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Game-based coordination control of multi-agent systems ☆

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ABSTRACT

In this paper, we are interested in how agents interact with each other in multi-agent systems. First, we model the interactions among agents in multi-agent systems as a multi-player game. The topology of the interactions among agents is a directed graph. We design cost functions for the game and assume that each agent in the systems tends to minimize its own cost. Then, the unique Nash equilibrium solution to the proposed multi-player game is obtained as the next state of the agent. A necessary and sufficient condition for achieving multi-agent consensus is established using the system transformation method and graph theory. Furthermore, we extend the result to the problem of containment control in the presence of leaders. The criterion for solving the containment control is also given in this paper. Finally, several simulation examples are given to verify the effectiveness of the theoretical results.

1. Introduction

As we know, various classic multi-agent coordination has been investigated, including coverage control [1], flocking [2,3], consensus [4–7], distributed estimation [8], formation control [9,10], containment control [11–13] and so on. Consensus (also known as synchronization or agreement in different scenarios) problems are the fundamental ones in coordination control. It means that a group of agents agree on a specific number of interest by designing appropriate control inputs. Consensus protocols have been studied extensively. Olfati-Saber et al. [14] considered consensus problems for the first-order continuous-time(CT) multi-agent systems(MASs) under a linear protocol. They established some necessary and sufficient conditions for achieving the average consensus of MASs. Munz et al. [15] investigated the robustness of consensus problems for first-order CT MASs under a preset protocol with coupling delays and switching topologies. Consensus of second-order CT MASs has been studied in [16] under a delayed output feedback control protocol. At the same time, attention also has been paid to discrete-time(DT) MASs. Jadbabaie et al. [17] investigated the consensus of first-order DT MASs. The protocol they considered for the MASs is expressed by the average of the states of agent i and its neighbors. By using graph theory and matrix theory, Ren and Beard [18] obtained several necessary and sufficient conditions for the consensus of first-order DT MASs under a given protocol. Consensus of second-order DT MASs has been studied in [19], where the protocol of MASs uses the local velocity information and the distributed relative state information.

The aforementioned consensus protocols were primarily studied under the assumption that there is no group reference state. However, there may be leaders in a system in practical scenarios. For instance, the male silkworm moth detects the pheromone released by the female and enters the bulge across which she spans. This kind of consensus problems is classified into containment control. Its main objective is to drive the followers eventually to enter the convex hull spanned by the leaders via appropriate distributed protocols. For DT MASs, the distributed containment control problems were studied in [12] under a switched directed graph. Li and Ren [20] considered the containment control problems for DT MASs with general linear dynamics. Based on routh criterion and z-transformation, the authors in [21] solved the containment control problem for the DT MASs with time delays.

Game theory can be used to model the strategic interactions of agents. In a game, each agent iteratively decides its action by minimizing its cost or maximizing its payoff, which depends the actions of itself and its neighbors. There is also a rich and still growing study on coordination control of MASs using game theory approaches. For example, Semsar-Kazerooni and Khorasanide [22] minimized the team cost function based on a combination of individual costs and obtained a set of Pareto-efficient solutions for a cooperative game. Mixing cooperative control, reinforcement learning and game theory, Vamvoudakis et al. [23] proposed an online solution formula of a team game

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based on MASs. Based on the competitive propagation model, Mei and Bullo [24] proposed two classes of games. Moreover, they characterized the quality-seeding trade-off and the Nash equilibrium for the games. In [25], a new attacker–detector game has been defined to study optimal sensor placement in networked control systems. Furthermore, Ma et al. [26] designed cost functions for hybrid MASs in light of game-theoretic approaches, and proved that the hybrid MASs can reach consensus. Ghaderi [27] showed that the best-response dynamics of MASs in a game can converge to consensus in the presence of stubborn agents. Ma et al. [28] employed a repeated bimatrix game to model the MASs and proved that agents belonging to the same unions can achieve consensus. Ye et al. [29] proved the MAS exponential convergence to the Nash equilibrium for games with first- and second-order players.

