Game-Based Consensus of Hybrid Multiagent Systems

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***Abstract*—This article considers consensus of first-order/second-order hybrid multiagent systems (MASs) based on game modeling. In the first-order hybrid MAS (HMAS), a subset of agents select the Nash equilibrium of a multiplayer game as their states at each game time and the others update their states with first-order continuous-time (C-T) dynamics. By graph theory and matrix theory, we establish sufficient and necessary conditions for consensus of the first-order HMAS with two proposed protocols. The second-order HMAS is composed of agents whose states are determined by the Nash equilibrium of a multiplayer game and agents whose states are governed by second-order C-T dynamics. Similarly, sufficient and necessary conditions are given for consensus of the second-order HMAS with two proposed protocols. Several numerical simulations are provided to verify the effectiveness of our theoretical results.**

***Index Terms*—Consensus, hybrid multiagent systems (MASs), multiplayer games, Nash equilibrium.**

I. Introduction

Coordination control of multiagent systems (MASs), including consensus [1], [2], [3], [4], [5], formation control [6], [7], containment control [8], [9], coverage control [10], and etcetera, has been widely concerned. Group consensus is an important problem in coordination control and has been attracting a lot of attention. Existing works suggest that, by using local information, a group of agents may reach consensus. In retrospect, consensus of homogeneous MASs has received a lot of attention, regardless of whether the dynamics of the agents are discrete-time (D-T) (e.g., [11] and [12]), or continuous-time (C-T) (e.g., [13], [14], and [15]).

However, the dynamics of agents in MASs are not always the same in practical applications. For example, pilot-controlled aircrafts cooperate with computer-controlled unmanned aircrafts in combats and human operators cooperate with robots in human–robot systems [16], [17]. Hence, much interest is focused on the consensus of hybrid MASs (HMASs). Hybrid generally refers to heterogeneity of nature or composition. Zheng et al. [18] first proposed the concept of HMASs, which consist of first-order C-T and D-T dynamic agents. For HMASs considered, they provided three kinds of consensus protocols. Moreover, sufficient and necessary conditions were obtained for solving consensus problems in light of matrix theory and graph theory. As an extension of [18], consensus of HMASs was studied in [19], where the agents with C-T dynamics can exploit their own states in real time. Zheng et al. [20] considered the consensus of HMASs composed of second-order C-T and D-T dynamic agents. For HMASs with heterogeneous dynamics, Zhao et al. [21] presented two effective protocols and established sufficient and necessary conditions for solving consensus problems. For networked games, the Nash equilibrium computation problems were solved in [22] for the HMASs which contain C-T and D-T dynamic agents. Consensus of HMASs which contain C-T and D-T dynamic agents was investigated in [23] by using a game-theoretic approach. Ye et al. [24] proved that HMASs with first- and second-order players converge exponentially to the Nash equilibrium by using game theory. It should be pointed out that game theory is not only an analysis tool, but also a powerful system modeling tool. Stackelberg games, aggregative games, multiplayer games, and bimatrix games, to name a few, have been extensively used to model the systems (e.g., [25], [26], [27], [28], [29], [30], and [31]).

Game theory can be used to model the strategy interactions of agents. In a game, each agent iteratively decides its action by minimizing its cost, which depends on the actions of itself and its neighbors. For example, each person's opinion in social networks is effected by many factors. Persons often make decisions independently and pursue cost minimization. Then, an opinion spread model is able to be established by game theory. The application of game theory to MASs modeling is abundant. Specifically, Ma et al. [26] leveraged a standard two-player zero-sum game to model the MASs. Etesami [27] provided a Stackelberg game to model the MASs and demonstrated convergence of optimal response dynamics.

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Ghaderi and Srikant [28] studied how opinions are formed in social networks. They modeled the spread of opinions in social networks as a network interaction game and proved that opinions can reach an agreement under optimal response dynamics. Parise et al. [29] employed network aggregative games to model multiagent populations and showed the convergence of optimal response dynamics. Kebriaei et al. [30] extended aggregative games to multipopulation games and modeled MASs. Ma et al. [31] studied MASs with two classes of agents. They employed a repeated bimatrix game to model the MASs, in which different agents form a union, and proved that agents belonging to the same unions can achieve consensus. Zhou et al. [32] employed multiplayer games to model the MASs and provided criteria for realizing coordination control.

Inspired by all the above analysis, this article focuses on studying the consensus problem for HMASs. Different from [18], [19], [20], and [21] that considered the dynamics of all agents in HMASs to be preset, the dynamics of some agents in the HMASs considered in this article are unknown. Therefore, we consider HMASs which are composed of agents with C-T dynamics and agents whose states are governed by a multiplayer game. In some of the existing literature, for example, Vamvoudakis et al. [33], Lopez et al. [34], and Niazi and Özgüler [35] considered the scenario that players prefer fewer control input and to be closer to their neighbors. The player's cost function they considered depends on not only the states of itself and its neighbors but also on control inputs. Different from them, we consider the scenario that all players are rational and selfish, and they prefer fewer states changes and to be closer to their neighbors. Therefore, in the multiplayer game models designed, each agent independently updates its state according to its own state at the last time and its neighbors' states at the current time, which is different from [23]. Regardless of whether the agents with C-T dynamics in HMASs are first- or second-order, we prove that a unique Nash equilibrium exists under rationality assumptions. Then, we establish sufficient and necessary conditions for the consensus of HMASs developed in this article, respectively. Difficulties come down to how to model and analyze HMASs considered in this article. The main contributions of this work are threefold.

1) A novel class of HMASs is developed based on game modeling, which consists of agents whose states are determined by a multiplayer game and agents whose states are governed by first-order/second-order C-T dynamics.
2) For first-order HMAS, we propose two protocols for agents whose states are governed by first-order C-T dynamics. In light of game theory and matrix theory, sufficient and necessary conditions are established separately to ensure consensus of the first-order HMAS with two proposed protocols.
3) For second-order HMAS, two protocols are provided for agents whose states are governed by second-order C-T dynamics. By utilizing game theory and matrix theory, sufficient and necessary conditions are separately established for consensus of the second-order HMAS with two protocols considered in this article.

The remainder of this article is structured as follows. Some notions in graph theory and lammas are listed in Section II. Our problem is formulated in Section III. The consensus problem for first- and second-order HMASs are solved in Sections IV and V, respectively. Section VI provides several numerical examples to illustrate the validity of our theoretical results. Conclusion is given in Section VII.

II. Preliminaries

A. Graph Theory

A weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ consists of a vertex set $\mathcal{V} = \{1, 2, \dots, n\}$, an edge set $\mathcal{E}$ and a weighted adjacency matrix $\mathcal{W} = (w_{ij}) \in \mathbb{R}^{n\times n}$. If $(j, i) \in \mathcal{E}$ for $i \neq j$, then $w_{ij} > 0$; otherwise $w_{ij} = 0$. Beside, $w_{ii} = 0$ for $i \in \mathcal{V}$. The weight of paths of length K from j to i is denoted as $(\mathcal{W}^K)_{ij}$, which is the (i, j) entry in the matrix $\mathcal{W}^K$. Let $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$ denote the neighbor set of vertex i. A finite-ordered sequence of distinct edges $(i, k_1), (k_1, k_2), \dots, (k_l, j)$ is called a directed path between two distinct vertices i and j of $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$. If $(i, j) \in \mathcal{E}$, we call that i is the parent of j. A directed graph is called a directed tree if every node in the graph has only one parent except the root node. A spanning tree is a directed tree composed of all vertices and some edges in the directed graph. The degree matrix $\mathcal{D} = (d_{ij}) \in \mathbb{R}^{n\times n}$ is a diagonal matrix with $d_{ii} = \sum_{j\in\mathcal{N}_i} w_{ij}$. Throughout this article, let $\mathcal{I}_n = \{1, 2, \dots, n\}$ is the set of the agents labeled 1 through n. Similarly, $\mathcal{I}_m = \{1, 2, \dots, m\}$ and $\mathcal{I}_n \backslash \mathcal{I}_m = \{m + 1, m + 2, \dots, n\}$.

If every entry of the matrix A is non-negative, A is said to be non-negative. A non-negative matrix with each row summing to 1 is said to be a (row) stochastic matrix. If a stochastic matrix $P = [p_{ij}]_{n\times n}$ satisfies that $\lim_{k\to\infty} P^k = \mathbf{1}_n y^T$, where y is some column vector. Then, it is called indecomposable and aperiodic (SIA). Given a graph $\mathcal{G}_P$ with n vertices, graph $\mathcal{G}_P$ is called the graph associated with P if $(j, i) \in \mathcal{E} \Leftrightarrow P_{ij} > 0$.

B. Game Theory

Next, we introduce some basic notions of multiplayer games (please refer to [36] for more details).

A multiplayer game consists of $n(> 2)$ players, a set of strategies available to those players and an individual cost function for each player. Each player in the multiplayer game selects a strategy that minimizes its own cost function. They make decisions independently and simultaneously. The precise description of a multiplayer game is provided below together with the notations to be used. Suppose that n players $v_1, v_2, \dots, v_n$ play a multiplayer game $\mathbb{G}$. Denote the set of all players as $\mathcal{V}$. Player v_i has strategies in $\Omega_i \subset \mathbb{R}$. Then, the strategy space of the multiplayer game $\mathbb{G}$ is $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$. If each $v_i \in \mathcal{V}$ corresponds to a strategy $x_i \in \Omega_i$, then $(x_1, x_2, \dots, x_n)$ is an n-tuple of strategies. The cost of v_i is $J_i(x_1, x_2, \dots, x_n)$. Players make their strategies to seek the minimum possible cost independently by considering the possible choices of the other rational players.

