more accurately, all agents should be divided into multiple clusters, and individuals between different clusters may have competition and cooperation. In this case, it becomes more realistic for the network to

[^0]achieve cluster (group) synchronization (CS) [18], [19], [20], [21], [22], [23], [24], [25]. CS only requires that nodes in each subnetwork arrive at the same trajectory, but the node trajectories may be different for each subnetwork. At present, the research on CS can be roughly divided into two frameworks: leaderless CS and leader-following CS. In [20], [25], and [26], the authors studied the issue of leaderless CS. In the existing literature, the pinning control is often used to study leaderfollowing CS [21], [22], [27]. In order to make the CS manifold invariant, cluster-input-equivalence (CIE) condition should be required [22]. However, in many references, it is often required that the nodes of each cluster receive the sum of the input degrees of the nodes of other clusters to be zero, which is a special case of CIE condition, called the IDB condition. Although we have studied the CS of coupled nonlinear systems with CIE condition in our recent papers [28], [29], the interaction graphs within each cluster need to satisfy the strong connectivity condition. Hence, it is very necessary to extend the strong connectivity condition to weaker conditions, such as spanning trees.

The topology of the coupled systems may change in practice [30], [31], [32]. Therefore, it is of practical significance to consider CS in the case of switching topologies. Currently, there are few studies on CS under switching topologies. In [21], by designing a virtual leader for each cluster, the CS problem for complex networks with fast switching topologies was analyzed based on the averaging method. The cluster lag consensus problem of a second-order nonlinear network with switching topologies was investigated in [33] under the framework of leader-following. For the situation of leader-following, the condition that a digraph has a spanning tree is often used to obtain a synchronization criterion. As far as we know, the leaderless CS problem with a switching topology has not been solved. For leading cases, synchronization error systems are easy to obtain, and generally satisfy better properties. For leaderless situations, however, the synchronization error system is more complex to represent and thus more cumbersome to handle. Therefore, in a switching topology, getting a leaderless CS is a bit harder than a leader-following CS.

Motivated by the above literature review, we will separately analyze the leaderless and leader-following CS problems for coupled nonlinear systems with fixed and switching topologies. Competitive and cooperative relationships exist between nodes, and only cooperative relationships exist between nodes within each cluster. For leaderless scenario, by using variable transformation, the CS problem is transformed into the stability problem of error system. For leader-following scenario, without
the requirement of the IDB condition and spanning tree in the interaction graph within each cluster, we design a pinning control to enable the system to achieve leader-following CS under fixed and switched topologies, respectively. Compared with the existing work, the main contributions can be summarized as follows.

1) Regardless of the situation where there is no leader or the situation where the leader follows, we do not need to use the IDB condition which is often required in most of the already existing work [20], [22], [24], [26], [27], [34]. Furthermore, the inherent nonlinear system of each node satisfies a more general condition, namely, the OSL condition [35].
2) In [20] and [26], for the leaderless case with fixed topology, due to the need to calculate the generalized algebraic connectivity of each strongly connected component, the calculation is extremely difficult for large networks. In contrast, our results are easier to calculate and are suitable for large networks.
3) For the leaderless situation with switching topologies, if each subdigraph has a directed spanning tree, which extends the condition in [28] and [29], then some sufficient conditions on dwell time are obtained to make the system achieve CS. When each subdigraph is strongly connected and the weights of the edges satisfy the balance condition, some sufficient conditions are obtained for the system to achieve CS under any switching.
4) In the case of leader-following, we design a new pinning scheme without the requirement of the IDB condition and spanning tree in the interaction graph within each cluster, which are often used in literatures [22], [24], [34]. For fixed and switching topologies, we get some sufficient conditions for CS of the nonlinear network, respectively. Unlike the fast switching topologies considered in [21], we have leader-following CS results, which are expressed in terms of average dwell time.
This article proceeds as follow. Section II gives the interaction graph studied in this article and problem statement. Section III shows the main results about leaderless and leader-following CS for nonlinear network with fixed and switching topologies. Section IV shows the results about leader-following CS for nonlinear network with fixed and switching topologies. Section V gives two numerical examples to illustrate the validity of the theoretical results. Section VI summarizes the article and describes future research plans.

Notation: The following symbols are used in this article. The operator $\otimes$ denotes the Kronecker product. $\operatorname{sgn}(\cdot)$ represents the sign function. $I_{n}$ represents an $n$-dimensional identity matrix, and $\mathbf{1}_{n}\left(\mathbf{0}_{n}\right)$ denotes a $n$ dimensional vector with entries 1 (0). Let $\|y\|$ denote the Euclidean norm of a finite-dimensional vector $y$. Superscript T means transpose of a matrix or vector. Let $E$ be a symmetric matrix, $E \succ 0$ indicates that the matrix $E$ is positive definite, the largest and smallest eigenvalues are represented by $\lambda_{\max }(E)$ and $\lambda_{\min }(E)$, respectively. A diagonal matrix with diagonal elements $b_{1}, \ldots, b_{n}$ is represented by $\operatorname{diag}\left\{b_{1}, \ldots, b_{n}\right\}$. Let $\mathcal{P}=\{1,2, \ldots, p\}$ and $\mathcal{K}=\{1,2, \ldots, K\}$.

## II. Preliminaries

## A. Directed Interaction Graph

The network studied in this article contains $N$ nodes, which are numbered as $\mathcal{V}:=\{1, \ldots, N\}$. Each node of the network constitutes a digraph that changes with time, namely $\mathcal{G}(t)=$ $(\mathcal{V}, \mathcal{E}(t), A(t))$, where $\mathcal{V}=\{1, \ldots, N\}$ is the nodes set, $\mathcal{E}(t) \subset$ $\mathcal{V} \times \mathcal{V}$ is the edge set at time $t, A(t)=\left[a_{i j}(t)\right] \in \mathbb{R}^{N \times N}$ is the adjacency matrix at time $t . a_{i j}(t) \neq 0$ if there is an interaction from nodes $j$ to $i$ at time $t, a_{i j}(t)=0$ otherwise (assume $\left.a_{i i}(t)=0\right) . a_{i j}(t)<0\left(a_{i j}(t)>0\right)$ represents that the relationship between nodes $j$ and $i$ is competitive (cooperative). The Laplacian matrix of $\mathcal{G}(t)$ is denoted by $L(t)=\left[l_{i j}(t)\right]$, where $l_{i j}(t)=-a_{i j}(t), i \neq j$ and $l_{i i}(t)=\sum_{j=1}^{N} a_{i j}(t)$. The digraph $\mathcal{G}(t)$ is called weight balanced when $\sum_{j=1}^{N} a_{i j}(t)=$ $\sum_{j=1}^{N} a_{j i}(t)$.

It is assumed that the considered network nodes $\mathcal{V}$ can be divided into $K$ subnetworks. There is no competition within each subnetwork. Suppose the $k$ th subnetwork contains $m_{k}$ nodes. In order to make it easier to handle later, assume that nodes $\left\{z_{k-1}+1, \ldots, z_{k-1}+m_{k}\right\}$ form the $k$ th subnetwork, satisfying the condition $z_{0}=0, z_{k}=\sum_{j=1}^{k} m_{j}$. Suppose that $\bar{i}$ represents the subnetwork to which the $i$ th node belongs, that is, $i \in \mathcal{V}_{\bar{i}}, \bar{i} \in \mathcal{K}$. Let $\mathcal{G}_{\mathcal{V}_{k}}(t)$ denote the interaction digraph within the $k$ th subnetwork. All directed edges in $\mathcal{G}_{\mathcal{V}_{k}}(t)$ have both end points inside $\mathcal{V}_{k}$.

For the digraph $\mathcal{G}(t)$ at time $t$, if there exists a sequence of edges $\left(i_{1}, i_{2}\right), \ldots,\left(i_{q-1}, i_{q}\right)$ with distinct nodes $i_{j}$, then it means that there is a directed path between nodes $i_{q}$ and $i_{1}$. A digraph $\mathcal{G}(t)$ at time $t$ is said to have a spanning tree when there exists a vertex from which there is a directed path to any other node. When there is a directed path between any pair of different nodes, the digraph $\mathcal{G}(t)$ is called a strongly connected graph.

## B. Problem Statements

Consider a nonlinear network defined over digraph $\mathcal{G}(t)$, the dynamics of the $i$ th agent is described as

$$
\begin{equation*}
\dot{x}_{i}=f_{i}\left(x_{i}\right)+u_{i}, \quad i \in \mathcal{V}_{k}, \quad k \in \mathcal{K} \tag{1}
\end{equation*}
$$

where $x_{i} \in \mathbb{R}^{n}$ and $u_{i} \in \mathbb{R}^{n}$ are the state and control input of $i$ th agent, respectively. $f_{i}\left(x_{i}\right): \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is a continuous nonlinear function that satisfies the following assumptions.