Note that the aforementioned studies of consensus and containment control problems mainly focused on the analysis of whether the coordination control goals can be achieved. The case where the consensus protocols of agents are unknown has also been considered. In [30–32], the authors considered the consensus of MASs under distributed model prediction control schemes. They used local predictive controllers to find an optimal control sequence and a consensus point. Combining consensus theory with approximate subgradient methods, Johansson et al. [33] solved the coupled optimization problems in a distributed way and studied the consensus of the proposed scheme. How to design interactive strategies that can make MASs achieve desired coordination control goals has gradually attracted attention. In this paper, we introduce game theory to solve the distributed consensus and containment control problems for MASs, respectively. Different from the literatures [30]–[33], we are interested in solving the coordination control problems for game-based MASs. We design a multi-player game to describe the interactions among agents. Each agent in the network updates its own state according to the given cost function and iteration rules. Regardless of whether there exist leaders, the game designed has a unique Nash equilibrium solution. Then, some criteria are obtained for solving the consensus and containment control problems. Difficulties come partially from how to design appropriate cost functions for agents and the proof of coordination for the game-based MASs. The reason is that compare to the coordination control problems in the existing literatures mentioned above, it is more difficult for game-based MASs to understand the interaction modes among agents. Additionally, designing games in accordance with the current states of agents increases the difficulty of systems analysis. Compared with the existing works, the contributions of this paper are summarized as follows.

- We develop a game-based framework for MASs. In addition, according to the multi-player game, we prove that the MASs under proposed interaction rules can achieve consensus without leaders and achieve containment control with leaders.
- We prove that a unique Nash equilibrium exists in our game model and it is chosen as the next state of agent.
- Based on the designed games, we provide the necessary and sufficient conditions for the establishment of the multi-agent coordination.

We organize the rest of this paper as follows. Section 2 presents preliminaries. Game-based interaction model is given in Section 3. We provide the theoretical analysis of multi-agent coordination according to the given iteration rules without leaders and with leaders in Section 4 and Section 5, respectively. In Section 6, the simulation results are given to show the effectiveness of the obtained results. Finally, we draw the main conclusions in Section 7. In Appendix, we state some lemmas and definitions which can be used in this paper.

2. Preliminaries

In this section, a number of basic concepts and notions are introduced.

First, we introduce the concepts associated with graphs. $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ is a weighted directed graph, which is composed of a vertex set $\mathcal{V}$, an edge set $\mathcal{E}$ and a weighted adjacency matrix $\mathcal{W} = (w_{ij}) \in \mathbb{R}^{n\times n}$. The neighbor set of vertex i is $\mathcal{N}_i = \{v_j \in \mathcal{V} \mid (v_j, v_i) \in \mathcal{E}\}$. If $(v_i, v_j) \in \mathcal{E}$, we call v_i the parent of v_j. A directed graph is called a directed tree if every node in the graph has only one parent except the root node. A spanning tree is a directed tree composed of all vertices and some edges in $\mathcal{G}$. One or more directed trees without common vertices consist a directed forest. A directed forest is called as a directed spanning forest, if it contains all the nodes and some edges in $\mathcal{G}$. The degree matrix $\mathcal{D} = (d_{ij}) \in \mathbb{R}^{n\times n}$ is a diagonal matrix with $d_{ii} = \sum_{j\in\mathcal{N}_i} w_{ij}$. Define the Laplacian matrix of a graph $\mathcal{G}$ by $L = \mathcal{D} - \mathcal{W} = (l_{ij}) \in \mathbb{R}^{n\times n}$. I_n is the $n \times n$ identity matrix and $\mathbf{1}_n$ is the n-dimensional column vector with all entries equal to one. $\mathcal{I}_n = \{1, 2, \ldots, n\}$ is an index set. $diag\{a_1, a_2, \ldots, a_n\}$ represents the diagonal matrix with diagonal elements being $a_1, a_2, \ldots, a_n$.

If every entry of the matrix A is nonnegative, A is said to be nonnegative. A nonnegative matrix with row sum 1 is said to be a (row) stochastic matrix. A stochastic matrix $P = [p_{ij}]_{n\times n}$ is called indecomposable and aperiodic (SIA) if $\lim_{k\to\infty} P^k = \mathbf{1}_n y^T$, where y is some column vector. Given a graph $\mathcal{G}_P$ with n vertices, graph $\mathcal{G}_P$ is called the graph associated with P if $(i, j) \in \mathcal{E} \Leftrightarrow P_{ij} > 0$.

Next we introduce some basic notions of multi-player games (please refer to [34] for more details). A multi-player game consists of $n(> 2)$ players, a set of strategies available to those players and an individual cost function for each player. Each player in the game selects a strategy that minimizes its own cost function, and makes decision independently. The precise description of a multi-player game is provided below together with the notations to be used. Suppose that n players $v_1, v_2, \ldots, v_n$ play a game $\mathbb{G}$. Let us denote the set of all players as $\mathcal{V}$. Player v_i has strategies in $\Omega_i \subset \mathbb{R}$. If each $v_i \in \mathcal{V}$ corresponds to a strategy $x_i \in \Omega_i$, then $(x_1, x_2, \ldots, x_n)$ is an n-tuple of strategies. The cost function of v_i is $C_i(x_1, x_2, \ldots, x_n)$. Players make their strategies to seek the minimum possible costs independently by considering the possible choices of the other rational players. Then, we introduce the Nash equilibrium solution to the multi-player game as follows.