The Nash equilibrium of the multiplayer game $\mathbb{G}$ with m players is defined as follows.

Definition 1: For any given $\mathcal{X}_c^*$, $(x_1^\star, x_2^\star, \ldots, x_m^\star)$ is said to be the Nash equilibrium of the m-player game $\mathbb{G}$ if the inequalities

$$\begin{cases} J_1\big(x_1^\star, x_2^\star, \ldots, x_m^\star, \mathcal{X}_c^*\big) \le J_1\big(x_1, x_2^\star, \ldots, x_m^\star, \mathcal{X}_c^*\big) \\ J_2\big(x_1^\star, x_2^\star, \ldots, x_m^\star, \mathcal{X}_c^*\big) \le J_2\big(x_1^\star, x_2, \ldots, x_m^\star, \mathcal{X}_c^*\big) \\ \qquad\qquad\vdots \\ J_m\big(x_1^\star, x_2^\star, \ldots, x_m^\star, \mathcal{X}_c^*\big) \le J_m\big(x_1^\star, x_2^\star, \ldots, x_m, \mathcal{X}_c^*\big) \end{cases}$$

hold for all $x_i \in \Omega_i$, $i \in \mathcal{I}_m$.

C. Some Lemmas

Lemma 1 [37]: $\mathcal{B} = (b_{ij}) \in R^{n\times n}$ is said to be strictly diagonally dominant (SDD) if $|b_{ii}| > \sum_{j=1, j\neq i}^{n} |b_{ij}|, i \in \mathcal{I}_n$, and the SDD matrix is invertible, that is, $\det \mathcal{B} \neq 0$.

Lemma 2 (Gersghorin Circle Theorem) [12]: Let $\mathcal{C} = (c_{ij}) \in \mathbb{R}^{n\times n}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be all eigenvalues of $\mathcal{C}$. Then, $\lambda_i \in \bigcup_{i=1}^{n} G_i, i \in \mathcal{I}_n$, where $G_i = \{z||z - c_{ii}| \le \sum_{j=1, j\neq i}^{N} |c_{ij}|\}$.

Lemma 3 [18]: Let $A = [a_{ij}]_{n\times n}$ be a stochastic matrix with positive diagonal elements. If $\mathcal{G}_A$ has a spanning tree, then A is SIA, that is, $\exists\ v$, s.t. $\lim_{m\to\infty} A^m = \mathbf{1}_n v^T$. Here, v is a non-negative vector and $A^T v = v$, $\mathbf{1}_n^T v = 1$.

Lemma 4 [21]: For $x \in (-\infty, 0)$, we have that $f(x) = e^{hx}$ and $g(x) = (e^{hx} - 1)/x$ are increasing functions, where $h > 0$ is a constant.

III. Problem Statement

We consider the consensus of an HMAS with n agents whose states are denoted by $x_i(t) \in \mathbb{R}, i \in \mathcal{I}_n$. The initial condition of agent $i \in \mathcal{I}_n$ is $x_i(0)$. The HMAS is composed of m $(m < n)$ agents in $\mathcal{I}_m$ and $n - m$ C-T dynamic agents in $\mathcal{I}_n \backslash \mathcal{I}_m$. The states of agents in $\mathcal{I}_m$ are governed by a multiplayer game $\mathbb{G}$ that will be given later. Moreover, the dynamics of the agents in $\mathcal{I}_n \backslash \mathcal{I}_m$ are governed by

$$x_i^{(s)}(t) = u_i(t) \tag{1}$$

where $x_i^{(s)}(t)$ denotes the sth-order time derivative of $x_i(t)$ and $u_i(t) \in \mathbb{R}$ is the control input of agent $i \in \mathcal{I}_n \backslash \mathcal{I}_m$. In the following, $s = 1$ and $s = 2$ will be investigated, respectively.

Remark 1: There are many examples that fall into the HMASs under consideration. A representative example is the Internet of Vehicles [38], in which the vehicles are in C-T dynamics while the drivers determine their states by a multiplayer game. One more example is the air traffic control systems [39], in which the computer-controlled unmanned aircrafts are governed by C-T dynamics while the pilot-controlled aircrafts update their states by a multiplayer game.

The definition of game $\mathbb{G}$ is given as follows.

1) *Players:* The set of all players is $\mathcal{I}_m$.
2) *Cost:* The cost function of agent $i \in \mathcal{I}_m$ is defined as follows:

$$J_i(x_1(t_{k+1}), \ldots, x_n(t_{k+1})) = \alpha_i (x_i(t_{k+1}) - x_i(t_k))^2 + \beta_i \left(x_i(t_{k+1}) - \sum_{j\in\mathcal{N}_i} w_{ij} x_j(t_{k+1}) \right)^2 \tag{2}$$

where $\sum_{j\in\mathcal{N}_i} w_{ij} = 1$, $\alpha_i + \beta_i = 1$, $0 < \alpha_i < 1$, $t_k = kh$ $(k \in \mathbb{N})$ is the game time and $h > 0$ is the game period. For simplicity, we use $J_i(t_{k+1})$ to denote the cost function of agent i. Note that the cost function of agent $i \in \mathcal{I}_m$ depends not only on the states of its neighbors in $\mathcal{I}_m$, but also on the states of its neighbors in $\mathcal{I}_n \backslash \mathcal{I}_m$. For notational convenience, let $x_i^*(\cdot) = x_i(\cdot)$ for the agent $i \in \mathcal{I}_n \backslash \mathcal{I}_m$ at every game time, and $\mathcal{X}_c^*(\cdot) = \mathcal{X}_c(\cdot) = \{x_{m+1}(\cdot), x_{m+2}(\cdot), \ldots, x_n(\cdot)\}$ be the set of states of agents in $\mathcal{I}_n \backslash \mathcal{I}_m$. Thus, we can represent the cost function of agent $i \in \mathcal{I}_m$ as $J_i(x_1(t_{k+1}), x_2(t_{k+1}), \ldots, x_m(t_{k+1}), \mathcal{X}_c^*(t_{k+1}))$.
3) *Strategy:* Player $i \in \mathcal{I}_m$ chooses its strategy in $\Omega_i \subset \mathbb{R}$. Suppose that it selects the strategy $x_i^*(t_{k+1}) \in \Omega_i$ as its state at time t_{k+1}, where

$$x_i^*(t_{k+1}) = \arg \min_{x_i(t_{k+1})} J_i\big(x_1^*(t_{k+1}), \ldots, x_{i-1}^*(t_{k+1}) \; x_i(t_{k+1}), x_{i+1}^*(t_{k+1}), \ldots, x_m^*(t_{k+1}), \mathcal{X}_c^*(t_{k+1})\big) \tag{3}$$

for $i \in \mathcal{I}_m$.

Remark 2: x_i^* is the state selected by agent i at game time. Formula (3) introduces the way for agent $i \in \mathcal{I}_m$ to determine x_i^*, which is different from that for agent $i \in \mathcal{I}_n \backslash \mathcal{I}_m$.

Assumption 1: The agent $i \in \mathcal{I}_m$ neither knows the specific form of the cost functions of other agents, nor the parameters α_j and β_j of other agents $j \in \mathcal{I}_m$.

Assumption 2: We assume that every state in the strategy set is reachable for all players. Moreover, if player i selects state $x_i^*(t_{k+1})$ at time t_{k+1}, it will reach the state $x_i^*(t_{k+1})$ before time t_{k+2}.

Then, we show that the game $\mathbb{G}$ is well defined.

For the agent $i \in \mathcal{I}_m$, if $x_j^*(t_{k+1}), j \in \mathcal{I}_n, j \neq i$ are fixed, it is obviously that

$$J_i\big(x_1^*(t_{k+1}), \ldots, x_i(t_{k+1}), \ldots, x_m^*(t_{k+1}), \mathcal{X}_c^*(t_{k+1})\big)$$

is a quadratic function of $x_i(t_{k+1}), i \in \mathcal{I}_m$. Therefore

$$J_i\big(x_1^*(t_{k+1}), \ldots, x_i(t_{k+1}), \ldots, x_m^*(t_{k+1}), \mathcal{X}_c^*(t_{k+1})\big)$$

has only one global minimum $x_i^*(t_{k+1}), i \in \mathcal{I}_m$, which satisfies

$$\begin{cases} \frac{\partial J_i(t_{k+1})}{\partial x_i(t_{k+1})}\Big|_{(x_1^*(t_{k+1}), \ldots, x_i^*(t_{k+1}), \ldots, x_m^*(t_{k+1}), \quad \mathcal{X}_c^*(t_{k+1}))} = 0 \\ \frac{\partial^2 J_i(t_{k+1})}{\partial^2 x_i(t_{k+1})}\Big|_{(x_1^*(t_{k+1}), \ldots, x_i^*(t_{k+1}), \ldots, x_m^*(t_{k+1}), \quad \mathcal{X}_c^*(t_{k+1}))} > 0. \end{cases} \tag{4}$$

That is to say, for $i \in \mathcal{I}_m$

$$\arg \min_{x_i(t_{k+1})} J_i\big(x_1(t_{k+1}), x_2(t_{k+1}), \ldots, x_m(t_{k+1}), \mathcal{X}_c^*(t_{k+1})\big)$$

has a unique solution. This implies that game $\mathbb{G}$ is well defined. Combining with (2)–(4), we know that the states $x_i^*(t_{k+1}), i \in \mathcal{I}_m$ satisfy the following formula:

$$x_i^*(t_{k+1}) - \beta_i \sum_{j\in\mathcal{N}_i} w_{ij} x_j^*(t_{k+1}) = \alpha_i x_i(t_k). \tag{5}$$

It is assumed that all agents in $\mathcal{I}_m$ are rational, that is, the following assumption holds.