Assumption 1: Suppose that $f_{i}\left(x_{i}\right), i \in \mathcal{V}_{k}$ are the same, that is, $f_{i}\left(x_{i}\right)=f_{\bar{i}}\left(x_{i}\right), i \in \mathcal{V}_{k}$, and there exists constant $\eta_{\bar{i}}$ such that for all $y_{1}, y_{2} \in \mathbb{R}^{n}$

$$
\left(y_{1}-y_{2}\right)^{\top}\left(f_{\bar{i}}\left(y_{1}\right)-f_{\bar{i}}\left(y_{2}\right)\right) \leq \eta_{\bar{i}}\left\|y_{1}-y_{2}\right\|^{2}
$$

Then, the nonlinear function $f_{i}\left(x_{i}\right)$ in (1) is said to satisfy the OSL condition [35].

Assumption 2: Suppose that there exist two constants $\sigma_{\bar{i}}$ and $\gamma_{\bar{i}}$ such that for all $y_{1}, y_{2} \in \mathbb{R}^{n}$

$$
\begin{aligned}
\left\|\left(f_{\bar{i}}\left(y_{1}\right)-f_{\bar{i}}\left(y_{2}\right)\right)\right\|^{2} \leq & \sigma_{\bar{i}}\left\|y_{1}-y_{2}\right\|^{2} \\
& +\gamma_{\bar{i}}\left(y_{1}-y_{2}\right)^{\top}\left(f_{\bar{i}}\left(y_{1}\right)-f_{\bar{i}}\left(y_{2}\right)\right) .
\end{aligned}
$$

Then, the nonlinear function $f_{i}\left(x_{i}\right)$ in (1) is said to satisfy the quadratic innerboundedness ( QIB ) condition.

Remark 1: In contrast to the Lipschitz constant, the OSL constant can be positive, zero, or even negative. Hence, if a function satisfies the OSL condition, then it does not necessarily satisfy Lipschitz condition. But the reverse must be true. As pointed out in [36], a function is continuous if it satisfies the QIB condition.

This article aims to design appropriate control inputs $u_{i}$ to achieve CS of the network (1). First, we will consider the leaderless case. When the directed interaction topology is fixed and switching, we will separately design appropriate control inputs $u_{i}$ such that the network (1) achieve CS. Second, we will consider the leader-following case, that is, designing a virtual leader for each cluster.

## III. LEADERLESS CS

In this section, under leaderless case, we will design the control input $u_{i}$ to synchronize the cluster of nonlinear network (1). The leaderless CS is defined as follows:

Definition 1: (Leaderless CS) The coupled nonlinear systems (1) is said to achieve CS when $\lim _{t \rightarrow \infty}\left\|x_{i}(t)-x_{j}(t)\right\|=$ $0, \forall \bar{i}=\bar{j}, i, j=1, \ldots, N$, and any initial states $x_{i}(0)$.

In order to achieve leaderless CS, we consider the following control protocol:

$$
\begin{equation*}
u_{i}=\sum_{j=1}^{N} c_{i j} a_{i j}(t) \Gamma\left(x_{j}-x_{i}\right) \tag{2}
\end{equation*}
$$

where $\Gamma$ is a positive definite diffusion matrix, $c_{i j}$ describes the mutual coupling between nodes, $c_{i j}=c_{k}>0$ when subscript $i$ and $j$ both belong to the $k$ th subnetwork, otherwise $c_{i j}=1$.

Assumption 3: Suppose that the interactive weights between clusters satisfy

$$
\mathcal{R}_{\mathcal{V}_{k_{1}} \mathcal{V}_{k_{2}}}(t):=\sum_{j \in \mathcal{V}_{k_{2}}} a_{i j}(t) \forall i \in \mathcal{V}_{k_{1}}
$$

where $k_{1}, k_{2} \in \mathcal{K}$ and $k_{1} \neq k_{2}$.
The Assumption 3 is called the CIE condition [26], [37], [38]. When the Assumption 3 is satisfied, the nonlinear network (1) under control protocol (2) has the following CS invariant manifold:
$S=\left\{x \in \mathbb{R}^{n N}\left\{x_{1}=\cdots=x_{m_{1}}, \ldots, x_{z_{K-1}+1}=\cdots=x_{N}\right\}\right.$.
Remark 2: Note that in [20] and [26], compared with CIE condition, a stronger condition (i.e., the IDB condition) was imposed in order to get the synchronization condition, namely,

$$
\sum_{j \in \mathcal{V}_{k}} a_{i j}=0 \quad \forall i=1, \ldots, N, i \in \mathcal{V} \backslash \mathcal{V}_{k}, k \in \mathcal{K}
$$

The IDB condition is somewhat restrictive in practical applications. In leaderless case, we will also only use the CIE condition in the derivation of the main results.

Let $L_{\mathcal{V}}(t)=\operatorname{diag}\left\{L_{\mathcal{V}_{1}}(t), \ldots, L_{\mathcal{V}_{K}}(t)\right\}$ and $\bar{L}(t)=L(t)-$ $L_{\mathcal{V}}(t)$. Then, $L_{\mathcal{V}}(t)$ and $\bar{L}(t)$ are Laplacian matrices formed by the topology of interactions within each cluster and between different clusters, respectively. Let $x=\left[x_{1}^{\top}, \ldots, x_{N}^{\top}\right]^{\top}$, the coupled
nonlinear systems (1) with control protocol (2) can be written in the following compact form:

$$
\begin{equation*}
\dot{x}=F(x)-\left(C L_{\mathcal{V}}(t) \otimes \Gamma\right) x-(\bar{L}(t) \otimes \Gamma) x \tag{3}
\end{equation*}
$$

where

$$
\begin{aligned}
F(x) & =\left[f_{\overline{1}}\left(x_{1}\right)^{\top}, \ldots, f_{\bar{N}}\left(x_{N}\right)^{\top}\right]^{\top} \\
C & =\operatorname{diag}\left\{c_{1} I_{m_{1}}, \ldots, c_{K} I_{m_{K}}\right\} .
\end{aligned}
$$

To derive the dynamics of CS errors, we introduce a variable transformation as follows:

$$
\begin{equation*}
y=\left(\Psi \otimes I_{n}\right) x \tag{4}
\end{equation*}
$$

where $\Psi=\operatorname{diag}\left\{\Psi_{1}, \ldots, \Psi_{K}\right\} \in \mathbb{R}^{N \times N}$ and the form of $\Psi_{k} \in$ $\mathbb{R}^{m_{k} \times m_{k}}$ is defined as

$$
\Psi_{k}=\left[\begin{array}{ccccc}
1 & 0 & 0 & \cdots & 0  \tag{5}\\
1 & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & -1
\end{array}\right]
$$

where $k \in \mathcal{K}$. It is easy to obtain that $\Psi=\Psi^{-1}$.
Denote $y=\left[y_{1}^{\top}, e_{\mathcal{V}_{1}}^{\top}, y_{z_{1}+1}^{\top}, e_{\mathcal{V}_{2}}^{\top}, \ldots, y_{z_{K-1}+1}^{\top}, e_{\mathcal{V}_{K}}^{\top}\right]^{\top} \quad$ with $e_{\mathcal{V}_{k}}=\left[e_{z_{k-1}+2}^{\top}, \ldots, e_{z_{k-1}+m_{k}}^{\top}\right]^{\top}$. By (4), we have $y_{z_{k-1}+1}^{\top}=$ $x_{z_{k-1}+1}^{\top} \quad$ and $e_{\mathcal{\nu}_{k}}=\left[\left(x_{z_{k-1}+1}-x_{z_{k-1}+2}\right)^{\top}, \ldots,\left(x_{z_{k-1}+1}-\right.\right.$ $\left.\left.x_{z_{k-1}+m_{k}}\right)^{\top}\right]^{\top}$.

It follows from (4) and (5) that

$$
\begin{align*}
\dot{y}= & \left(\Psi \otimes I_{n}\right) F(x)-\left(\Psi C L_{\mathcal{V}}(t) \Psi^{-1} \otimes \Gamma\right) y \\
& -\left(\Psi \bar{L}(t) \Psi^{-1} \otimes \Gamma\right) y \tag{6}
\end{align*}
$$

Define

$$
\begin{aligned}
E_{k} & =\left[\begin{array}{ll}
\mathbf{1}_{m_{k}-1} & -I_{m_{k}-1}
\end{array}\right] \in \mathbb{R}^{\left(m_{k}-1\right) \times m_{k}} \\
F_{k} & =\left[\begin{array}{ll}
\mathbf{0}_{m_{k}-1} & -I_{m_{k}-1}
\end{array}\right]^{\top} \in \mathbb{R}^{m_{k} \times\left(m_{k}-1\right)}
\end{aligned}
$$

Let

$$
\begin{aligned}
& E=\operatorname{diag}\left\{E_{1}, \ldots, E_{K}\right\} \in \mathbb{R}^{(N-K) \times N} \\
& F=\operatorname{diag}\left\{F_{1}, \ldots, F_{K}\right\} \in \mathbb{R}^{N \times(N-K)}
\end{aligned}
$$

Based on the above definitions of $E$ and $F$, we can rewrite (6) into the following two subsystems:

$$
\begin{align*}
\dot{y}_{z_{k-1}+1}= & f_{k}\left(x_{z_{k-1}+1}\right)-\left(c_{k} l_{\mathcal{V}_{k-1}+1}(t) \Psi \otimes \Gamma\right) y \\
& -\left(\bar{l}_{z_{k-1}+1}(t) \Psi \otimes \Gamma\right) y  \tag{7}\\
\dot{e}= & \bar{F}(x)-\left(E C L_{\mathcal{V}}(t) F \otimes \Gamma\right) e-(E \bar{L}(t) F \otimes \Gamma) e \tag{8}
\end{align*}
$$

where

$$
e=\left[e_{\mathcal{V}_{1}}^{\top}, \ldots, e_{\mathcal{V}_{K}}^{\top}\right]^{\top}
$$

$l_{\mathcal{V}_{k-1}+1}$ is the $z_{k-1}+1$ th row of $L_{\mathcal{V}}$ and $\bar{F}(x)=\left[\bar{F}_{1}(x)^{\top}\right.$, $\left.\ldots, \bar{F}_{K}(x)^{\top}\right]^{\top}$ with $\bar{F}_{k}(x)=\left[\left(f_{k}\left(x_{z_{k-1}+1}\right)-f_{k}\left(x_{z_{k-1}+2}\right)\right)^{\top}\right.$, $\left.\ldots,\left(f_{k}\left(x_{z_{k-1}+1}\right)-f_{k}\left(x_{z_{k-1}+m_{k}}\right)\right)^{\top}\right]^{\top}$.