Definition 1 (*[34]*)**.** Consider an n-tuple of strategies $(x_1^*, x_2^*, \ldots, x_n^*)$ with $x_i^* \in \Omega_i$, $i \in \mathcal{I}_n$. $(x_1^*, x_2^*, \ldots, x_n^*)$ is said to be a Nash equilibrium of the n-player nonzero-sum game if the following inequalities

$$\begin{cases} C_i(x_1^*, x_2^*, \ldots, x_n^*) & \leq C_i(x_1, x_2^*, \ldots, x_n^*) \\ C_i(x_1^*, x_2^*, \ldots, x_n^*) & \leq C_i(x_1^*, x_2, \ldots, x_n^*) \\ & \vdots \\ C_i(x_1^*, x_2^*, \ldots, x_n^*) & \leq C_i(x_1^*, x_2^*, \ldots, x_n) \end{cases} \tag{1}$$

are satisfied for all $x_i \in \Omega_i$, $i \in \mathcal{I}_n$.

3. Game-based interaction

Consider an MAS with n agents $v_1, v_2, \ldots, v_n$, where each agent has state $x_i(t) \in \mathbb{R}$ at time t. The initial condition of agent v_i, $i \in \mathcal{I}_n$ is $x_i(0)$. In this section, we model the interaction among them as a multi-player game, represented by $\mathbb{G}(\mathcal{V}, \Omega_i, C_i)$.

The definition of game $\mathbb{G}(\mathcal{V}, \Omega_i, C_i)$ is given as follows:

- **Players** : The set of all players can be denoted as $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$.
- **Strategy** : Each player $v_i \in \mathcal{V}$ chooses its strategy in $\Omega_i \subset \mathbb{R}$. Players interact with their neighbors and choose their strategies to minimize their costs $C_i(x_1(t+1), \ldots, x_i(t+1), \ldots, x_n(t+1))$ at time $t+1$. For simplicity, let $C_i(t+1) \triangleq C_i(x_1(t+1), \ldots, x_i(t+1), \ldots, x_n(t+1))$. Suppose that all players choose strategies independently and simultaneously. If a player chooses strategy $x_i(t+1) \in \Omega_i$, it means that this player will decide $x_i(t+1)$ as its state at time $t+1$.
- **Cost** : The cost of each player is determined by the cost in changing its own state and the disagreement cost with neighbor players. For player $v_i \in \mathcal{V}$, the cost of changing its own state is $(x_i(t+1) - x_i(t))^2$, and the disagreement cost with neighbor players is $(x_i(t+1) - \frac{1}{d_i}\sum_{j\in\mathcal{N}_i} w_{ij}x_j(t+1))^2$. Therefore, the cost function of player $v_i \in \mathcal{V}$ is

$$C_i(t+1) = \alpha_i(x_i(t+1)-x_i(t))^2+\beta_i(x_i(t+1)-\frac{1}{d_i}\sum_{j\in\mathcal{N}_i} w_{ij}x_j(t+1))^2, \quad (2)$$

when $d_i = \sum_{j\in\mathcal{N}_i} w_{ij} > 0$, $\alpha_i + \beta_i = 1$, $0 < \alpha_i < 1$, and

$$C_i(t+1) = (x_i(t+1) - x_i(t))^2, \quad (3)$$

when $d_i = 0$.

In the game $\mathbb{G}(\mathcal{V}, \Omega_i, C_i)$, every player $v_i \in \mathcal{V}$ decides its strategy to minimize its cost, which uses information from its neighbors and itself. Note that each player's best strategy depends on the decisions of the other players. Hence, no player can reduce its cost by unilaterally changing itself action at Nash equilibrium.

Remark 1. α_i and β_i are the weights of the costs in changing states and the disagreement costs among the players, respectively. The higher the weight is, the higher the importance of the corresponding term is.

Throughout this paper, we make the following assumption:

Assumption 1. We assume that every state in the strategy set is reachable for all the players. Moreover, if the player selects state $x_i(t+1)$ at time $t+1$, it will reach the state $x_i(t+1)$ before time $t+2$.

Assumption 2. All agents are rational, i.e., they consistently make decisions to minimize their costs without making mistakes and update their states independently. Besides, every agent believes that all of its neighbors are rational.