Assumption 3: All agents in $\mathcal{I}_m$ are rational, that is, they consistently make decisions to minimize their costs and update their states independently. Besides, each agent $i \in \mathcal{I}_m$ believes that all of its neighbors are rational. In other words, each agent in $\mathcal{I}_m$ knows that the strategies of its neighbors are to make decisions to minimize their costs.

Remark 3: Note that α_i is the weight of the cost of agent $i \in \mathcal{I}_m$ to change its own state. The larger α_i is, the more opinionated agent $i \in \mathcal{I}_m$ is. If $\alpha_i = 1$, then the agent $i \in \mathcal{I}_m$ will insist its state without changing, that is, $x_i(t_{k+1}) = x_i(t_k) = x_i(0)$. Such agent is called as a leader. Consider an HMAS* which is composed of agents in $\mathcal{I}_m$ whose states are determined by the game $\mathbb{G}$ with $\alpha_i = 1$ and C-T dynamic agents in $\mathcal{I}_n \backslash \mathcal{I}_m$. If $m = 1$, consensus problem for HMAS* becomes a leader–follower control problem for MAS. And if $m > 1$, the consensus problem for HMAS* becomes a containment control problem for MAS.

It is easy to check that at each game time, a game $\mathbb{G}$ has a unique Nash equilibrium for any given $\mathcal{X}_c^*$. Under Assumption 3, the state of agent i in $\mathcal{I}_m$ is the ith component of the Nash equilibrium at each game time t_{k+1}. Without causing confusion, we directly use $x_i(\cdot)$ to denotes the state of agent $i \in \mathcal{I}_m$ corresponding to the Nash equilibrium.

IV. First-Order HMASs

In this section, we investigate the consensus of an HMAS, in which the states of agents in $\mathcal{I}_m$ are governed by a multiplayer game $\mathbb{G}$ defined above and the dynamics of agent $i \in \mathcal{I}_n \backslash \mathcal{I}_m$ are governed by

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{I}_n \backslash \mathcal{I}_m \tag{6}$$

where the control protocol $u_i(t)$ will be given later.

Definition 2: The first-order HMAS is said to reach consensus if for any initial states

$$\lim_{t_k \to \infty} \|x_i(t_k) - x_j(t_k)\| = 0, \quad i, j \in \mathcal{I}_n \tag{7}$$

and

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\ \| = 0, \quad i, j \in \mathcal{I}_n \backslash \mathcal{I}_m. \tag{8}$$

In the following, we will consider two scenarios for agent $i \in \mathcal{I}_n \backslash \mathcal{I}_m$.

1) Agent $i \in \mathcal{I}_n \backslash \mathcal{I}_m$ only exploits its own state at the game time.
2) Agent $i \in \mathcal{I}_n \backslash \mathcal{I}_m$ exploits its own state in real time.

A. Case of C-T Dynamic Agents Only Using Their Own Game Time States

Consider the following distributed protocol for agent $i \in \mathcal{I}_n \backslash \mathcal{I}_m$:

$$u_i(t) = \sum_{j \in \mathcal{N}_i} w_{ij} \big(x_j(t_k) - x_i(t_k)\big), \quad t \in (t_k, t_{k+1}]. \tag{9}$$

Devise an HMAS in which the states of agents in $\mathcal{I}_m$ are governed by game $\mathbb{G}$ defined above and the states of agents in $\mathcal{I}_n \backslash \mathcal{I}_m$ are governed by first-order C-T dynamics (6) with protocol (9). For convenience, we use HMAS_1 to denote such an HMAS. Besides, let $x(\cdot) = [x_1(\cdot), x_2(\cdot), \ldots, x_n(\cdot)]^T$.

The following theorem establishes the relation between the states of two adjacent game times for HMAS_1.

Theorem 1: For HMAS_1, one has $x(t_{k+1}) = B_1^{-1} A_1 x(t_k)$, where

$$\begin{aligned} A_1 &= \text{diag}\Big\{\alpha_1, \alpha_2, \ldots, \alpha_m, \underbrace{1-h, 1-h, \ldots, 1-h}_{n-m}\Big\} \\ &\quad + h\text{diag}\Big\{\underbrace{0, 0, \ldots, 0}_{m}, \underbrace{1, 1, \ldots, 1}_{n-m}\Big\}\mathcal{W} \\ B_1 &= I_n - \text{diag}\Big\{\beta_1, \beta_2, \ldots, \beta_m, \underbrace{0, 0, \ldots, 0}_{n-m}\Big\}\mathcal{W}. \end{aligned} \tag{10}$$

Proof: When $t = t_{k+1}$, it follows from (6) and (9) that:

$$x_i(t_{k+1}) = (1-h)x_i(t_k) + h \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t_k) \tag{11}$$

for $i \in \mathcal{I}_n \backslash \mathcal{I}_m$.

Since $(x_1(t_{k+1}), x_2(t_{k+1}), \ldots, x_m(t_{k+1}))$ is the unique Nash equilibrium of game $\mathbb{G}$, it follows that:

$$\begin{cases} x_i(t_{k+1}) - \beta_i \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t_{k+1}) = \alpha_i x_i(t_k), & i \in \mathcal{I}_m \\ x_i(t_{k+1}) = (1-h)x_i(t_k) + h \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t_k), & i \in \mathcal{I}_n \backslash \mathcal{I}_m. \end{cases} \tag{12}$$

By the definitions of A_1 and B_1, the compact form of (12) is $B_1 x(t_{k+1}) = A_1 x(t_k)$.

Let b_{ij} denote the (i, j)th entry of matrix B_1. Then, it is easy to verify that

$$\begin{cases} \sum_{j=1, j\neq i}^{n} |b_{ij}| = \sum_{j=1, j\neq i}^{n} |-\beta_i w_{ij}| = \beta_i < 1 = |b_{ii}|, i \in \mathcal{I}_m \\ \sum_{j=1, j\neq i}^{n} |b_{ij}| = 0 < 1 = |b_{ii}|, i \in \mathcal{I}_n \backslash \mathcal{I}_m. \end{cases}$$

This implies that B_1 is SDD. By Lemma 1, one obtains that B_1 is invertible.

Therefore, $x(t_{k+1}) = B_1^{-1} A_1 x(t_k)$.

The following theorem gives a sufficient and necessary condition to ensure consensus of HMAS_1.

Theorem 2: Assume that $\mathcal{G}$ is directed and $h < 1$. HMAS_1 reaches consensus if and only if $\mathcal{G}$ has a directed spanning tree.

Proof (Sufficiency): First, we prove that $B_1^{-1} A_1$ is a stochastic matrix. It is obviously that

$$\begin{aligned} B_1 \mathbf{1}_n &= \left(I_n - \text{diag}\Big\{\beta_1, \beta_2, \ldots, \beta_m, \underbrace{0, 0, \ldots, 0}_{n-m}\Big\}\mathcal{W}\right)\mathbf{1}_n \\ &= \Bigg[1 - \beta_1 \sum_{j \in \mathcal{N}_1} a_{1j}, \ldots, 1 - \beta_m \sum_{j \in \mathcal{N}_m} a_{mj}, \underbrace{1, 1, \ldots, 1}_{n-m}\Bigg]^T \\ &= \Big[1 - \beta_1, 1 - \beta_2, \ldots, 1 - \beta_m, \underbrace{1, 1, \ldots, 1}_{n-m}\Big]^T \\ &= \Big[\alpha_1, \alpha_2, \ldots, \alpha_m, \underbrace{1, 1, \ldots, 1}_{n-m}\Big]^T \end{aligned}$$

$$= \left(\mathrm{diag}\left\{\alpha_1, \alpha_2, \ldots, \alpha_m, \underbrace{1-h, 1-h, \ldots, 1-h}_{n-m}\right\}\right.$$
$$\left. + h\mathrm{diag}\left\{\underbrace{0, 0, \ldots, 0}_{m}, \underbrace{1, 1, \ldots, 1}_{n-m}\right\}\mathcal{W}\right)\mathbf{1}_n = A_1\mathbf{1}_n.$$

Note that B_1 is invertible. Then, one has

$$B_1^{-1}A_1\mathbf{1}_n = \mathbf{1}_n. \tag{13}$$

Let $Q_1 = \mathrm{diag}\{\beta_1, \beta_2, \ldots, \beta_m, \underbrace{0, 0, \ldots, 0}_{n-m}\}$. It is easy to verify that $B_1 = I_n - Q_1\mathcal{W}$. It is assumed that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all eigenvalues of $Q_1\mathcal{W} = (c_{ij}) \in \mathbb{R}^{n\times n}$. Then, one has $\lambda_i \in \bigcup_{i=1}^n G_i, i \in \mathcal{I}_n$ by utilizing Lemma 2, where

$$G_i = \left\{z \middle| |z - c_{ii}| \le \sum_{j=1, j\neq i}^{n} |c_{ij}|\right\}$$
$$= \begin{cases} \left\{z \middle| |z-0| \le \sum_{j=1,j\neq i}^n |\beta_i w_{ij}| = |\beta_i| < 1\right\}, & i \in \mathcal{I}_m \\ \{z \mid |z-0| \le 0 < 1\}, & i \in \mathcal{I}_n \backslash \mathcal{I}_m. \end{cases}$$

This means that $|\lambda_i| < 1$ for $i \in \mathcal{I}_n$. Therefore

$$B_1^{-1} = (I_n - Q_1\mathcal{W})^{-1} = \sum_{n=0}^{\infty}(Q_1\mathcal{W})^n.$$

As matrix $Q_1\mathcal{W}$ is non-negative, B_1^{-1} is non-negative with its diagonal elements being positive. Since $h < 1$, A_1 is non-negative with its diagonal elements being positive. Combining with (19) yields that $B_1^{-1}A_1$ is a (row) stochastic matrix whose diagonal elements are positive.