Remark 3: Note that by variable transformation (4), the CS of coupled nonlinear systems (1) and control protocols (2) can be achieved when the system (8) is asymptotically stable. Hence,
we will obtain appropriate conditions such that the system (8) is asymptotically stable in the sequel. The variable transformation (4) is not unique. If we choose

$$
\Psi_{k}=\left[\begin{array}{cccccc}
1 & 0 & 0 & \cdots & 0 & 0 \\
1 & -1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -1
\end{array}\right]
$$

then we can also obtain the system (8).

## A. Leaderless CS Under a Fixed Topology

Based on the previous analysis, in this section, we study the leaderless CS problem of nonlinear network (1) under fixed topology. In this case, $L(t) \equiv L$.

Theorem 1: Suppose that all subdigraphs $\mathcal{G}_{\mathcal{V}_{k}}, k \in \mathcal{K}$ have a directed spanning tree and the Assumptions 1-3 hold. The CS of the nonlinear network (1) with control protocol (2) can be achieved if there exist positive constant $\theta_{k}$ and $c_{k}$ satisfying

$$
\begin{align*}
\gamma_{k}+2 \theta_{k} & \geq 0  \tag{9}\\
\lambda_{\max }\left(P_{k}\right)\left(\theta_{k}^{2}-1\right) & \leq \lambda_{\min }\left(P_{k}\right) \theta_{k}^{2} \tag{10}
\end{align*}
$$

and

$$
\begin{equation*}
c_{k}>\frac{\lambda_{\max }\left(P_{k}\right) \delta_{k}-\lambda_{\min }\left(P_{k}\right)-\theta_{k} \varsigma}{\theta_{k} \lambda_{\min }(\Gamma)} \tag{11}
\end{equation*}
$$

where $P_{k} \in \mathbb{R}^{\left(m_{k}-1\right) \times\left(m_{k}-1\right)}$ is a positive definite matrix satisfying $\left(E_{k} L \mathcal{V}_{k} F_{k}\right)^{\top} P_{k}+P_{k}\left(E_{k} L \mathcal{V}_{k} F_{k}\right)=I_{m_{k}-1}$,

$$
\begin{aligned}
& P=\operatorname{diag}\left\{P_{1}, \ldots, P_{K}\right\}, \delta_{k}=\sigma_{k}+1+\eta_{k}\left(2 \theta_{k}+\gamma_{k}\right) \\
& \zeta=\lambda_{\min }\left((E \bar{L} F)^{\top} P+P(E \bar{L} F)\right) \\
& \varsigma=(1+\operatorname{sgn}(\zeta)) \lambda_{\min }(\Gamma) \zeta+(1-\operatorname{sgn}(\bar{\zeta})) \lambda_{\max }(\Gamma) \zeta
\end{aligned}
$$

Proof: Let $\widehat{L}_{\mathcal{V}_{k}}=E_{k} L_{\mathcal{V}_{k}} F_{k}$. Because $\Psi_{k}^{-1}=\Psi_{k}$, the eigenvalues of $\Psi_{k} L_{\mathcal{V}_{k}} \Psi_{k}$ and $L_{\mathcal{V}_{k}}$ are the same. One can obtain the following equation by direct calculation:

$$
\Psi_{k} L \mathcal{V}_{k} \Psi_{k}=\left[\begin{array}{cc}
0 & l F_{k} \\
\mathbf{0}_{\left(m_{k}-1\right)} & \widehat{L}_{\mathcal{V}_{k}}
\end{array}\right]
$$

where $l$ is the row vector formed by the first row of matrix $L_{\mathcal{V}_{k}}$. If all digraphs $\mathcal{G}_{\mathcal{V}_{k}}, k \in \mathcal{K}$ have a directed spanning tree, then the corresponding Laplacian matrix $L \mathcal{V}_{k}$ has the following properties: 1) 0 is a simple eigenvalue of $L_{\mathcal{V}_{k}} ; 2$ ) the other $m_{k}-$ 1 eigenvalues have positive real parts. Hence, all eigenvalues of $\widehat{L} \mathcal{V}_{k}$ are equal to the nonzero eigenvalues of $L_{\mathcal{V}_{k}}$, and there exists a positive definite matrix $P_{k}$ such that $\left(E_{k} L \mathcal{V}_{k} F_{k}\right)^{\top} P_{k}+$ $P_{k}\left(E_{k} L_{\mathcal{V}_{k}} F_{k}\right)=I_{m_{k}-1}$.

Let

$$
V(t)=e^{\top}\left(P \otimes I_{n}\right) e
$$

Differentiating $V(t)$ along the system (8) gives the following expression:

$$
\begin{align*}
\dot{V}(t)= & \sum_{k=1}^{K}-c_{k} e_{\mathcal{V}_{k}}^{\top}\left[\left(\widehat{L}_{\mathcal{V}_{k}}^{\top} P_{k}+P_{k} \widehat{L}_{\mathcal{V}_{k}}\right) \otimes \Gamma\right] e_{\mathcal{V}_{k}} \\
& -e^{\top}\left[\left((E \bar{L} F)^{\top} P+P(E \bar{L} F)\right) \otimes \Gamma\right] e \\
& +\sum_{k=1}^{K} 2 e_{\mathcal{V}_{k}}^{\top}\left(P_{k} \otimes I_{n}\right) \bar{F}_{k}(x) . \tag{12}
\end{align*}
$$

It is easy to obtain

$$
\begin{equation*}
e_{\mathcal{V}_{k}}^{\top}\left[\left(\widehat{L}_{\mathcal{V}_{k}}^{\top} P_{k}+P_{k} \widehat{L}_{\mathcal{V}_{k}}\right) \otimes \Gamma\right] e_{\mathcal{V}_{k}} \geq \lambda_{\min }(\Gamma)\left\|e_{\mathcal{V}_{k}}\right\|^{2} \tag{13}
\end{equation*}
$$

Since $\Gamma$ is a positive definite matrix, one has

$$
\begin{equation*}
e^{\top}\left[\left(P(E \bar{L} F)+(E \bar{L} F)^{\top} P\right) \otimes \Gamma\right] e \geq \varsigma\|e\|^{2} \tag{14}
\end{equation*}
$$

Since $P_{k}$ a positive definite matrix, thus

$$
\begin{align*}
& 2 e_{\mathcal{V}_{k}}^{\top}\left(P_{k} \otimes I_{n}\right) \bar{F}_{k}(x) \\
& \quad=\frac{1}{\theta_{k}}\left(e_{\mathcal{V}_{k}}+\theta_{k} \bar{F}_{k}(x)\right)^{\top}\left(P_{k} \otimes I_{n}\right)\left(e_{\mathcal{V}_{k}}+\theta_{k} \bar{F}_{k}(x, t)\right) \\
& \quad-\frac{1}{\theta_{k}} e_{\mathcal{V}_{k}}^{\top}\left(P_{k} \otimes I_{n}\right) e_{\mathcal{V}_{k}}-\theta_{k} \bar{F}_{k}(x)^{\top}\left(P_{k} \otimes I_{n}\right) \bar{F}_{k}(x) \\
& \quad \leq \frac{1}{\theta_{k}} \lambda_{\max }\left(P_{k}\right)\left\|e_{\mathcal{V}_{k}}+\theta_{k} \bar{F}_{k}(x)\right\|^{2} \\
& \quad-\frac{1}{\theta_{k}} \lambda_{\min }\left(P_{k}\right)\left(\left\|e_{\mathcal{V}_{k}}\right\|^{2}+\theta_{k}^{2}\left\|\bar{F}_{k}(x)\right\|^{2}\right) \tag{15}
\end{align*}
$$

By the Assumption 2, one has

$$
\begin{aligned}
\left\|e_{\mathcal{V}_{k}}+\theta_{k} \bar{F}_{k}(x)\right\|^{2} \leq & \left(\sigma_{k}+1\right)\left\|e_{\mathcal{V}_{k}}\right\|^{2}+\left(\theta_{k}^{2}-1\right)\left\|\bar{F}_{k}(x)\right\|^{2} \\
& +\left(2 \theta_{k}+\gamma_{k}\right) e_{\mathcal{V}_{k}}^{\top} \bar{F}_{k}(x)
\end{aligned}
$$