Remark 2. We assume that all the players are rational and selfish. They want fewer states changes and closer to their neighbors. It is practicable in many realistic scenarios. Take tariffs for example, each country intends to not only stick to its own tariff but also reach consensus with the others, which requires a balance between its own interest and the differences with the others. Moreover, in social networks, people want to both insist on their own opinions and ultimately reach an agreement with others. Therefore, each player must make a compromise to reach consensus between maintaining its own state unchanged and closing the gap with other players.

4. Game-based consensus of MASs

Consider a directed network of agents with digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$. Assume that $d_i = \sum_{j\in\mathcal{N}_i} w_{ij} > 0$ for all agents in the MAS. We model the interaction among these agents as a game $\mathbb{G}(\mathcal{V}, \Omega_i, C_i)$ which follows the definition in Section 3. The cost function of agent $v_i \in \mathcal{V}$ is given in (2). Let $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$, we have the following results.

4.1. Existence and characterization of Nash equilibrium

Theorem 1. *Game $\mathbb{G}(\mathcal{V}, \Omega_i, C_i)$ has a unique Nash equilibrium solution $(x_1^*(t+1), \ldots, x_i^*(t+1), \ldots, x_n^*(t+1))$ with $x^*(t+1) = B^{-1}Ax(t)$ holds, where $A = diag\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, $B = I_n - diag\{\frac{\beta_1}{d_1}, \frac{\beta_2}{d_2}, \ldots, \frac{\beta_n}{d_n}\}\mathcal{W}$.*

Proof. For agent $i \in \mathcal{I}_n$, if $x_j^*(t+1), j \in \mathcal{I}_n, j \neq i$ are fixed, it is obvious that $C_i(x_1^*(t+1), \ldots, x_i(t+1), \ldots, x_n^*(t+1))$ is a quadratic function of $x_i(t+1)$. Therefore, $C_i(x_1^*(t+1), \ldots, x_i(t+1), \ldots, x_n^*(t+1))$ has only one global minimum $x_i^*(t+1)$, which satisfies

$$\begin{cases} \frac{\partial C_i(t+1)}{\partial x_i(t+1)}|_{(x_1^*(t+1),\ldots,x_i^*(t+1),\ldots,x_n^*(t+1))} = 0, \\ \frac{\partial^2 C_i(t+1)}{\partial^2 x_i(t+1)}|_{(x_1^*(t+1),\ldots,x_i^*(t+1),\ldots,x_n^*(t+1))} > 0. \end{cases} \quad (4)$$

Combining with (2) and (4), we know that $(x_1^*(t+1), \ldots, x_i^*(t+1), \ldots, x_n^*(t+1))$ satisfies

$$\begin{aligned} &C_i(x_1^*(t+1), \ldots, x_i^*(t+1), \ldots, x_n^*(t+1)) \\ &\quad \leq C_i(x_1^*(t+1), \ldots, x_i(t+1), \ldots, x_n^*(t+1)) \end{aligned}$$

with the following formula holds:

$$x_i^*(t+1) - \frac{\beta_i}{d_i}\sum_{j\in\mathcal{N}_i} w_{ij}x_j^*(t+1) = \alpha_1 x_i(t).$$

According to Definition 1, $(x_1^*(t+1), \ldots, x_i^*(t+1), \ldots, x_n^*(t+1))$ is the Nash equilibrium solution if and only if

$$\begin{cases} x_1^*(t+1) - \frac{\beta_1}{d_1}\sum_{j\in\mathcal{N}_1} w_{1j}x_j^*(t+1) = \alpha_1 x_1(t), \\ x_2^*(t+1) - \frac{\beta_2}{d_2}\sum_{j\in\mathcal{N}_2} w_{2j}x_j^*(t+1) = \alpha_2 x_2(t), \\ \quad\vdots \\ x_n^*(t+1) - \frac{\beta_n}{d_n}\sum_{j\in\mathcal{N}_n} w_{nj}x_j^*(t+1) = \alpha_n x_n(t). \end{cases} \quad (5)$$

The matrix-form of (5) is $Bx^*(t+1) = Ax(t)$.

Let $B = I_n - diag\{\frac{\beta_1}{d_1}, \frac{\beta_2}{d_2}, \ldots, \frac{\beta_n}{d_n}\}\mathcal{W} = (b_{ij}) \in \mathbb{R}^{n\times n}$. It is obvious that

$$\sum_{j=1,j\neq i}^{n} |b_{ij}| = \sum_{j=1,j\neq i}^{n} |-\frac{\beta_i}{d_i}w_{ij}| = \frac{\beta_i}{d_i}\sum_{j\in\mathcal{N}_i} w_{ij} = \beta_i < 1 = |b_{ii}|, i \in \mathcal{I}_n.$$

This implies that B is SDD. By Lemma 1 in the Appendix, it is easy to know that $\det B \neq 0$. Therefore, we have $x^*(t+1) = B^{-1}Ax(t)$.