Next, we prove that graph $\mathcal{G}_{B_1^{-1}A_1}$ has a directed spanning tree. Denote the directed spanning tree of graph $\mathcal{G}$ as $T_{\mathcal{G}}$. For each edge $(j, i) \in T_{\mathcal{G}}$.

1) If $i \in \mathcal{I}_m$, one has $(Q_1\mathcal{W})_{ij} > 0, j \in \mathcal{I}_n$ from $\beta_i > 0$ and $w_{ij} > 0, j \in \mathcal{I}_n$. Then, we can deduce that $(B_1^{-1})_{ij} \neq 0, j \in \mathcal{I}_n$. Therefore, the following formulas hold:

$$\left(B_1^{-1}A_1\right)_{ij} = \sum_{k=1}^{n}\left(B_1^{-1}\right)_{ik}(A_1)_{kj}$$
$$= \left(B_1^{-1}\right)_{ij}(A_1)_{jj} + \sum_{k=1,k\neq j}^{n}\left(B_1^{-1}\right)_{ik}(A_1)_{kj}$$
$$> \left(B_1^{-1}\right)_{ij}(A_1)_{jj} > 0.$$

2) If $i \in \mathcal{I}_n\backslash\mathcal{I}_m$, one has $(B_1^{-1}A_1)_{ij} = (A_1)_{ij} > 0, j \in \mathcal{I}_n$. Therefore, graph $\mathcal{G}_{B_1^{-1}A_1}$ also has a directed spanning tree.

Thus, $B_1^{-1}A_1$ is an SIA matrix according to Lemma 3. Then, there exists v such that

$$\lim_{m\to\infty}\left(B_1^{-1}A_1\right)^m = \mathbf{1}_n v^T \tag{14}$$

where $(B_1^{-1}A_1)^T v = v$, $\mathbf{1}_n^T v = 1$. Therefore, a simple manipulation yields that

$$\lim_{t_k\to\infty} x(t_k) = \lim_{t_k\to\infty}\left(B_1^{-1}A_1\right)^{t_k} x(0) = \mathbf{1}_n v^T x(0)$$

which means that (7) holds. Similar to the analysis in [21], (8) holds. This completes the proof of sufficiency.

Necessity: Suppose that graph $\mathcal{G}$ does not have a directed spanning tree. Then, there exist at least two vertices i and j $(i \neq j)$ such that there is no path between them. By $B_1^{-1} = \sum_{n=0}^{\infty}(Q_1\mathcal{W})^n$ and $\beta_i > 0$, we know that there is also no path between vertices i and j in graph $\mathcal{G}_{B_1}$. Through analysis yields the following.

1) If $i \in \mathcal{I}_m$ and $j \in \mathcal{I}_m$, then

$$\left(B_1^{-1}A_1\right)_{ij} = \alpha_j\left(B_1^{-1}\right)_{ij} + \sum_{k=m+1}^{n}\left(B_1^{-1}\right)_{ik} hw_{kj} = 0.$$

2) If $i \in \mathcal{I}_m$ and $j \in \mathcal{I}_n\backslash\mathcal{I}_m$, then

$$\left(B_1^{-1}A_1\right)_{ij} = (1-h)\left(B_1^{-1}\right)_{ij} + \sum_{\substack{k=m+1 \\ k\neq j}}^{n}\left(B_1^{-1}\right)_{ik} hw_{kj} = 0.$$

3) If $i \in \mathcal{I}_n\backslash\mathcal{I}_m$ and $j \in \mathcal{I}_n$, then

$$\left(B_1^{-1}A_1\right)_{ij} = hw_{ij} = 0.$$

Thus, we can draw the conclusion that $\mathcal{G}_{B_1^{-1}A_1}$ also does not have a directed spanning tree, which means that (14) does not hold. Therefore, HMAS_1 cannot reach consensus. This completes the proof of necessity.

Remark 4: The cost (2) is based on the current states of the neighboring players without any time delay. When the time delay $1 \le \tau \le \tau_{\max}, \tau \in N^+$ exists, we can modify the cost function of agent $i \in \mathcal{I}_m$ as follows:

$$J_i(x_1(t_{k+1}), \ldots, x_n(t_{k+1})) = \alpha_i(x_i(t_{k+1}) - x_i(t_k))^2$$
$$+ \beta_i\left(x_i(t_{k+1}) - \sum_{j\in\mathcal{N}_i} w_{ij}x_j(t_{k+1} - \tau h)\right)^2 \tag{15}$$

where $\tau_{\max}$ denotes a upper bound of τ.

Similar to the analysis in Section III, one has

$$x_i(t_{k+1}) = \beta_i\sum_{j\in\mathcal{N}_i} w_{ij}x_j(t_k) + \alpha_i x_i(t_{k+1} - \tau h)$$
$$= x_i(t_k) + \beta_i\sum_{j\in\mathcal{N}_i} w_{ij}\big(x_j(t_{k+1} - \tau h) - x_i(t_k)\big)$$

for agent $i \in \mathcal{I}_m$, where $\alpha_i + \beta_i = 1$.

Consider an HMAS*, in which the states of agents in $\mathcal{I}_m$ are governed by game $\mathbb{G}$ defined above with cost function (15) and the states of agents in $\mathcal{I}_n\backslash\mathcal{I}_m$ are governed by the first-order C-T dynamics (6) with protocol (9). When $\tau = 1$ in (15), the HMAS* is the HMAS which is considered in [21]. When $1 < \tau \le \tau_{\max}$ in (15), the HMAS* is an HMAS with time delays.

B. Case of C-T Dynamic Agents Using Their Own Real-Time States

Consider the following distributed protocol for the agent $i \in \mathcal{I}_n\backslash\mathcal{I}_m$:

$$u_i(t) = \sum_{j\in\mathcal{N}_i} w_{ij}\big(x_j(t_k) - x_i(t)\big), \quad t \in (t_k, t_{k+1}]. \tag{16}$$

Devise an HMAS in which the states of agents in $\mathcal{I}_m$ are governed by game $\mathbb{G}$ defined above and the states of agents in $\mathcal{I}_n \backslash \mathcal{I}_m$ are governed by first-order C-T dynamics (6) with protocol (16). For convenience, we use HMAS$_2$ to denote such an HMAS.

The following theorem establishes the relation between the states of two adjacent game times for HMAS$_2$.

Theorem 3: For HMAS$_2$, one has $x(t_{k+1}) = B_1^{-1}A_2x(t_k)$, where B_1 is defined by (10) and

$$A_2 = \mathrm{diag}\left\{\alpha_1, \alpha_2, \ldots, \alpha_m, \underbrace{e^{-h}, e^{-h}, \ldots, e^{-h}}_{n-m}\right\} + \mathrm{diag}\left\{\underbrace{0, 0, \ldots, 0}_{m}, \underbrace{1-e^{-h}, \ldots, 1-e^{-h}}_{n-m}\right\}\mathcal{W}.$$

Proof: From (6), (16) and $\sum_{j\in\mathcal{N}_i} w_{ij} = 1$, one has

$$x_i(t_{k+1}) = e^{-h}x_i(t_k) + \left(1-e^{-h}\right)\sum_{j\in\mathcal{N}_i} w_{ij}x_j(t_k) \tag{17}$$

for $i \in \mathcal{I}_n \backslash \mathcal{I}_m$.

Since $(x_1(t_{k+1}), x_2(t_{k+1}), \ldots, x_m(t_{k+1}))$ is the unique Nash equilibrium solution of game $\mathbb{G}$, it follows that:

$$\begin{cases} x_i(t_{k+1}) - \beta_i \sum_{j\in\mathcal{N}_i} w_{ij}x_j(t_{k+1}) = \alpha_i x_i(t_k), \quad i \in \mathcal{I}_m \\ x_i(t_{k+1}) = \left(1-e^{-h}\right)\sum_{j\in\mathcal{N}_i} w_{ij}x_j(t_k) \\ +e^{-h}x_i(t_k), \quad i \in \mathcal{I}_n \backslash \mathcal{I}_m. \end{cases} \tag{18}$$

The compact form of (18) is $B_1x(t_{k+1}) = A_2x(t_k)$. Since B_1 is invertible, one obtains that $x(t_{k+1}) = B_1^{-1}A_2x(t_k)$.

The following theorem gives a sufficient and necessary condition to ensure consensus of HMAS$_2$.

Theorem 4: Assume that $\mathcal{G}$ is directed. HMAS$_2$ reaches consensus if and only if $\mathcal{G}$ has a directed spanning tree.