Furthermore, using Assumption 1 and condition (9), we can get

$$
\begin{align*}
\left\|e_{\mathcal{V}_{k}}+\theta_{k} \bar{F}_{k}(x)\right\|^{2} \leq & \left(\sigma_{k}+1+\eta_{k}\left(2 \theta_{k}+\gamma_{k}\right)\right)\left\|e_{\mathcal{V}_{k}}\right\|^{2} \\
& +\left(\theta_{k}^{2}-1\right)\left\|\bar{F}_{k}(x)\right\|^{2} \tag{16}
\end{align*}
$$

Then, combining (10), (15), and (16), one obtains

$$
\begin{align*}
2 e_{\mathcal{V}_{k}}^{\top}\left(P_{k} \otimes I_{n}\right) \bar{F}_{k}(x) \leq & \frac{1}{\theta_{k}}\left(\lambda_{\max }\left(P_{k}\right) \delta_{k}\right. \\
& \left.-\lambda_{\min }\left(P_{k}\right)\right)\left\|e_{\mathcal{V}_{k}}\right\|^{2} \tag{17}
\end{align*}
$$

where $q(k)=\lambda_{\text {max }}\left(P_{k}\right)\left(\theta_{k}^{2}-1\right)-\lambda_{\min }\left(P_{k}\right) \theta_{k}^{2}$.
It can be obtained by combining (13), (14), and (17) that

$$
\begin{aligned}
\dot{V}(t) \leq & -\sum_{k=1}^{K}\left(c_{k} \lambda_{\min }(\Gamma)+\lambda_{0}\right)\left\|e_{\mathcal{V}_{k}}\right\|^{2} \\
& +\sum_{k=1}^{K} \frac{1}{\theta_{k}}\left(\lambda_{\max }\left(P_{k}\right) \delta_{k}-\lambda_{\min }\left(P_{k}\right)\right)\left\|e_{\mathcal{V}_{k}}\right\|^{2} .
\end{aligned}
$$

Thus, when the condition (11) holds, then for any $e \neq 0, V(t)<$ 0 . This means that the system (8) is asymptotically stable, that is, the coupled nonlinear systems (1) with control protocol (2) can achieve CS.

Remark 4: Compared with the results in [26], we allow $\mathcal{R}_{\mathcal{V}_{k_{1}} \nu_{k_{2}}}$ to be an arbitrary constant, and the nonlinear dynamics satisfy OSL condition. Therefore, the system discussed in [26] can be regarded as a special case of this article. Moreover, in [20]
and [26], the authors partition a directed graph containing spanning trees into strongly connected subgraphs and compute the generalized algebraic connectivity of each subgraph. However, when the coupled topology is complex and the number of nodes is large, the calculation is extremely difficult and obviously reduces the validity of the results.

## B. Leaderless CS Under Switching Topologies

In this section, we assume that the communication topology is time-varying, and there exist $p$ possible digraphs $\left\{\mathcal{G}^{1}, \ldots, \mathcal{G}^{p}\right\}$. The communication topology switches at the time instants $t_{1}, t_{2}, \ldots$ Consider a switching signal $\sigma(t):[0,+\infty) \rightarrow \mathcal{P}$ and let $\mathcal{G}^{\sigma(t)}$ be the communication topology for the coupled nonlinear systems (1) at time $t \geq 0$. Then, $\mathcal{G}^{\sigma(t)} \in \widehat{\mathcal{G}}$ for all $t \geq 0$, where $\widehat{\mathcal{G}}=\left\{\mathcal{G}^{1}, \ldots, \mathcal{G}^{p}\right\}$. Then, $\mathcal{G}_{\mathcal{V}_{k}}^{\sigma(t)} \in\left\{\mathcal{G}_{\mathcal{V}_{k}}^{1}, \ldots, \mathcal{G}_{\mathcal{V}_{k}}^{p}\right\}, k \in$ $\mathcal{K}$ describes the communication topology within each cluster $\mathcal{V}_{k}$ at time $t \geq 0$.

Assumption 4: Assume every subdigraph $\mathcal{G}_{\mathcal{V}_{k}}^{j}, j \in \mathcal{P}, k \in \mathcal{K}$ contains a directed spanning tree.

Definition 2: Suppose that the switching of the digraph in the time period $\left[t_{0}, t\right), \forall t>t_{0}$ is $N_{\sigma}\left[t_{0}, t\right)$. If the following inequality holds:

$$
N_{\sigma}\left[t_{0}, t\right) \leq N_{0}+\frac{t-t_{0}}{\tau_{a}}
$$

where $N_{0}$ is a nonnegative integer, then $\tau_{a}$ is called the average dwell time.

Theorem 2: Suppose that the Assumptions 1-4 hold. CS of nonlinear network (1) with control protocol (2) can be achieved if there are constants $\theta_{k}, c_{k}$ and an average dwell time $\tau_{a}$ satisfying (9)

$$
\begin{align*}
\bar{\lambda}\left(P_{k}\right)\left(\theta_{k}^{2}-1\right) & \leq \underline{\lambda}\left(P_{k}\right) \theta_{k}^{2}  \tag{18}\\
\tau_{a} & >\frac{\bar{\lambda}(P) \ln \mu}{\beta}  \tag{19}\\
c_{k} & >\frac{\bar{\lambda}\left(P_{k}\right) \delta_{k}-\underline{\lambda}\left(P_{k}\right)-\theta_{k} \underline{\varsigma}}{\theta_{k} \lambda_{\min }(\Gamma)} \tag{20}
\end{align*}
$$

where $\quad \mu=\bar{\lambda}(P) / \underline{\lambda}(P) \quad$ with $\quad \beta=\min _{k \in \mathcal{K}} \beta_{k} \quad$ and $\quad \beta_{k}=$ $c_{k} \lambda_{\min }(\Gamma)+\underline{\varsigma}-\frac{1}{\theta_{k}}\left(\bar{\lambda}\left(P_{k}\right) \delta_{k}-\underline{\lambda}\left(P_{k}\right)\right), \quad \underline{\varsigma}=0.5 \min _{i \in \mathcal{P}} \varsigma^{i}$, $\varsigma^{i}=\zeta^{i}\left(\lambda_{\max }(\Gamma)+\lambda_{\min }(\Gamma)\right)+\operatorname{sgn}(\zeta) \zeta^{i}\left(\lambda_{\min }(\Gamma)-\lambda_{\max }(\Gamma)\right)$, $\zeta^{i}=\lambda_{\min }\left(\left(E \bar{L}^{i} F\right)^{\top} P+P\left(E \bar{L}^{i} F\right)\right), \quad \delta_{k}=\left(\gamma_{k}+2 \theta_{k}\right) \eta_{k}$ $+\sigma_{k}+1, \quad \underline{\lambda}\left(P_{k}\right)=\min _{i \in \mathcal{P}} \lambda_{\text {min }}\left(P_{k}^{i}\right), \quad \bar{\lambda}\left(P_{k}\right)=$ $\max _{i \in \mathcal{P}} \lambda_{\max }\left(P_{k}^{i}\right), \quad \bar{\lambda}(P)=\max _{k \in \mathcal{K}} \bar{\lambda}\left(P_{k}\right), \quad \underline{\lambda}(P)=$ $\min _{k \in \mathcal{K}} \underline{\lambda}\left(P_{k}\right), \quad P_{k}^{i} \in \mathbb{R}^{\left(m_{k}-1\right) \times\left(m_{k}-1\right)}$ is a positive definite matrix satisfying $\left(E_{k} L_{\mathcal{V}_{k}}^{i} F_{k}\right)^{\top} P_{k}^{i}+P_{k}^{i}\left(E_{k} L_{\mathcal{V}_{k}}^{i} F_{k}\right)=I_{m_{k}-1}$, $i \in \mathcal{P}$.