As a consequence, the game $\mathbb{G}(\mathcal{V}, \Omega_i, C_i)$ has the unique Nash equilibrium solution $(x_1^*(t+1), \ldots, x_i^*(t+1), \ldots, x_n^*(t+1))$ with $x^*(t+1) = B^{-1}Ax(t)$ holds. □

4.2. Consensus analysis

The agents play the game $\mathbb{G}(\mathcal{V}, \Omega_i, C_i)$ which is mentioned in the above subsection. Moreover, they choose the Nash equilibrium solution as their states at time $t+1$. Through these interaction rules among agents, the algorithm of the MAS is formulated as

$$x(t+1) = B^{-1}Ax(t). \quad (6)$$

Remark 3. The cost (2) is based on the current states of the neighboring players without any time delay. When the time delay $1 \leq \tau \leq \tau_{max}$, $\tau \in N^+$ exists, we can modify the cost as

$$C_i(t+1) = \alpha_i(x_i(t+1) - x_i(t))^2 + \beta_i(x_i(t+1) - \frac{1}{d_i}\sum_{j\in\mathcal{N}_i} w_{ij}x_j(t+1-\tau))^2, \ i,j \in \mathcal{I}_n, \tag{7}$$

where τ_{max} denotes an upper bound of τ.

Similar to the analysis above, we have

$$x_i(t+1) = \alpha_i x_i(t) + \frac{\beta_i}{d_i}\sum_{j\in\mathcal{N}_i} w_{ij}x_j(t+1-\tau), \ i \in \mathcal{I}_n, \tag{8}$$

where $\alpha_i + \frac{\beta_i}{d_i}\sum_{j\in\mathcal{N}_i} w_{ij} = 1$.

When $\tau = 1$ in (7), system (8) is a typical DT MAS which achieves consensus asymptotically if and only if the associated interaction graph $\mathcal{G}$ has a spanning tree [18]. When $1 < \tau \leq \tau_{max}$ in (7), system (8) is a DT MAS with time delays.

In this subsection, we consider the consensus problem for MAS (6). Some criteria are obtained to solve the consensus problem.

Definition 2. If for any initial states, $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0$ holds for all $i, j \in \mathcal{I}_n$. The consensus of MAS (6) is achieved.

Theorem 2. *Consider a directed network of agents with the communication topology $\mathcal{G}$. MAS (6) reaches consensus if and only if the topology $\mathcal{G}$ has a directed spanning tree. Furthermore, the consensus state is $\upsilon^T x(0)$, where $(B^{-1}A)^T\upsilon = \upsilon$, $\mathbf{1}_n^T\upsilon = 1$.*

Proof. *(Sufficiency)* First we will prove that $B^{-1}A$ is a stochastic matrix. It is easy to obtain

$$B\mathbf{1}_n = \begin{pmatrix} 1 & -\frac{\beta_1}{d_1}w_{12} & \cdots & -\frac{\beta_1}{d_1}w_{1n} \\ -\frac{\beta_2}{d_2}w_{21} & 1 & \cdots & -\frac{\beta_2}{d_2}w_{2n} \\ \vdots & \vdots & & \vdots \\ -\frac{\beta_n}{d_n}w_{n1} & -\frac{\beta_n}{d_n}w_{n2} & \cdots & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\beta_1}{d_1}\sum_{j\in\mathcal{N}_1} w_{1j} \\ 1 - \frac{\beta_2}{d_2}\sum_{j\in\mathcal{N}_2} w_{2j} \\ \vdots \\ 1 - \frac{\beta_n}{d_n}\sum_{j\in\mathcal{N}_n} w_{nj} \end{pmatrix} = \begin{pmatrix} 1-\beta_1 \\ 1-\beta_2 \\ \vdots \\ 1-\beta_n \end{pmatrix}$$

and $A\mathbf{1}_n = diag\{\alpha_1, \alpha_2, \ldots, \alpha_n\}\mathbf{1}_n = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T = [1-\beta_1, 1-\beta_2, \ldots, 1-\beta_n]^T$. According to Theorem 1, we have

$$B^{-1}A\mathbf{1}_n = \mathbf{1}_n. \tag{9}$$

Recall that $B = I_n - Q\mathcal{W}$, where $Q = diag\{\frac{\beta_1}{d_1}, \frac{\beta_2}{d_2}, \ldots, \frac{\beta_n}{d_n}\}$. Assume $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all eigenvalues of $Q\mathcal{W} = (c_{ij}) \in \mathbb{R}^{n\times n}$. Then resort to Lemma 2 in the Appendix, we have $\lambda_i \in \bigcup_{i=1}^n G_i$, $i \in \mathcal{I}_n$, where