Proof (Sufficiency): First, we prove that $B_1^{-1}A_2$ is a stochastic matrix. It is obviously that

$$\begin{aligned} B_1\mathbf{1}_n &= \left[1-\beta_1, 1-\beta_2, \ldots, 1-\beta_m, \underbrace{1, 1, \ldots, 1}_{n-m}\right]^T \\ &= \left[\alpha_1, \alpha_2, \ldots, \alpha_m, \underbrace{1, 1, \ldots, 1}_{n-m}\right]^T \\ &= \left(\mathrm{diag}\left\{\alpha_1, \alpha_2, \ldots, \alpha_m, \underbrace{e^{-h}, e^{-h}, \ldots, e^{-h}}_{n-m}\right\}\right. \\ &\quad \left. + \mathrm{diag}\left\{\underbrace{0, 0, \ldots, 0}_{m}, \underbrace{1-e^{-h}, \ldots, 1-e^{-h}}_{n-m}\right\}\mathcal{W}\right)\mathbf{1}_n \\ &= A_2\mathbf{1}_n. \end{aligned}$$

Note that B_1 is invertible. Then, one has

$$B_1^{-1}A_2\mathbf{1}_n = \mathbf{1}_n. \tag{19}$$

From the proof of Theorem 2, B_1^{-1}, whose diagonal elements are positive, is non-negative since $0 < \alpha_i < 1$ for all $i \in \mathcal{I}_m$ and $0 < e^{-h} < 1$, $0 < 1 - e^{-h} < 1$ for $h > 0$, A_2 is non-negative with its diagonal elements being positive. Therefore, $B_1^{-1}A_2$ is a (row) stochastic matrix whose diagonal elements are positive.

The subsequent derivation is similar to the proof in Theorem 2, which is omitted here.

Remark 5: Theorem 2 indicates that if agent $i \in \mathcal{I}_n \backslash \mathcal{I}_m$ only exploits its own state at the game time, then the game period h has an upper bound. Theorem 4 demonstrates that there is no upper bound on the game period h when the agent $i \in \mathcal{I}_n \backslash \mathcal{I}_m$ exploits its own state in real time. In practical applications, the energy of each agent is limited. Monitoring own state and communicating with neighbors both result in energy consumption for the agent. Therefore, by comparing the results of Theorems 2 and 4, a compromise should be made between the frequencies of state monitoring and neighboring communication.

V. Second-Order HMASs

In this section, consensus problems are investigated for an HMAS, in which the states of agents in $\mathcal{I}_m$ are governed by a multiplayer game $\mathbb{G}$ defined above and the dynamics of agent $i \in \mathcal{I}_n \backslash \mathcal{I}_m$ are given as follows:

$$\dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = u_i(t), \quad i \in \mathcal{I}_n \backslash \mathcal{I}_m \tag{20}$$

where $v_i \in \mathbb{R}$ is the velocity of agent i and the control protocol $u_i(t)$ will be given later.

Definition 3: The second-order HMAS is said to reach consensus if for any initial states

$$\lim_{t_k\to\infty} \|x_i(t_k) - x_j(t_k)\| = 0, \; i, j \in \mathcal{I}_n \tag{21}$$

and

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \; \lim_{t\to\infty} \|v_i(t) - v_j(t)\| = 0 \tag{22}$$

for $i, j \in \mathcal{I}_n \backslash \mathcal{I}_m$.

We also consider two scenarios for agent $i \in \mathcal{I}_n \backslash \mathcal{I}_m$. Agent $i \in \mathcal{I}_n \backslash \mathcal{I}_m$ in one scenario only exploits its own state at game time, while in the other scenario can exploit its own state in real time.

A. Case of C-T Dynamic Agents Only Using Their Own Game Time States

Consider the following distributed protocol for the agent $i \in \mathcal{I}_n \backslash \mathcal{I}_m$:

$$u_i(t) = \sum_{j\in\mathcal{N}_i} w_{ij}\big(x_j(t_k) - x_i(t_k)\big) - k_1v_i(t_k) \tag{23}$$

where $t \in (t_k, t_{k+1}]$ and $k_1 > 0$ is the feedback gain.

Devise an HMAS in which the states of agents in $\mathcal{I}_m$ are governed by game $\mathbb{G}$ defined above and the states of agents in $\mathcal{I}_n \backslash \mathcal{I}_m$ are governed by first-order C-T dynamics (20) with protocol (23). For convenience, we use HMAS$_3$ to denote such an HMAS.

Let $y_{i'}(t_k) = x_i(t_k) + (2/k_1)v_i(t_k)$, which is called the state of the nominal agent i'. Then, for $i \in \mathcal{I}_n \backslash \mathcal{I}_m$ and $i' = n-m+i$, it follows from (20) and (23) that:

$$\begin{cases} x_i(t_{k+1}) = \frac{h^2}{2}\sum_{j\in\mathcal{N}_i} w_{ij}x_j(t_k) + k_2 y_{i'}(t_k) \\ +\left(1-k_2-\frac{h^2}{2}\right)x_i(t_k) \\ y_{i'}(t_{k+1}) = (1-k_4)y_{i'}(t_k) + k_3\sum_{j\in\mathcal{N}_i} w_{ij}x_j(t_k) \\ +(k_4-k_3)x_i(t_k) \end{cases}$$

where $k_2 = (hk_1/2) - (h^2k_1^2/4)$, $k_3 = (h^2/2) + (2h/k_1)$, and $k_4 = (hk_1/2) + (h^2k_1^2/4)$.

Let $x'(\cdot) = [x_1(\cdot), x_2(\cdot), \ldots, x_n(\cdot), y_{m+1'}(\cdot), \ldots, y_{n'}(\cdot)]^T$, which is called the augmented state of the HMAS$_3$. The following theorem establishes the relation between the states of two adjacent game times for the HMAS$_3$.

Theorem 5: For HMAS$_3$, one has $x'(t_{k+1}) = B_2^{-1}A_3x'(t_k)$, where B_2 is defined by (24), at the bottom of the next page and

$$A_3 = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$\mathcal{W} = \begin{pmatrix} (\mathcal{W}_{11})_{m\times m} & (\mathcal{W}_{12})_{m\times(n-m)} \\ (\mathcal{W}_{21})_{(n-m)\times m} & (\mathcal{W}_{22})_{(n-m)\times(n-m)} \end{pmatrix}$$

$$A_{11} = \mathrm{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$$

$$A_{12} = A_{13} = \mathbf{0}_{m\times(n-m)}, A_{21} = \frac{h^2}{2}\mathcal{W}_{21}$$

$$A_{22} = \mathrm{diag}\left\{\underbrace{1-\frac{h^2}{2}-k_2, \ldots, 1-\frac{h^2}{2}-k_2}_{n-m}\right\} + \frac{h^2}{2}\mathcal{W}_{22}$$

$$A_{23} = \mathrm{diag}\left\{\underbrace{k_2, k_2, \ldots, k_2}_{n-m}\right\}, A_{31} = k_3\mathcal{W}_{21}$$

$$A_{32} = \mathrm{diag}\left\{\underbrace{k_4-k_3, \ldots, k_4-k_3}_{n-m}\right\} + k_3\mathcal{W}_{22}$$

$$A_{33} = \mathrm{diag}\left\{\underbrace{1-k_4, 1-k_4, \ldots, 1-k_4}_{n-m}\right\}.$$

Proof: Since $(x_1(t_{k+1}), x_2(t_{k+1}), \ldots, x_m(t_{k+1}))$ is the Nash equilibrium of game $\mathbb{G}$, it follows that:

$$\begin{cases} x_i(t_{k+1}) - \beta_i\sum_{j\in\mathcal{N}_i} w_{ij}x_j(t_{k+1}) = \alpha_i x_i(t_k), \quad i\in\mathcal{I}_m \\ x_i(t_{k+1}) = \left(1-k_2-\frac{h^2}{2}\right)x_i(t_k) + \frac{h^2}{2}\sum_{j\in\mathcal{N}_i} w_{ij}x_j(t_k) \\ +k_2y_{i'}(t_k), \quad i\in\mathcal{I}_n\backslash\mathcal{I}_m \\ y_{i'}(t_{k+1}) = (1-k_4)y_{i'}(t_k) + k_3\sum_{j\in\mathcal{N}_i} w_{ij}x_j(t_k) \\ +(k_4-k_3)x_i(t_k), \quad i'\in\mathcal{I}_n\backslash\mathcal{I}_m. \end{cases} \tag{25}$$

The compact form of (25) is $B_2x'(t_{k+1}) = A_3x'(t_k)$.

Let b'_{ij} denote the (i,j)th entry of the matrix B_2. Then, a direct computation yields that

$$\begin{cases} \sum_{j=1,j\neq i}^{2n-m} |b'_{ij}| = \sum_{j=1,j\neq i}^{2n-m} |-\beta_i w_{ij}| = \beta_i < 1 = |b'_{ii}|, \; i\in\mathcal{I}_m \\ \sum_{j=1,j\neq i}^{2n-m} |b'_{ij}| = 0 < 1 = |b'_{ii}|, \; i\in\mathcal{I}_n\backslash\mathcal{I}_m \\ \sum_{j=1,j\neq i'}^{2n-m} |b'_{i'j}| = 0 < 1 = |b'_{i'i'}|, \; i'\in\mathcal{I}_n\backslash\mathcal{I}_m. \end{cases}$$

Therefore, B_2 is SDD. According to Lemma 1, B_2 is invertible. Thus, $x'(t_{k+1}) = B_2^{-1}A_3x'(t_k)$.

The following theorem gives a sufficient and necessary condition to ensure consensus of HMAS$_3$.

Theorem 6: Assume that $\mathcal{G}$ is directed, $0 < h < (\sqrt{5}-1)/2$ and $2 < k_1 < (\sqrt{5}-1)/h$. HMAS$_3$ reaches consensus if and only if $\mathcal{G}$ has a directed spanning tree.