Proof: Let

$$
V(t)=e^{\top}\left(P^{\sigma(t)} \otimes I_{n}\right) e
$$

Differentiating $V(t)$ along the system (8) gives the following expression:

$$
\begin{align*}
\dot{V}(t) \leq & \sum_{k=1}^{K}-c_{k} \lambda_{\min }(\Gamma)\left\|e_{\mathcal{V}_{k}}\right\|^{2}-\varsigma\|e\|^{2} \\
& +\sum_{k=1}^{K} 2 e_{\mathcal{V}_{k}}^{\top}\left(P_{k}^{\sigma(t)} \otimes I_{n}\right) \bar{F}_{k}(x) . \tag{21}
\end{align*}
$$

Using the same proof method as (15) in Theorem 1, we have

$$
\begin{align*}
& 2 e_{\mathcal{V}_{k}}^{\top}\left(P_{k}^{\sigma(t)} \otimes I_{n}\right) \bar{F}_{k}(x) \\
& \quad \leq \frac{1}{\theta_{k}} \bar{\lambda}\left(P_{k}\right)\left\|e_{\mathcal{V}_{k}}+\theta_{k} \bar{F}_{k}(x)\right\|^{2} \\
& \quad-\frac{1}{\theta_{k}} \underline{\lambda}\left(P_{k}\right)\left(\left\|e_{\mathcal{V}_{k}}\right\|^{2}+\theta_{k}^{2}\left\|\bar{F}_{k}(x)\right\|^{2}\right) \tag{22}
\end{align*}
$$

Thus,

$$
\begin{equation*}
2 e_{\mathcal{V}_{k}}^{\top}\left(P_{k}^{\sigma(t)} \otimes I_{n}\right) \bar{F}_{k}(x) \leq \frac{1}{\theta_{k}}\left(\bar{\lambda}\left(P_{k}\right) \delta_{k}-\underline{\lambda}\left(P_{k}\right)\right)\left\|e_{\mathcal{V}_{k}}\right\|^{2} \tag{23}
\end{equation*}
$$

By (21) and (23), it can be thus derived from the foregoing analysis that

$$
\begin{aligned}
\dot{V}(t) \leq & -\sum_{k=1}^{K}\left(c_{k} \lambda_{\min }(\Gamma)+\underline{\varsigma}\right)\left\|e_{\mathcal{V}_{k}}\right\|^{2} \\
& +\sum_{k=1}^{K} \frac{1}{\theta_{k}}\left(\bar{\lambda}\left(P_{k}\right) \delta_{k}-\underline{\lambda}\left(P_{k}\right)\right)\left\|e_{\mathcal{V}_{k}}\right\|^{2} \\
\leq & -\frac{\beta}{\bar{\lambda}(P)} V(t) .
\end{aligned}
$$

Let $\quad \beta_{0}=\frac{\beta}{\bar{\lambda}(P)}, \quad$ obviously $\quad \beta_{0}>0$. Then, when $t \in$ $\left[t_{k}, t_{k+1}\right), k=1,2, \ldots$, we have

$$
\begin{equation*}
V(t) \leq e^{-\beta_{0}\left(t-t_{k}\right)} V\left(t_{k}\right) \tag{24}
\end{equation*}
$$

Let $V\left(t_{k}^{-}\right)=\lim _{t \rightarrow t_{k}} V(t)$. Then

$$
\begin{equation*}
V(t) \leq \mu V\left(t_{k}^{-}\right) \tag{25}
\end{equation*}
$$

Furthermore, we can deduce that

$$
\begin{align*}
& V\left(t_{1}\right) \leq \mu V\left(t_{1}^{-}\right) \leq \mu e^{-\beta_{0}\left(t_{1}-t_{0}\right)} V\left(t_{0}\right) \\
& V\left(t_{2}\right) \leq \mu e^{-\beta_{0}\left(t_{2}-t_{1}\right)} V\left(t_{1}\right) \leq \mu^{2} e^{-\beta_{0}\left(t_{2}-t_{0}\right)} V\left(t_{0}\right) \tag{26}
\end{align*}
$$

By analogy, it is not hard to come out

$$
\begin{equation*}
V\left(t_{k}\right) \leq \mu^{k} e^{-\beta_{0}\left(t_{k}-t_{0}\right)} V\left(t_{0}\right) \tag{27}
\end{equation*}
$$

For $\forall t \geq t_{0}$, combining (24) and (27), one has

$$
V(t) \leq \mu^{N_{\sigma}\left[t_{0}, t\right)} e^{-\beta_{0}\left(t-t_{0}\right)} V\left(t_{0}\right)
$$

According to the Definition 1, we can get $N_{\sigma}\left[t_{0}, t\right) \leq N_{0}+$ $\frac{t-t_{0}}{\tau_{a}}$, so we have

$$
V(t) \leq \mu^{N_{0}} e^{-\left(\beta_{0}-\frac{\ln \mu}{\tau_{a}}\right)\left(t-t_{0}\right)} V\left(t_{0}\right)
$$

In view of condition $\tau_{a}>\frac{\bar{\lambda}(P) \ln \mu}{\beta}$, it means that $V(t) \rightarrow 0$ as $t \rightarrow \infty$. Thus, $e \rightarrow 0$ as $t \rightarrow \infty$ and the nonlinear network (1) with control protocol (2) can achieve CS.

Remark 5: In [39] and [40], the problem of leaderless synchronization under the switching topology was investigated, but

CS was not discussed. In fact, Theorem 2 can be seen as an extension of the result in [40], that is, the special case when the system has only one cluster.

Assumption 5: Assume all subgraphs $\mathcal{G}_{\mathcal{V}_{k}}^{1}, \ldots, \mathcal{G}_{\mathcal{V}_{k}}^{p}, k \in \mathcal{K}$ are strongly connected, and the edge weights of each graph satisfy the balance condition.

When the Assumption 5 is satisfied, we have the following results.

Theorem 3: Suppose that the Assumptions 1-3 and 5 hold. The CS of the nonlinear network (1) with protocol (2) can be achieved if there exist positive constant $\theta_{k}$ and $c_{k}$ satisfying (9) and

$$
\begin{equation*}
c_{k}>\frac{\delta_{k}-1-\theta_{k} \underline{\varsigma}}{\theta_{k} \lambda_{\min }(\Gamma) \varrho_{k}} \tag{28}
\end{equation*}
$$

where $\quad \varsigma=0.5 \min _{i \in \mathcal{P}} \varsigma^{i}, \quad \varsigma^{i}=\zeta^{i}\left(\lambda_{\min }(\Gamma)+\lambda_{\max }(\Gamma)\right)+$ $\operatorname{sgn}(\zeta) \zeta^{i}\left(\bar{\lambda}_{\min }(\Gamma)-\lambda_{\max }(\Gamma)\right), \zeta^{i}=\lambda_{\text {min }}\left(\left(E \bar{L}^{i} F\right)^{\top}+\left(E \bar{L}^{i} F\right)\right)$, $\delta_{k}=\sigma_{k}+1+\eta_{k}\left(2 \theta_{k}+\gamma_{k}\right), \varrho_{k}=\min _{i \in \mathcal{P}} \lambda_{\min }\left(F_{k}^{\top}\left(L_{\mathcal{V}_{k}}^{i}+\right.\right.$ $\left.\left.\left(L_{\mathcal{V}_{k}}^{i}\right)^{\top}\right) F_{k}\right)$.

Proof: Since $\mathcal{G}_{\mathcal{V}_{k}}(t), k \in \mathcal{K}$ are not only strongly connected, but also weight balanced, then $\mathbf{1}_{m_{k}}^{\top} L \mathcal{V}_{k}(t)=0$. Thus, the system (8) can be replaced by

$$
\begin{equation*}
\dot{e}=\bar{F}(x)-\left(F^{\top} C L_{\mathcal{V}}(t) F \otimes \Gamma\right) e-(E \bar{L}(t) F \otimes \Gamma) e \tag{29}
\end{equation*}
$$

Let

$$
V(t)=e^{\top}\left(I_{N-K} \otimes I_{n}\right) e
$$

Differentiating $V(t)$ along the system (29) gives the following expression:

$$
\begin{equation*}
\dot{V}(t) \leq \sum_{k=1}^{K}\left(\frac{\delta_{k}-1}{\theta_{k}}-c_{k} \varrho_{k} \lambda_{\min }(\Gamma)\right)\left\|e_{\mathcal{V}_{k}}\right\|^{2}-\varsigma\|e\|^{2} \tag{30}
\end{equation*}
$$

Thus, $\dot{V}(t)<0$ for any $e \neq 0$ when the condition (28) holds. Thus, $e \rightarrow 0$ when $t \rightarrow \infty$ and the system (1) with control protocol (2) can achieve CS.

Remark 6: In Theorem 2, we require the average dwell time greater than a threshold. This means that the switching speed must not be too fast. In Theorem 3, CS under arbitrary switching topology is solved by a common Lyapunov function. However, the connection topology of each cluster is more conservative.

## IV. LEADER-FOLLOWING CS

In this section, we consider a more general interaction topology that the connection topology between different clusters may not satisfy Assumption 3 and the interaction graph within each cluster may not contain a directed spanning tree. Note that the symbols in the previous section still apply to this section without special declaration. Obviously, the control protocol (2) is no longer feasible. First, we choose $K$ special solutions $s_{k}(t)$ satisfying

$$
\begin{equation*}
\dot{s}_{k}=f_{k}\left(s_{k}\right), \quad k \in \mathcal{K} \tag{31}
\end{equation*}
$$

as virtual leaders of the $k$ th cluster.
Definition 3: (Leader-Following CS) The nonlinear network (1) is said to achieve leader-following CS when
$\lim _{t \rightarrow \infty}\left\|x_{i}(t)-s_{\bar{i}}(t)\right\|=0 \quad \forall i \in \mathcal{V}$, and for any initial states $x_{i}(0)$.