$$\begin{aligned} G_i &= \{z \mid |z - c_{ii}| \leq \sum_{j=1,j\neq i}^n |c_{ij}|\} \\ &= \{z \mid |z - 0| \leq \sum_{j=1,j\neq i}^n |\frac{\beta_i}{d_i}w_{ij}|\} \\ &= \{z \mid |z| \leq |\beta_i| < 1\}, \end{aligned}$$

which means $|\lambda_i| < 1$. Therefore,

$$B^{-1} = (I_n - Q\mathcal{W})^{-1} = \sum_{n=0}^{\infty}(Q\mathcal{W})^n = I_n + Q\mathcal{W} + (Q\mathcal{W})^2 + \cdots. \tag{10}$$

Obviously, every entry of matrix $Q\mathcal{W}$ is nonnegative. Thus, from (10), it can be found that B^{-1} is nonnegative and its diagonal elements are positive. With the fact that $\alpha_i > 0$ and (9), we know that $B^{-1}A = (e_{ij}) \in \mathbb{R}^{n\times n}$ is a (row) stochastic matrix with positive diagonal elements.

From (10) and $\alpha_i \neq 0, i \in \mathcal{I}_n$, we know that $w_{ij} \neq 0, i, j \in \mathcal{I}_n \Rightarrow e_{ij} \neq 0, i, j \in \mathcal{I}_n$. Therefore, if there is a directed edge in $\mathcal{G}$ from υ_j to υ_i, then there is also a directed edge in $\mathcal{G}_{B^{-1}A}$ from υ_j to υ_i. As a consequence, $\mathcal{G}_{B^{-1}A}$ has a spanning tree when $\mathcal{G}$ has a directed spanning tree.

Thus, by Lemma 3 in the Appendix, $B^{-1}A$ is an SIA matrix, i.e. $\exists\ \upsilon$ such that

$$\lim_{m\to\infty}(B^{-1}A)^m = \mathbf{1}_n\upsilon^T, \tag{11}$$

where $(B^{-1}A)^T\upsilon = \upsilon$, $\mathbf{1}_n^T\upsilon = 1$.

Then, $\lim_{t\to\infty} x(t) = \lim_{t\to\infty}(B^{-1}A)^t x(0) = \mathbf{1}_n\upsilon^T x(0)$ holds, which means that $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0$, $i, j \in \mathcal{I}_n$. This completes the proof of sufficiency.

(Necessity) Suppose that graph $\mathcal{G}$ does not have a directed spanning tree. It follows from Lemma 4 in the Appendix and $\frac{\beta_i}{d_i} > 0$ that the graph $\mathcal{G}_{Q\mathcal{W}+(Q\mathcal{W})^2+(Q\mathcal{W})^3+\cdots}$ does not have a directed spanning tree. Then, it can be derived that $\mathcal{G}_{B^{-1}A}$ does not have a directed spanning tree with the fact (10) and $\alpha_i > 0$, which means that (11) does not hold. Consequently, MAS (6) cannot reach consensus. This completes the proof of necessity. □

Remark 4. $B^{-1}A$ is a stochastic matrix reveals that each player needs to seek a balance between its own interest and the differences with the others.

Remark 5. Because B^{-1} is nonnegative and $0 < \alpha_i < 1, i \in \mathcal{I}_n$, it is evident that the higher weight α_i is, the greater the influence of υ_i on its neighbors is.

5. Game-based containment control of MASs

In this section, the game-based containment control problem is investigated. When $d_i = 0$, it means the player υ_i has no neighbors. We call such player as a leader with the cost function (3). Meanwhile, followers are the players with $d_i > 0$, whose cost functions are defined in (2).

Consider an MAS with communication topology $\mathcal{G}' = (\mathcal{V}, \mathcal{E}, \mathcal{W}')$. Assume that there are $m(m < n)$ followers and $n - m$ leaders in the MAS. The set of followers is $\{\upsilon_1, \upsilon_2, \ldots, \upsilon_m\}$ and the set of leaders is $\{\upsilon_{m+1}, \upsilon_{m+2}, \ldots, \upsilon_n\}$. Let $F = \{1, 2, \ldots, m\}$ and $R = \{m+1, m+2, \ldots, n\}$. We model the interaction among agents of the MAS by a multi-player game $\mathbb{G}'(\mathcal{V}, \Omega_i, C_i)$. The definition of game $\mathbb{G}'(\mathcal{V}, \Omega_i, C_i)$ is similar to that in Section 3. The cost functions of leaders and followers are given in (2) and (3), respectively.