Proof (Sufficiency): First, we prove that $B_2^{-1}A_3$ is a stochastic matrix. It is obviously that

$$\begin{aligned} B_2\mathbf{1}_{2n-m} &= \Big[1-\beta_1, 1-\beta_2, \ldots, 1-\beta_m, \underbrace{1, 1, \ldots, 1}_{2n-2m}\Big]^T \\ &= \Big[\alpha_1, \alpha_2, \ldots, \alpha_m, \underbrace{1, 1, \ldots, 1}_{2n-2m}\Big]^T \\ A_3\mathbf{1}_{2n-m} &= [\alpha_1, \alpha_2, \ldots, \alpha_m \\ &\quad \underbrace{\frac{h^2}{2}+1-\frac{h^2}{2}-k_2+k_2, \ldots, \frac{h^2}{2}+1-\frac{h^2}{2}-k_2+k_2}_{n-m} \\ &\quad \underbrace{k_3+k_4-k_3+1-k_4, \ldots, k_3+k_4-k_3+1-k_4}_{n-m}\Big]^T \\ &= \Big[\alpha_1, \alpha_2, \ldots, \alpha_m, \underbrace{1, 1, \ldots, 1}_{2n-2m}\Big]^T. \end{aligned}$$

Therefore, one has $B_2\mathbf{1}_{2n-m} = A_3\mathbf{1}_{2n-m}$. Since B_2 is invertible, it follows that:

$$B_2^{-1}A_3\mathbf{1}_{2n-m} = \mathbf{1}_{2n-m}. \tag{27}$$

Let

$$C = \mathrm{diag}\left\{\beta_1, \ldots, \beta_m, \underbrace{0, \ldots, 0}_{2n-2m}\right\}\begin{pmatrix} \mathcal{W} & \mathbf{0}_{n\times(n-m)} \\ \mathbf{0}_{(n-m)\times n} & \mathbf{0}_{(n-m)\times(n-m)} \end{pmatrix}.$$

It is easy to derive that $B_2 = I_{2n-m} - C$. Assume that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all eigenvalues of $C = (c_{ij}) \in \mathbb{R}^{n\times n}$. Then, one has $\lambda_i \in \bigcup_{i=1}^n G_i$, $i\in\mathcal{I}_n$ by utilizing Lemma 2, where

$$\begin{aligned} G_i &= \left\{z \,\Big|\, |z-c_{ii}| \le \sum_{j=1,j\neq i}^n |c_{ij}|\right\} \\ &= \begin{cases} \left\{z \,\big|\, |z-0| \le \sum_{j=1,j\neq i}^n |\beta_i w_{ij}| = |\beta_i| < 1\right\}, & i\in\mathcal{I}_m \\ \{z \,|\, |z-0| \le 0 < 1\}, & i\in\mathcal{I}_n\backslash\mathcal{I}_m, i'\in\mathcal{I}_n\backslash\mathcal{I}_m. \end{cases} \end{aligned}$$

This means that $|\lambda_i| < 1$. Therefore, $B_2^{-1} = (I_{2n-m} - C)^{-1} = \sum_{n=0}^{\infty}(C)^n$ is non-negative with positive diagonal elements because matrix C is non-negative.

As $0 < h < (\sqrt{5}-1)/2$ and $2 < k_1 < (\sqrt{5}-1)/h$, we can obtain that $k_2 = (hk_1/2) - (h^2k_1^2/4) > 0$, $k_3 = (h^2/2) + (2h/k_1) > 0$, and $1-k_4 = 1-(hk_1/2) - (h^2k_1^2/4) > 0$. It is worth noting that $(h^2/2)k_1^2 - (h^2/2) > 0$, $(hk_1/2) - (2h/k_1) > 0$, and $(h^2k_1^2/4) - (h^2/2) > 0$. Thus, $1-(h^2/2)-k_2 = (1-k_4) + (h^2/2)k_1^2 - (h^2/2) > 0$, $k_4 - k_3 = ([hk_1/2] - [2h/k_1]) + ([h^2k_1^2/4] - [h^2/2]) > 0$. Therefore, A_3, whose diagonal elements are positive, is non-negative. Together with (27), $B_2^{-1}A_3$ is a (row) stochastic matrix whose diagonal elements are positive.

It is easy to obtain that

$$C=\begin{pmatrix} (C_{11})_{m\times m} & (C_{12})_{m\times(n-m)} & \mathbf{0}_{m\times(n-m)} \\ \mathbf{0}_{(n-m)\times m} & \mathbf{0}_{(n-m)\times(n-m)} & \mathbf{0}_{(n-m)\times(n-m)} \\ \mathbf{0}_{(n-m)\times m} & \mathbf{0}_{(n-m)\times(n-m)} & \mathbf{0}_{(n-m)\times(n-m)} \end{pmatrix}.$$

One has

$$B_2^{-1}A_3=\begin{pmatrix} *_{11} & *_{12} & *_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

where $*_{11}=(I_m+C_{11}+C_{11}^2+\cdots)(A_{11}+C_{12}A_{21})$, $*_{12}=(I_m+C_{11}+C_{11}^2+\cdots)C_{12}A_{22}$, $*_{13}=(I_m+C_{11}+C_{11}^2+\cdots)C_{12}A_{23}$.

Next, we prove that graph $\mathcal{G}_{B_2^{-1}A_3}$ has a directed spanning tree. Denote the directed spanning tree of graph $\mathcal{G}$ as $T_{\mathcal{G}}$. For each edge $(j,i)\in T_{\mathcal{G}}$.

1) If $i\in\mathcal{I}_m, j\in\mathcal{I}_m$, then $(B_2^{-1}A_3)_{ij}=(*_{11})_{ij}\geq(C_{11}A_{11})_{ij}\geq(C_{11})_{ij}(A_{11})_{ij}=\beta_i w_{ij}\alpha_j>0$.
2) If $i\in\mathcal{I}_m, j\in\mathcal{I}_n\backslash\mathcal{I}_m$ and $j'=n-m+j$, then $(B_2^{-1}A_3)_{ij}\geq\beta_i w_{ij}\alpha_j>0$ and $(B_2^{-1}A_3)_{j'j}>0$.
3) If $i\in\mathcal{I}_n\backslash\mathcal{I}_m, j\in\mathcal{I}_m$, then $(B_2^{-1}A_3)_{ij}=W_{ij}>0$.
4) If $i\in\mathcal{I}_n\backslash\mathcal{I}_m, j\in\mathcal{I}_n\backslash\mathcal{I}_m$ and $j'=n-m+j$, then $(B_2^{-1}A_3)_{ij}=W_{ij}>0$ and $(B_2^{-1}A_3)_{j'j}>0$.

Thus, graph $\mathcal{G}_{B_2^{-1}A_3}$ also has a directed spanning tree.

The above analysis leads to that $B_2^{-1}A_3$ is an SIA matrix. Then, there exists v such that

$$\lim_{m\to\infty}\left(B_2^{-1}A_3\right)^m=\mathbf{1}_n v^T \tag{28}$$

where $(B_2^{-1}A_3)^T v=v$, and $\mathbf{1}_n^T v=1$. Similar to the analysis in [21], we obtain (21) and (22) hold.

This completes the proof of sufficiency.

Necessity: Denote the directed spanning tree of graph $\mathcal{G}_{B_2^{-1}A_3}$ as $T'_{\mathcal{G}}$. First, one has

$$W^2=\begin{pmatrix} W_{11}^2+W_{12}W_{21} & W_{11}W_{12}+W_{12}W_{22} \\ W_{21}W_{11}+W_{22}W_{21} & W_{22}^2+W_{21}W_{12}\end{pmatrix}$$

where $W_{ij}^2>0$ denotes that there is a path from j to i.

For each edge $(j,i)\in T'_{\mathcal{G}}$ $(i\neq j)$.

1) If $i\in\mathcal{I}_m, j\in\mathcal{I}_m$, then

$$(*_{11})_{ij}=\left(\sum_{k=0}^{\infty}\Lambda^{k+1}(W_{11})^k\left(\Lambda A_{11}W_{11}^2+\frac{h^2}{2}W_{12}W_{21}\right)+A_{11}+\Lambda A_{11}W_{11}\right)_{ij}>0$$

where $A_{11}=\mathrm{diag}\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$ and $\Lambda=\mathrm{diag}\{\beta_1,\beta_2,\ldots,\beta_m\}$. Thus, at least one of $(W_{11})_{ij}>0$, $((W_{11})^k(W_{11}^2+W_{12}W_{21}))_{ij}>0$, $k=0,1,\ldots,$ holds, which means that there is a path from j to i in $\mathcal{G}$.

2) If $i\in\mathcal{I}_m, j\in\mathcal{I}_n\backslash\mathcal{I}_m$, then

$$(*_{12})_{i,(j-m)}=\left(\sum_{k=0}^{\infty}\Lambda^{k+1}(W_{11})^k\left(\Lambda\Lambda_2W_{11}W_{12}+\frac{h^2}{2}W_{12}W_{22}\right)+\Lambda\Lambda_2W_{12}\right)_{i,(j-m)}>0$$

in which

$$\Lambda_2=\mathrm{diag}\Bigg\{\underbrace{1-\frac{h^2}{2}-k_2,\ldots,1-\frac{h^2}{2}-k_2}_{n-m}\Bigg\}$$

and $\Lambda=\mathrm{diag}\{\beta_1,\beta_2,\ldots,\beta_m\}$. Thus, at least one of $(W_{12})_{i,(j-m)}>0$, $((W_{11})^k(W_{11}W_{12}+W_{12}W_{22}))_{i,(j-m)}>0$, $k=0,1,\ldots,$ holds, which means that there is a path from j to i in $\mathcal{G}$.