We design the pinning control protocol as follows:

$$
\begin{align*}
u_{i}= & \sum_{j=1}^{N} c_{i j} a_{i j}(t) \Gamma\left(x_{j}-x_{i}\right)+c_{\bar{i}} d_{i}(t) \Gamma\left(s_{\bar{i}}-x_{i}\right) \\
& +\sum_{j=1}^{N} a_{i j}(t) \Gamma\left(s_{\bar{i}}-s_{\bar{j}}\right) \tag{32}
\end{align*}
$$

where $d_{i}(t)>0$ if node $i$ is pinned at time $t$, otherwise, $d_{i}(t)=$ 0 . Then, it follows from (1) and (32) that

$$
\begin{align*}
\dot{x}_{i}= & f_{i}\left(x_{i}\right)+\sum_{j=1}^{N} c_{i j} a_{i j}(t) \Gamma\left(x_{j}-x_{i}\right)+c_{\bar{i}} d_{i}(t) \Gamma\left(s_{\bar{i}}-x_{i}\right) \\
& +\sum_{j=1}^{N} a_{i j}(t) \Gamma\left(s_{\bar{i}}-s_{\bar{j}}\right) \tag{33}
\end{align*}
$$

If node $i$ keeps IDB condition with the cluster $\mathcal{V}_{\bar{j}}(\bar{i} \neq \bar{j})$, i.e, $\sum_{j \in V / \nu_{\bar{i}}} a_{i j}(t)=0$, then the fourth part of (32) equals zero. Let $\widehat{e}_{i}=x_{i}-s_{\bar{i}}$, it follows that

$$
\begin{equation*}
\dot{\widehat{e}}_{i}=f_{\bar{i}}\left(x_{i}\right)-f_{\bar{i}}\left(s_{\bar{i}}\right)-\sum_{j=1}^{N} c_{i j} l_{i j}(t) \Gamma \widehat{e}_{j}-c_{\bar{i}} d_{i}(t) \Gamma \widehat{e}_{i} . \tag{34}
\end{equation*}
$$

Rewrite (34) as a compact form, we have

$$
\begin{equation*}
\dot{\hat{e}}=\widehat{F}(x)-\left(C\left(L_{\mathcal{V}}(t)+D(t)\right) \otimes \Gamma\right) \widehat{e}-(\bar{L}(t) \otimes \Gamma) \widehat{e} \tag{35}
\end{equation*}
$$

where

$$
\begin{aligned}
D(t) & =\operatorname{diag}\left\{D_{1}(t), \ldots, D_{K}(t)\right\} \\
D_{k}(t) & =\operatorname{diag}\left\{d_{z_{k-1}+1}(t), \ldots, d_{z_{k-1}+m_{k}}(t)\right\}
\end{aligned}
$$

$\widehat{F}(x)=\left[\widehat{F}_{1}(x)^{\top}, \ldots, \widehat{F}_{K}(x)^{\top}\right]^{\top}$ with $\widehat{F}_{k}(x)=\left[\left(f_{k}\left(x_{z_{k-1}+1}\right)\right.\right.$ $\left.\left.-f_{k}\left(s_{k}\right)\right)^{\top}, \ldots,\left(f_{k}\left(x_{z_{k-1}+m_{k}}\right)-f_{k}\left(s_{k}\right)\right)^{\top}\right]^{\top}$.

## A. Leader-Following CS Under a Fixed Topology

In this section, we discuss leader-following CS under fixed topology. Without losing generality, let $\breve{\mathcal{G}} \mathcal{V}_{k}$ be an augmented graph formed by graph $\mathcal{G} \mathcal{V}_{k}$ and the virtual leader node $s_{k}$, and regard leader node as the 0th node. In order to obtain the main results, the following assumptions are required.

Assumption 6: The augmented graph $\breve{\mathcal{G}} \mathcal{V}_{k}, k \in \mathcal{K}$ has a directed spanning tree. Moreover, the pinning point is the root of this spanning tree.

Lemma 1: [41, Lemma 4] If the Assumption 6 holds, then there exists a positive diagonal matrix $\Xi_{k}$ such that

$$
\tilde{L}_{\mathcal{V}_{k}}^{\top} \Xi_{k}+\Xi_{k} \tilde{L}_{\mathcal{V}_{k}} \succ 0
$$

where $\tilde{L}_{\mathcal{V}_{k}}=L \mathcal{V}_{k}+D_{k}, \Xi_{k}=\operatorname{diag}\left\{\xi_{z_{k-1}+1}, \ldots, \xi_{z_{k-1}+m_{k}}\right\}$, and $\tilde{L}_{\mathcal{V}_{k}}^{-1} \mathbf{1}_{m_{k}}=\left[1 / \xi_{z_{k-1}+1}, \ldots, 1 / \xi_{z_{k-1}+m_{k}}\right]^{\top}$.

Theorem 4: Suppose that the Assumptions 1-2 and 6 hold. The leader-following CS of nonlinear network (1) with input protocol (32) can be achieved if there exist positive constants $\theta_{k}$
and $c_{k}$ satisfying (9) and

$$
\begin{equation*}
c_{k}>\max \left\{\frac{\bar{\xi}_{k}\left(\delta_{k}-1\right)-\theta_{k} \varsigma_{0}}{\theta_{k} \lambda_{\min }\left(Q_{k}\right) \lambda_{\min }(\Gamma)}, \frac{\underline{\xi}_{k}\left(\delta_{k}-1\right)-\theta_{k} \varsigma_{0}}{\theta_{k} \lambda_{\min }\left(Q_{k}\right) \lambda_{\min }(\Gamma)}\right\} \tag{36}
\end{equation*}
$$

where

$$
\varsigma_{0}=\left(1+\operatorname{sgn}\left(\zeta_{0}\right)\right) \lambda_{\min }(\Gamma) \zeta_{0}+\left(1-\operatorname{sgn}\left(\zeta_{0}\right)\right) \lambda_{\max }(\Gamma) \zeta_{0}
$$

$\zeta_{0}=\lambda_{\min }\left(\bar{L}^{\top} \Xi+\Xi \bar{L}\right), \quad \bar{\xi}_{k}=\max \left\{\xi_{z_{k-1}+1}, \ldots, \xi_{z_{k-1}+m_{k}}\right\}$, $\underline{\xi}_{k}=\min \left\{\xi_{z_{k-1}+1}, \ldots, \xi_{z_{k-1}+m_{k}}\right\}, \quad \delta_{k}=\sigma_{k}+1+\eta_{k}\left(2 \theta_{k}+\right.$ $\left.\gamma_{k}\right), Q_{k}=\tilde{L}_{\mathcal{V}_{k}}^{\top} \Xi_{k}+\Xi_{k} \tilde{L} \mathcal{V}_{k}, \Xi=\operatorname{diag}\left\{\Xi_{1}, \ldots, \Xi_{K}\right\}$.

Proof: Let

$$
V(t)=\widehat{e}^{\top}\left(\Xi \otimes I_{n}\right) \widehat{e}
$$

Differentiating $V(t)$ along the system (35) gives the following expression:

$$
\begin{align*}
\dot{V}(t)= & \sum_{k=1}^{K}-c_{k} \widehat{e}_{\mathcal{V}_{k}}^{\top}\left[\left(\tilde{L}_{\mathcal{V}_{k}}^{\top} \Xi_{k}+\Xi_{k} \tilde{L}_{\mathcal{V}_{k}}\right) \otimes \Gamma\right] \widehat{e}_{\mathcal{V}_{k}} \\
& -\widehat{e}^{\top}\left[\left(\bar{L}^{\top} \Xi+\Xi \bar{L}\right) \otimes \Gamma\right] \widehat{e} \\
& +\sum_{k=1}^{K} 2 \widehat{e}_{\mathcal{V}_{k}}^{\top}\left(\Xi_{k} \otimes I_{n}\right) \widehat{F}_{k}(x) . \tag{37}
\end{align*}
$$

It is easy to obtain

$$
\begin{align*}
& \widehat{e}_{\mathcal{V}_{k}}^{\top}\left[\left(\tilde{L}_{\mathcal{V}_{k}}^{\top} \Xi_{k}+\Xi_{k} \tilde{L}_{\mathcal{V}_{k}}\right) \otimes \Gamma\right] \widehat{e}_{\mathcal{V}_{k}} \\
& \quad \geq \lambda_{\min }\left(Q_{k}\right) \lambda_{\min }(\Gamma)\left\|\widehat{e}_{k}\right\|^{2} \tag{38}
\end{align*}
$$

and

$$
\begin{align*}
\widehat{e}^{\top}\left[\left(\bar{L}^{\top} \Xi+\Xi \bar{L}\right) \otimes \Gamma\right] \widehat{e} & \geq \zeta^{\top}\left(I_{N} \otimes \Gamma\right) \widehat{e} \\
& \geq \varsigma_{0}\|\widehat{e}\|^{2} \tag{39}
\end{align*}
$$

Similar to Theorem 1, using Assumption 1, 2, and (9), we obtain

$$
\begin{align*}
2 \widehat{e}_{\mathcal{V}_{k}}^{\top}\left(\Xi_{k} \otimes I_{n}\right) \widehat{F}_{k}(x) & =\sum_{i \in \mathcal{V}_{k}} \xi_{i} \widehat{e}_{i}^{\top}\left(f_{k}\left(x_{i}\right)-f_{k}\left(s_{k}\right)\right) \\
& \leq \sum_{i \in \mathcal{V}_{k}} \frac{\xi_{i}}{\theta_{k}}\left(\delta_{k}-1\right)\left\|\widehat{e}_{i}\right\|^{2} \tag{40}
\end{align*}
$$