5.1. Existence and characterization of Nash equilibrium

Every player in the game $\mathbb{G}'(\mathcal{V}, \Omega_i, C_i)$ decides $x_i(t+1)$ to minimize its cost $C_i(t+1)$. From the definition of Nash equilibrium solution, every agent of the MAS will choose the Nash equilibrium solution as its next time state when a unique Nash equilibrium exists.

Let $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$, $x_F(t) = [x_1(t), x_2(t), \ldots, x_m(t)]^T$ and $x_R(t) = [x_{m+1}(t), x_{m+2}(t), \ldots, x_n(t)]^T$.

Theorem 3. *Game $\mathbb{G}'(\mathcal{V}, \Omega_i, C_i)$ has a unique Nash equilibrium solution $(x_1^*(t+1), \ldots, x_i^*(t+1), \ldots, x_n^*(t+1))$ with $x^*(t+1) = \bar{B}^{-1}\bar{A}x(t)$ holds, where*

$$\bar{A} = \begin{pmatrix} \bar{A}_{11} & 0_{m\times(n-m)} \\ \mathbf{0}_{(n-m)\times m} & I_{n-m} \end{pmatrix}, \bar{B} = \begin{pmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \mathbf{0}_{(n-m)\times m} & I_{n-m} \end{pmatrix},$$

with $\bar{A}_{11} = diag\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$,

$$\bar{B}_{11} = \begin{pmatrix} 1 & -\frac{\beta_1}{d_1}w'_{12} & \cdots & -\frac{\beta_1}{d_1}w'_{1m} \\ -\frac{\beta_2}{d_2}w'_{21} & 1 & \cdots & -\frac{\beta_2}{d_2}w'_{2m} \\ \vdots & \vdots & & \vdots \\ -\frac{\beta_m}{d_m}w'_{m1} & -\frac{\beta_m}{d_m}w'_{m2} & \cdots & 1 \end{pmatrix},$$

$$\bar{B}_{12} = \begin{pmatrix} -\frac{\beta_1}{d_1}w'_{1(m+1)} & -\frac{\beta_1}{d_1}w'_{1(m+2)} & \cdots & -\frac{\beta_1}{d_1}w'_{1n} \\ -\frac{\beta_2}{d_2}w'_{2(m+1)} & -\frac{\beta_2}{d_2}w'_{2(m+2)} & \cdots & -\frac{\beta_2}{d_2}w'_{2n} \\ \vdots & \vdots & & \vdots \\ -\frac{\beta_m}{d_m}w'_{m(m+1)} & -\frac{\beta_m}{d_m}w'_{m(m+2)} & \cdots & -\frac{\beta_m}{d_m}w'_{mn} \end{pmatrix}.$$

Proof. Similar to the analysis in Theorem 1, we can get that game $\mathbb{G}'$ has a unique Nash equilibrium solution $(x_1^*(t+1), \ldots, x_i^*(t+1), \ldots, x_n^*(t+1))$ if and only if

$$\begin{cases} x_i^*(t+1) - \frac{\beta_i}{d_i}\sum_{j\in\mathcal{N}_i} w'_{ij}x_j^*(t+1) = \alpha_i x_i(t), i \in F, \\ x_i^*(t+1) = x_i(t), i \in R. \end{cases} \tag{12}$$

The matrix-form of (12) is

$$\begin{pmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \mathbf{0}_{(n-m)\times m} & I_{n-m} \end{pmatrix}\begin{pmatrix} x_F^*(t+1) \\ x_R^*(t+1) \end{pmatrix} = \begin{pmatrix} \bar{A}_{11} & \mathbf{0}_{m\times(n-m)} \\ \mathbf{0}_{(n-m)\times m} & I_{n-m} \end{pmatrix}\begin{pmatrix} x_F(t) \\ x_R(t) \end{pmatrix},$$

i.e., $\bar{B}x^*(t+1) = \bar{A}x(t)$. Let $\bar{B} = (\bar{b}_{ij}) \in \mathbb{R}^{n\times n}$, it is easy to obtain

$$\sum_{j=1,j\neq i}^{N} |\bar{b}_{ij}| = \begin{cases} \sum_{j=1,j\neq i}^{N} |-\frac{\beta_i}{d_i}w'_{ij}| = \frac{\beta_i}{d_i}\sum_{j\in\mathcal{N}_i} w'_{ij} = \beta_i < 1 = |\bar{b}_{ii}|, i \in F, \\ 0 < 1 = |\bar{b}_{ii}|, i \in R, \end{cases}$$

which implies that $\bar{B}$ is SDD. According to Lemma 1 in the Appendix, it can be derived that $\det \bar{B} \neq 0$.