3) If $i\in\mathcal{I}_n\backslash\mathcal{I}_m, j\in\mathcal{I}_m$, then $(A_{21})_{(i-m),j}=(h^2/2)(W_{21})_{(i-m),j}>0$.
4) If $i\in\mathcal{I}_n\backslash\mathcal{I}_m, j\in\mathcal{I}_n\backslash\mathcal{I}_m$, then

$$(A_{22})_{(i-m),(j-m)}=\left(\mathrm{diag}\Bigg\{\underbrace{1-\frac{h^2}{2}-k_2,\ldots,1-\frac{h^2}{2}-k_2}_{n-m}\Bigg\}+\frac{h^2}{2}\mathcal{W}_{22}\right)_{(i-m),(j-m)}>0.$$

For each edge $(j,i')\in T'_{\mathcal{G}}$, $j\in\mathcal{I}_n$, $i\in\mathcal{I}_n\backslash\mathcal{I}_m$, and $i'=n-m+i$, one has $W_{ij}>0$.

Through the above analysis, we draw the conclusion that $\mathcal{G}$ also has a directed spanning tree. Thus, if $\mathcal{G}$ does not have a directed spanning tree, then $\mathcal{G}_{B_2^{-1}A_3}$ does not have a directed spanning tree, which means that (28) does not hold. Therefore, HMAS_3 cannot reach consensus. This completes the proof of necessity.

B. Case of C-T Dynamic Agents Using Their Own Real-Time States

Consider the following distributed protocol for agent $i\in\mathcal{I}_n\backslash\mathcal{I}_m$:

$$u_i(t)=\sum_{j\in\mathcal{N}_i}w_{ij}\big(x_j(t_k)-x_i(t)\big)-k_1v_i(t) \tag{29}$$

where $t\in(t_k,t_{k+1}]$ and $k_1>0$ is feedback gain.

$$B_2=I_{2n-m}-\mathrm{diag}\Bigg\{\beta_1,\beta_2,\ldots,\beta_m,\underbrace{0,0,\ldots,0}_{2n-2m}\Bigg\}\begin{pmatrix}\mathcal{W} & \mathbf{0}_{n\times(n-m)} \\ \mathbf{0}_{(n-m)\times n} & \mathbf{0}_{(n-m)\times(n-m)}\end{pmatrix} \tag{24}$$

$$A'_{32}=\mathrm{diag}\Bigg\{\underbrace{k_5-k_7-\frac{k_1k_6}{2}-\frac{2k_6}{k_1},\ldots,k_5-k_7-\frac{k_1k_6}{2}-\frac{2k_6}{k_1}}_{n-m}\Bigg\}+\left(1-k_5+\frac{2k_6}{k_1}\right)\mathcal{W}_{22} \tag{26}$$

Devise an HMAS in which the states of agents in $\mathcal{I}_m$ are governed by game $\mathbb{G}$ defined above and the states of agents in $\mathcal{I}_n \backslash \mathcal{I}_m$ are governed by first-order C-T dynamics (20) with protocol (29). For convenience, we use HMAS$_4$ to denote such an HMAS.

Let $y_{i'}(t_k) = x_i(t_k) + (2/k_1)v_i(t_k)$. Then, for $i \in \mathcal{I}_n \backslash \mathcal{I}_m$ and $i' = n - m + i$, it follows from (20) and (29) that:

$$\begin{cases} x_i(t_{k+1}) = (1-k_5)\sum_{j\in\mathcal{N}_i} w_{ij}x_j(t_k) + \frac{k_1k_6}{2}y_{i'}(t_k) \\ +\left(k_5 - \frac{k_1k_6}{2}\right)x_i(t_k) \\ y_{i'}(t_{k+1}) = \left(k_5 - k_7 - \frac{k_1k_6}{2} - \frac{2k_6}{k_1}\right)x_i(t_k) \\ +\left(1 - k_5 + \frac{2k_6}{k_1}\right)\sum_{j\in\mathcal{N}_i} w_{ij}x_j(t_k) + \left(\frac{k_1k_6}{2} + k_7\right)y_{i'}(t_k) \end{cases}$$

for $i \in \mathcal{I}_n \backslash \mathcal{I}_m$ and $t = t_{k+1}$, where $k_5 = (r_2e^{r_1h} - r_1e^{r_2h})/(r_2 - r_1)$, $k_6 = (e^{r_2h} - e^{r_1h})/(r_2 - r_1)$, $k_7 = (r_2e^{r_2h} - r_1e^{r_1h})/(r_2 - r_1)$, $r_1 = (-k_1 + \sqrt{k_1^2 - 4})/2$ and $r_2 = (-k_1 - \sqrt{k_1^2 - 4})/2$.

Let $x'(\cdot) = [x_1(\cdot), x_2(\cdot), \ldots, x_n(\cdot), y_{m+1'}(\cdot), \ldots, y_{n'}(\cdot)]^T$, which is called the augmented state of the HMAS$_4$. The following theorem establishes the relation between the states of two adjacent game times for HMAS$_4$.

Theorem 7: For HMAS$_4$, one has $x'(t_k + 1) = B_2^{-1}A_4x'(t_k)$, where B_2 is defined by (24), A'_{32} is defined by (26), at the bottom of the previous page and

$$\begin{aligned} A_4 &= \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A'_{21} & A'_{22} & A'_{23} \\ A'_{31} & A'_{32} & A'_{33} \end{pmatrix} \\ \mathcal{W} &= \begin{pmatrix} (\mathcal{W}_{11})_{m\times m} & (\mathcal{W}_{12})_{m\times(n-m)} \\ (\mathcal{W}_{21})_{(n-m)\times m} & (\mathcal{W}_{22})_{(n-m)\times(n-m)} \end{pmatrix} \\ A_{11} &= \text{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_m\} \\ A_{12} &= A_{13} = \mathbf{0}_{m\times(n-m)} \\ A'_{21} &= (1-k_5)\mathcal{W}_{21} \\ A'_{22} &= \text{diag}\Bigg\{\underbrace{k_5 - \frac{k_1k_6}{2}, \ldots, k_5 - \frac{k_1k_6}{2}}_{n-m}\Bigg\} + (1-k_5)\mathcal{W}_{22} \\ A'_{23} &= \text{diag}\Bigg\{\underbrace{\frac{k_1k_6}{2}, \frac{k_1k_6}{2}, \ldots, \frac{k_1k_6}{2}}_{n-m}\Bigg\} \\ A'_{31} &= \left(1 - k_5 + \frac{2k_6}{k_1}\right)\mathcal{W}_{21} \\ A'_{33} &= \text{diag}\Bigg\{\underbrace{\frac{k_1k_6}{2} + k_7, \frac{k_1k_6}{2} + k_7, \ldots, \frac{k_1k_6}{2} + k_7}_{n-m}\Bigg\}. \end{aligned}$$

Proof: The proof is similar to that in Theorem 5, which is omitted here.

The following theorem gives a sufficient and necessary condition to ensure consensus of HMAS$_4$.

Theorem 8: Assume that $\mathcal{G}$ is directed and $k_1 > 2$. HMAS$_4$ reaches consensus if and only if $\mathcal{G}$ has a directed spanning tree.

Proof: First, we prove that $B_2^{-1}A_4$ is a stochastic matrix

$$\begin{aligned} A_4\mathbf{1}_{2n-m} &= \Big[\alpha_1, \alpha_2, \ldots, \alpha_m, \underbrace{\gamma_1, \gamma_1, \ldots, \gamma_1}_{n-m}, \underbrace{\gamma_2, \gamma_2, \ldots, \gamma_2}_{n-m}\Big]^T \\ &= \Big[\alpha_1, \alpha_2, \ldots, \alpha_m, \underbrace{1, 1, \ldots, 1}_{2n-2m}\Big]^T = B_2\mathbf{1}_{2n-m} \end{aligned}$$

where $\gamma_1 = 1 - k_5 + k_5 - (k_1k_6/2) + (k_1k_6/2)$ and $\gamma_2 = 1 - k_5 + (2k_6/k_1) + k_5 - k_7 - (k_1k_6/2) - (2k_6/k_1) + (k_1k_6/2) + k_7$. Therefore, one has

$$B_2^{-1}A_4\mathbf{1}_{2n-m} = \mathbf{1}_{2n-m}. \tag{30}$$

From the proof in Theorem 6, one has that $B_2^{-1} = (I_{2n-m} - C)^{-1} = \sum_{n=0}^{\infty}(C)^n = I_{2n-m} + C + C^2 + \cdots$ and B_2^{-1} is non-negative with positive diagonal elements.

As $k_1 > 2$ and $r_2 < r_1 < 0$, we can draw the conclusion that $k_5 > 0$ and $k_6 > 0$ from Lemma 4. Moreover, it is easy to obtain that $1 - k_5 = [(e^{r_2h} - 1)/r_2 - (e^{r_1h} - 1)/r_1]/(r_2 - r_1) > 0$, $k_5 - (k_1k_6)/2 = (k_1k_6)/2 + k_7 = (e^{r_1h} + e^{r_2h})/2 > 0$ and $1 - k_5 + 2k_6/k_1 > 0$. It is obvious that $(k_1/2) > (2/k_1)$, thus $k_5 - k_7 - (k_1k_6/2) - (2k_6/k_1) > k_5 - k_7 - (k_1k_6/2) - (k_1k_6/2) = 0$. Therefore, A_4, whose diagonal elements are positive, is non-negative. Together with (30), $B_2^{-1}A_4$ is a (row) stochastic matrix whose diagonal elements are positive.