When $\delta_{k}-1 \geq 0$, it thus follows from (38)-(40) that

$$
\begin{aligned}
\dot{V}(t) \leq & -\sum_{k=1}^{K}\left(c_{k} \lambda_{\min }\left(Q_{k}\right) \lambda_{\min }(\Gamma)+\varsigma_{0}\right)\left\|\widehat{e}_{\mathcal{V}_{k}}\right\|^{2} \\
& +\sum_{k=1}^{K} \frac{\bar{\xi}_{k}}{\theta_{k}}\left(\delta_{k}-1\right)\left\|\widehat{e}_{\mathcal{V}_{k}}\right\|^{2} .
\end{aligned}
$$

On the other hand, when $\delta_{k}-1<0$, we have

$$
\begin{aligned}
\dot{V}(t) \leq & -\sum_{k=1}^{K}\left(c_{k} \lambda_{\min }\left(Q_{k}\right) \lambda_{\min }(\Gamma)+\varsigma_{0}\right)\left\|\widehat{e}_{\mathcal{V}_{k}}\right\|^{2} \\
& +\sum_{k=1}^{K} \frac{\underline{\xi}_{k}}{\theta_{k}}\left(\delta_{k}-1\right)\left\|\widehat{e}_{\mathcal{V}_{k}}\right\|^{2} .
\end{aligned}
$$

Using condition (36), one has $\dot{V}(t)<0$, then $\left\|\widehat{e}_{\mathcal{V}_{k}}\right\| \longrightarrow 0$ as $t \longrightarrow+\infty$. This means that the nonlinear network (1) with input protocol (32) achieves leader-following CS.

Remark 7: Note that nodes of different clusters in Theorem 1 may have the same trajectory. In Theorem 4, we can solve this problem by selecting $K$ leaders with different trajectories. However, in practical applications, not all systems can be equipped with leaders.

Remark 8: Although the cluster consensus problem was also solved by pinning control in [22], [24], and [34]. In [22] and [24], however, the interaction topology of the system is required to satisfy Assumption 3 and $\mathcal{R}_{\mathcal{V}_{k_{1}}} \mathcal{V}_{k_{2}}=0$. In [34], the authors do not consider competitive relationships, and the dynamics of each node are the same.

## B. Leader-Following CS Under Switching Topologies

Based on the analysis in the previous section, we consider the case of switching topologies in this section.

Assumption 7: Suppose that each possible augmented graph $\breve{\mathcal{G}}_{\mathcal{V}_{k}}^{\sigma(t)} \in\left\{\breve{\mathcal{G}}_{\mathcal{V}_{k}}^{1}, \ldots, \breve{\mathcal{G}}_{\mathcal{V}_{k}}^{p}\right\}, k \in \mathcal{K}$ contain a directed spanning tree.

By analogy Lemma 1, when the Assumption 7 holds, the following inequality holds:

$$
\left(\tilde{L}_{\mathcal{V}_{k}}^{i}\right)^{\top} \Xi_{k}^{i}+\Xi_{k}^{i} \tilde{L}_{\mathcal{V}_{k}}^{i} \succ 0, i \in \mathcal{P}
$$

where $\tilde{L}_{\mathcal{V}_{k}}^{i}=L_{\mathcal{V}_{k}}^{i}+D_{k}, \quad \Xi_{k}^{i}=\operatorname{diag}\left\{\xi_{z_{k-1}+1}^{i}, \ldots, \xi_{z_{k-1}+m_{k}}^{i}\right\}$ with $\quad \boldsymbol{\xi}_{k}^{i}=\left[1 / \xi_{z_{k-1}+1}^{i}, \ldots, 1 / \xi_{z_{k-1}+m_{k}}^{i}\right]^{\top}=\left(\tilde{L}_{\mathcal{V}_{k}}^{i}\right)^{-1} \mathbf{1}_{m_{k}} \in$ $\mathbb{R}^{m_{k}}$.

Theorem 5: Suppose that the Assumptions 1-2 and 7 hold. The leader-following CS of nonlinear network (1) with input protocol (32) can be achieved if there exist positive constants $\theta_{k}, c_{k}$ and average dwell time $\tau_{a}$ satisfying (9) and

$$
\begin{align*}
& \tau_{a}>\frac{\bar{\xi} \ln \tilde{\mu}}{\tilde{\beta}}  \tag{41}\\
& c_{k}>\max \left\{\frac{\bar{\xi}_{k}^{0}\left(\delta_{k}-1\right)-\theta_{k} \underline{\varsigma}_{0}}{\theta_{k} \lambda_{0}\left(Q_{k}\right) \lambda_{\min }(\Gamma)}, \frac{\xi_{k}^{0}\left(\delta_{k}-1\right)-\theta_{k} \underline{\varsigma}_{0}}{\theta_{k} \lambda_{0}\left(Q_{k}\right) \lambda_{\min }(\Gamma)}\right\} \tag{42}
\end{align*}
$$

where

$$
\begin{aligned}
Q_{k}^{i} & =\left(\tilde{L}_{\mathcal{V}_{k}}^{i}\right)^{\top} \Xi_{k}^{i}+\Xi_{k}^{i} \tilde{L}_{\mathcal{V}_{k}}^{i}, \lambda_{0}\left(Q_{k}\right)=\min _{i \in \mathcal{P}} \lambda_{\min }\left(Q_{k}^{i}\right) \\
\underline{\varsigma}_{0} & =\min _{i \in \mathcal{P}} \varsigma_{0}^{i}, \bar{\xi}=\max _{k \in \mathcal{K}} \bar{\xi}_{k}^{0}, \underline{\xi}=\min _{k \in \mathcal{K}} \bar{\xi}_{k}^{0}, \tilde{\mu}=\bar{\xi} / \underline{\xi} \\
\varsigma_{0}^{i} & =\frac{\zeta_{0}^{i}}{2}\left[\left(1+\operatorname{sgn}\left(\zeta_{0}^{i}\right)\right) \lambda_{\min }(\Gamma)+\left(1-\operatorname{sgn}\left(\zeta_{0}^{i}\right)\right) \lambda_{\max }(\Gamma)\right] \\
\zeta_{0}^{i} & =\lambda_{\min }\left(\left(\bar{L}^{i}\right)^{\top} \Xi^{i}+\Xi^{i} \bar{L}^{i}\right), \delta_{k}=\sigma_{k}+1+\eta_{k}\left(2 \theta_{k}+\gamma_{k}\right) \\
\bar{\xi}_{k}^{0} & =\max _{i \in \mathcal{P}}\left\{\xi_{z_{k-1}+1}^{i}, \ldots, \xi_{z_{k-1}+m_{k}}^{i}\right\}, \tilde{\beta}=\min _{k \in \mathcal{K}} \tilde{\beta}_{k} \\
\underline{\xi}_{k}^{0} & =\min _{i \in \mathcal{P}}\left\{\xi_{z_{k-1}+1}^{i}, \ldots, \xi_{z_{k-1}+m_{k}}^{i}\right\}, \tilde{\beta}_{k}=\min \left\{\Delta_{k}^{1}, \Delta_{k}^{2}\right\} \\
\Delta_{k}^{1} & =c_{k} \lambda_{0}\left(Q_{k}\right) \lambda_{\min }(\Gamma)+\underline{\varsigma}_{0}-\frac{\bar{\xi}_{k}^{0}}{\theta_{k}}\left(\delta_{k}-1\right) \\
\Delta_{k}^{2} & =c_{k} \lambda_{0}\left(Q_{k}\right) \lambda_{\min }(\Gamma)+\underline{\varsigma}_{0}-\frac{\xi_{k}^{0}}{\theta_{k}}\left(\delta_{k}-1\right) .
\end{aligned}
$$

Proof: Let

$$
V(t)=\widehat{e}^{\top}\left(\Xi^{\sigma(t)} \otimes I_{n}\right) \widehat{e}
$$

Differentiating $V(t)$ gives the following expression:

$$
\begin{align*}
\dot{V}(t) \leq & \sum_{k=1}^{K}-c_{k} \lambda_{0}\left(Q_{k}\right) \lambda_{\min }(\Gamma)\left\|\widehat{e}_{k}\right\|^{2}-\varsigma_{0}\|\widehat{e}\|^{2} \\
& +\sum_{k=1}^{K} 2 \widehat{e}_{\mathcal{V}_{k}}^{\top}\left(\Xi_{k}^{\sigma(t)} \otimes I_{n}\right) \widehat{F}_{k}(x) . \tag{43}
\end{align*}
$$

When $\delta_{k}-1 \geq 0$, we have

$$
\begin{equation*}
2 \widehat{e}_{\mathcal{V}_{k}}^{\top}\left(\Xi_{k}^{\sigma(t)} \otimes I_{n}\right) \widehat{F}_{k}(x) \leq \frac{\bar{\xi}_{k}^{0}}{\theta_{k}}\left(\delta_{k}-1\right)\left\|\widehat{e}_{\mathcal{V}_{k}}\right\|^{2} \tag{44}
\end{equation*}
$$

When $\delta_{k}-1<0$, the following inequality holds:

$$
\begin{equation*}
2 \widehat{e}_{\mathcal{V}_{k}}^{\top}\left(\Xi_{k}^{\sigma(t)} \otimes I_{n}\right) \widehat{F}_{k}(x) \leq \frac{\xi_{k}^{0}}{\theta_{k}}\left(\delta_{k}-1\right)\left\|\widehat{e}_{\mathcal{V}_{k}}\right\|^{2} \tag{45}
\end{equation*}
$$

Based on the above analysis, we obtain

$$
\dot{V}(t) \leq-\frac{\tilde{\beta}}{\bar{\xi}} V(t)
$$

Thus, follow the steps in the proof of the Theorem 3 to get the conclusion of this theorem.