The subsequent derivation is similar to the proof in Theorem 6, which is omitted here.

Remark 6: This article establishes a unified framework for agents whose states are governed by a multiplayer game and the agents with C-T dynamics. More specifically, if $m = 0$, the HMAS becomes a first-order/second-order C-T MAS, which reaches consensus if and only if the network contains a directed spanning tree [12], [40]. If $m = n$, the HMAS becomes a game-based MAS, in which the interactions among agents are described as a multiplayer game. The game-based MAS reaches consensus if and only if the network has a directed spanning tree [32]. It can be seen that the obtained results in this article include the above-mentioned two scenarios.

VI. SIMULATIONS

In the following, numerical simulations for our theoretical results are presented.

We consider an HMAS with eight agents. The digraph $\mathcal{G}$, which has a directed spanning tree, is shown in Fig. 1. In the subsequent simulations, $x(0) = [5, 10, -5, 15, -10, 20, 25, -15]^T$, $\alpha = [0.2, 0.3, 0.4, 0.5]^T$, and $\beta = [0.8, 0.7, 0.6, 0.5]^T$.

Example 1: Consider HMAS$_1$ under digraph $\mathcal{G}$, in which agents 1–4 decide their states by multiplayer game $\mathbb{G}$ and agents 5–8 update their states with first-order C-T dynamics (6) under protocol (9). Figs. 2 and 3 depict the trajectories of agents' states in HMAS$_1$ for $h = 0.25 < 1$ and $h = 0.7 < 1$, respectively. They are both in accordance with Theorem 2. Consider the HMAS$'_1$ which is similar to the HMAS$_1$ except that $\alpha = [0.8, 0.85, 0.86, 0.9]^T$ and

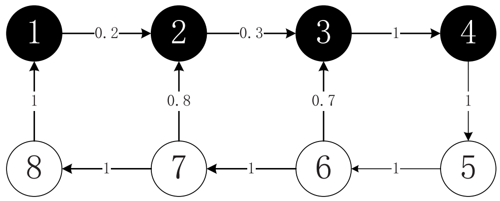

Fig. 1. Digraph $\mathcal{G}$.

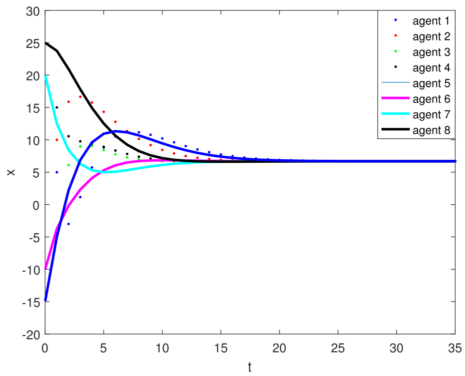

Fig. 2. Trajectories of agents' states in HMAS_1, where $h = 0.25$.

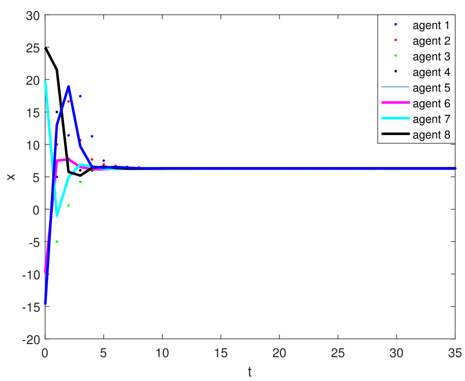

Fig. 3. Trajectories of agents' states in HMAS_1, where $h = 0.7$.

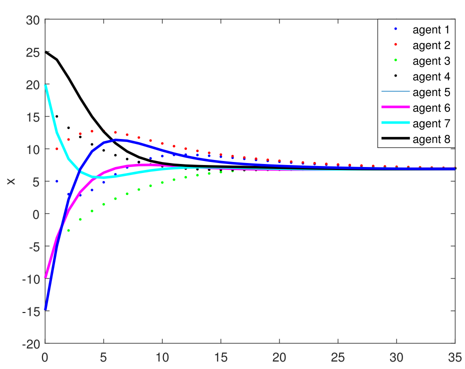

Fig. 4. Trajectories of agents' states in HMAS_1'.

$\beta = [0.2, 0.15, 0.14, 0.1]^T$. Let $h = 0.25 < 1$. Then, the trajectories of agents' states in HMAS_1' are drawn in Fig. 4. It also verifies the result of Theorem 2.

Example 2: Consider HMAS_2 under digraph $\mathcal{G}$, in which agents 1–4 decide their states by multiplayer game $\mathbb{G}$ and

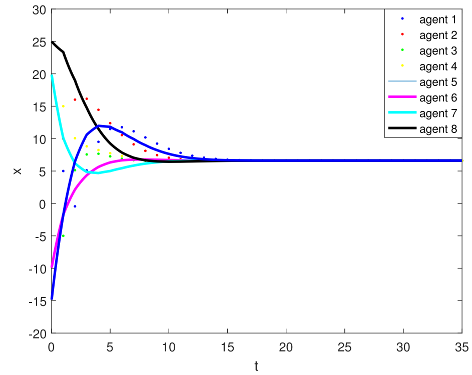

Fig. 5. Trajectories of agents' states in HMAS_2.

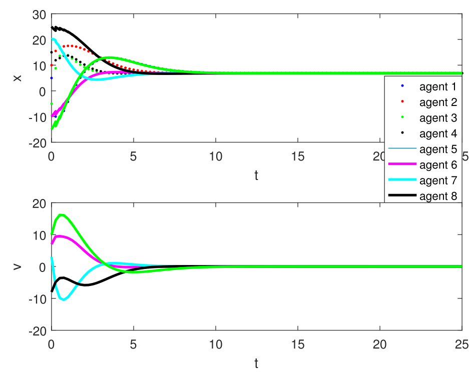

Fig. 6. Trajectories of agents' states and C-T dynamic agents' velocities in HMAS_3.

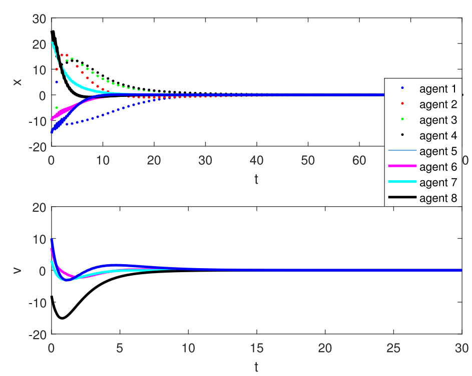

Fig. 7. Trajectories of agents' states and C-T dynamic agents' velocities in HMAS_4.

agents 5–8 update their states with first-order C-T dynamics (6) under protocol (16). Let $h = 0.4$. Fig. 5 depicts the trajectories of agents' states in the HMAS_2, which shows the result of Theorem 4.

Example 3: Consider HMAS_3 under digraph $\mathcal{G}$, in which agents 1–4 decide their states by multiplayer game $\mathbb{G}$ and agents 5–8 update their states with second-order C-T dynamics (20) under protocol (23). Let $0 < h = 0.25 < (\sqrt{5}-1)/2$ and $2 < k_1 = 2.2 < (\sqrt{5}-1)/h$. Theorem 6 is illustrated in Fig. 6 by the trajectories of agents' states and C-T dynamic agents' velocities in HMAS_3.

Example 4: Consider HMAS_4 under digraph $\mathcal{G}$, in which agents 1–4 decide their states by multiplayer game $\mathbb{G}$ and agents 5–8 update their states with second-order C-T dynamics (20) under protocol (29). Let $h = 0.35$, $k_1 = 2.5 > 2$. Fig. 7 depicts the trajectories of agents' states and C-T dynamic agents' velocities in HMAS_4, which shows the result of Theorem 8.

Remark 7: In Examples 1–4, $t = k$ represents time t_k. Example 1 indicates that α and β will affect consensus convergence rate of agents in $\mathcal{I}_m$, and h will affect the consensus convergence rate of agents in $\mathcal{I}_n$. Agents 5 and 6 interact with agents 1–4 at every time t_k, which leads to the nonsmoothness of trajectories in Figs. 2 and 3. Large $|u_5| - |u_8|$ are produced as a result of the initial states of agents 5–8 deviating significantly from those of their neighbors. Therefore, it is shown in Figs. 4 and 5 that the slope of tangent line of trajectories is relatively large at the beginning.

VII. Conclusion

This article investigates the consensus of HMASs based on game modeling, in which some agents select the Nash equilibrium of a multiplayer game as their states at each game time and the others update their states by first-order/second-order C-T dynamics. Moreover, we consider two protocols for agents with C-T dynamics. In one protocol, they only exploit their own states at game time, while in the other protocol, they exploit their own states in real time. Then, sufficient and necessary conditions are separately obtained for the consensus of first-order/second-order HMAS. Future work will focus on establishing different cost functions of games to model MASs.

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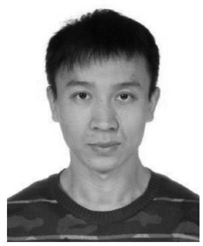

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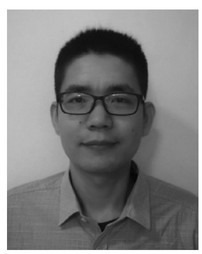

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