Remark 9: Note that there is a common condition in Theorems $1-5$, namely $\gamma_{k}+2 \theta_{k} \geq 0$. If $\gamma_{k} \geq 0$, then $\theta_{k}$ can be chosen as an arbitrarily small positive number. When $\gamma_{k}<0$, $\theta_{k}$ must satisfy $\theta_{k} \geq-0.5 \gamma_{k}$. It is easy to see that the coupling strength $c_{k}$ depends on the choice of $\theta_{k}$. However, the coupling strength $c_{k}$ has no direct monotonic dependence on $\theta_{k}$. Therefore, this is a very interesting research point, we hope that future research can give some meaningful results.

## V. Numerical Examples

Example 1: In this example, consider a nonlinear network of two cluster interactions described by (1), where $\Gamma=I_{2}$ and the nonlinear function is expressed as

$$
\begin{equation*}
f_{i}\left(x_{i}\right)=M_{i}\left[-x_{i 1}\left(x_{i 1}^{2}+x_{i 2}^{2}\right),-x_{i 2}\left(x_{i 1}^{2}+x_{i 2}^{2}\right)\right]^{\top} \tag{46}
\end{equation*}
$$

where $\quad M_{i}=0.005 I_{2}, i=1, \ldots, 4$, and $M_{i}=0.006 I_{2}, i=$ $5, \ldots, 8$. Let the interaction topology of system (1) be timevarying, and the topology switches among $\mathcal{G}^{1}$ and $\mathcal{G}^{2}$ as shown in Fig. 1. In Fig. 1, all nodes can be partitioned into two clusters with $\mathcal{V}_{1}=\{1,2,3,4\}$ and $\mathcal{V}_{2}=\{5,6,7,8\}$. The interaction topology of each cluster contains a directed spanning tree, and the interaction topologies between different clusters satisfy Assumptions 3 and 4.

Hence, $m_{1}=m_{2}=4, E_{1}=E_{2}=\left[\begin{array}{ll}\mathbf{1}_{3} & -I_{3}\end{array}\right], F_{1}=F_{2}=$ $\left[\begin{array}{ll}\mathbf{0}_{3} & -I_{3}\end{array}\right]^{\top}$. One can obtain positive definite matrices

$$
P_{1}^{1}=\left[\begin{array}{ccc}
0.5 & 0 & 0 \\
0 & 0.96 & 0.29 \\
0 & 0.29 & 0.56
\end{array}\right] \quad P_{2}^{1}=\left[\begin{array}{ccc}
1.48 & 0.49 & 0.17 \\
0.49 & 0.66 & 0.25 \\
0.17 & 0.25 & 0.56
\end{array}\right]
$$



Fig. 1. Directed interaction graph in Example 1. All nodes can be partitioned into two clusters with $\mathcal{V}_{1}=\{1,2,3,4\}$ and $\mathcal{V}_{2}=\{5,6,7,8\}$.


Fig. 2. Trajectories of the coupled nonlinear systems (1) with control protocol (2) when $c_{1}=c_{2}=10$ and $\tau_{a}=0.7$.


Fig. 3. Directed interaction graph in Example 2. All nodes can be partitioned into two clusters with $\mathcal{V}_{1}=\{1,2,3,4\}$ and $\mathcal{V}_{2}=\{5,6,7,8\}$. The pinning nodes of each cluster are $s_{1}$ and $s_{2}$, respectively.


Fig. 4. Trajectories of the coupled nonlinear systems (1) with control protocol (32) when $c_{1}=c_{2}=10.1$.

$$
P_{1}^{2}=\left[\begin{array}{ccc}
0.45 & 0 & 0 \\
0 & 0.75 & 0.09 \\
0 & 0.09 & 0.17
\end{array}\right] \quad P_{2}^{2}=\left[\begin{array}{ccc}
1.18 & 0.25 & 0 \\
0.25 & 0.38 & 0 \\
0 & 0 & 0.5
\end{array}\right]
$$

satisfying $\quad\left(E_{k} L_{\nu_{k}}^{i} F_{k}\right)^{\top} P_{k}^{i}+P_{k}^{i}\left(E_{k} L_{\nu_{k}}^{i} F_{k}\right)=I_{3}, \quad i=1,2$, $k=1,2$. Hence, $\hat{\lambda}(P)=1.77, \underline{\lambda}(P)=0.15$, and $\mu=11.45$. By inequality (18), we have $\theta_{k} \leq 1.05$. We choose $\theta_{k}=1, k=$ 1,2 . By simple calculation, one can get $\eta_{k}=0, k=1,2$. Moreover, one can get $\sigma_{1}=-1, \gamma_{1}=-0.57$, and $\sigma_{2}=-1.2, \gamma_{2}=$ -0.68 [42]. Hence, $\delta_{1}=0$ and $\delta_{2}=-0.2$, and $\varsigma=-2.71$. One has $c_{1}>2.6$ and $c_{2}>2.2$. If we choose $c_{1}=c_{2}=10$, then $\beta=7.09$ and $\tau_{a}>0.61$.
We apply control protocol (2) to system (1) and choose $c_{1}=$ $c_{2}=10$ and $\tau_{a}=0.7$ to satisfy Theorem 2. Fig. 2 shows the trajectories of all nodes when $c_{1}=c_{2}=10$ and $\tau_{a}=0.7$.
Note that the two digraphs in this example do not satisfy the IDB condition in [20], [22], [24], [26], [27], [34]. Therefore, the results in [20], [22], [24], [26], [27], and [34], cannot deal with this example. Moreover, since each subdigraph has a directed spanning tree, one cannot use the results obtained in [28] and [29].
Example 2: This example will design some leaders to make the system (1) with input protocol (32) achieve CS. Except for the different topology, other assumptions are the same as in Example 1. The possible interaction topologies are shown in Fig. 3 . All nodes are divided into two clusters, $\mathcal{V}_{1}=\{1,2,3,4\}$ and
$\mathcal{V}_{2}=\{5,6,7,8\}$. The pinning nodes of each cluster are $s_{1}$ and $s_{2}$, respectively. It is clear that $\mathcal{G}^{3}$ and $\mathcal{G}^{4}$ satisfy Assumption 7 .

One can get from the Example 1 that $\eta_{k}=0, k=1,2, \sigma_{1}=$ $-1, \gamma_{1}=-0.57, \sigma_{2}=-1.2$, and $\gamma_{2}=-0.68$ [42]. Hence, $\delta_{1}=0$ and $\delta_{2}=-0.2$. We can choose $\theta_{k}=0.4, k=1,2$ satisfying the inequality (9). We can get

$$
\begin{aligned}
& \Xi_{1}^{1}=\operatorname{diag}\{1,0.52,0.44,0.5\} \\
& \Xi_{2}^{1}=\operatorname{diag}\{1,0.25,0.5,0.35\} \\
& \Xi_{1}^{2}=\operatorname{diag}\{1,0.47,0.26,0.35\} \\
& \Xi_{2}^{2}=\operatorname{diag}\{1,0.34,0.5,0.34\}
\end{aligned}
$$

$\bar{\xi}_{1}^{0}=\bar{\xi}_{2}^{0}=1, \underline{\xi}_{1}^{0}=0.26, \underline{\xi}_{2}^{0}=0.25$. Hence, $\bar{\xi}=1, \underline{\xi}=0.25$, and $\tilde{\mu}=4$. By simple calculation, one can obtain $\lambda_{0}\left(Q_{1}\right)=$ $0.25, \lambda_{0}\left(Q_{2}\right)=0.27, \quad \varsigma_{0}^{1}=\zeta_{0}^{1}=-3.13, \varsigma_{0}^{2}=\zeta_{0}^{2}=-2.31$. One has $c_{1}>10$ and $c_{2}>6.7$. If we choose $c_{1}=c_{2}=10.1$, then $\tilde{\beta}=1.87$ and $\tau_{a}>0.74$. Suppose that the interaction topology switches between $\mathcal{G}^{3}$ and $\mathcal{G}^{4}$ every 0.8 s . The state trajectories of system (1) with control protocol (32) is shown in Fig. 4 , which indicates that leader-following CS is realized. Note that since all digraphs do not satisfy the IDB condition and subdigraphs just satisfy spanning tree condition, the pinning control strategies proposed in [22], [24], and [34], cannot be used to deal with this example.

## VI. CONCLUSION

In the grouping case, this article investigated the leaderless and leader-following CS for nonlinear networks with fixed and switching directed topologies, respectively. It is assumed that nodes belonging to the same cluster can only have a cooperative relationship, and the inherently nonlinear system of each node satisfies the OSL condition. In the case of leaderless, we used a variable transformation to make the CS problem become as a stability problem. For the case of leader-following, we design a class of leader-following protocols, which can be used to achieve CS in nonlinear networks. Future research will consider CS when competing relationships exist in each cluster.

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