Therefore, we have $x^*(t+1) = \bar{B}^{-1}\bar{A}x(t)$. We can draw the conclusion that the game $\mathbb{G}'(\mathcal{V}, \Omega_i, C_i)$ has the unique Nash equilibrium solution $(x_1^*(t+1), \ldots, x_i^*(t+1), \ldots, x_n^*(t+1))$ with $x^*(t+1) = \bar{B}^{-1}\bar{A}x(t)$ holds. □

5.2. Containment control analysis

The interaction among agents of the MAS is modeled by multi-player game $\mathbb{G}'(\mathcal{V}, \Omega_i, C_i)$. Moreover, the agents choose the Nash equilibrium solution as their states $x(t+1)$ at time $t+1$. By repeating this process, the algorithm of the MAS can be described by

$$x(t+1) = \bar{B}^{-1}\bar{A}x(t). \tag{13}$$

Then, we show the necessary and sufficient conditions of solving the containment control problem for MAS(13) in the next theorem.

Definition 3. In any initial states, if the states of all followers converge asymptotically to the convex hull spanned by the leaders' states, the containment control problem of MAS (13) is solved.

Theorem 4. *Consider a directed network of agents with the communication topology $\mathcal{G}'$. MAS (13) can solve the containment control problem if and only if the topology $\mathcal{G}'$ has a directed spanning forest and all root nodes are leaders. Moreover, the final position of the followers is $-(I_m - \bar{B}_{11}^{-1}\bar{A}_{11})^{-1}\bar{B}_{11}^{-1}\bar{B}_{12}x_R(0)$, where $x_R(0)$ is the initial value of the leader set.*

Proof. *(Sufficiency)* Let $\bar{B} = (\bar{b}_{ij}) \in \mathbb{R}^{n\times n}$. It is obvious that

$$\sum_{j=1,j\neq i}^{m} |\bar{b}_{ij}| \leq \sum_{j=1,j\neq i}^{n} |\bar{b}_{ij}| = \sum_{j=1,j\neq i}^{n} |-\frac{\beta_i}{d_i}w'_{ij}| = \frac{\beta_i}{d_i}\sum_{j\in\mathcal{N}_i} w'_{ij} = \beta_i < 1 = |\bar{b}_{ii}|, i \in F,$$

which means that $\bar{B}_{11}$ is SDD. Then, we have $\det \bar{B}_{11} \neq 0$. It follows from Lemma 6 in the Appendix that

$$\bar{B}^{-1} = \begin{pmatrix} (\bar{B}_{11} - \bar{B}_{12}\bar{B}_{22}^{-1}\bar{B}_{21})^{-1} & -\bar{B}_{11}^{-1}\bar{B}_{12}F_2 \\ -F_2\bar{B}_{21}\bar{B}_{11}^{-1} & F_2 \end{pmatrix} = \begin{pmatrix} \bar{B}_{11}^{-1} & -\bar{B}_{11}^{-1}\bar{B}_{12} \\ \mathbf{0}_{(n-m)\times m} & I_{n-m} \end{pmatrix}, \tag{14}$$

where $F_2 = (\bar{B}_{22} - \bar{B}_{21}\bar{B}_{11}^{-1}\bar{B}_{12})^{-1} = I_{n-m}$.

From (13) and (14), we can obtain that

$$\begin{aligned} \begin{pmatrix} x_F(t+1) \\ x_R(t+1) \end{pmatrix} &= \begin{pmatrix} \bar{B}_{11}^{-1} & -\bar{B}_{11}^{-1}\bar{B}_{12} \\ \mathbf{0}_{(n-m)\times m} & I_{n-m} \end{pmatrix} \times \begin{pmatrix} \bar{A}_{11} & \mathbf{0}_{m\times(n-m)} \\ \mathbf{0}_{(n-m)\times m} & I_{n-m} \end{pmatrix}\begin{pmatrix} x_F(t) \\ x_R(t) \end{pmatrix} \\ &= \begin{pmatrix} \bar{B}_{11}^{-1}A_{11} & -\bar{B}_{11}^{-1}\bar{B}_{12} \\ \mathbf{0}_{(n-m)\times m} & I_{n-m} \end{pmatrix}\begin{pmatrix} x_F(t) \\ x_R(t) \end{pmatrix} \\ &\vdots \\ &= \begin{pmatrix} (\bar{B}_{11}^{-1}\bar{A}_{11})^{t+1} & -\sum_{n=0}^{t}(\bar{B}_{11}^{-1}\bar{A}_{11})^n\bar{B}_{11}^{-1}\bar{B}_{12} \\ \mathbf{0}_{(n-m)\times m} & I_{n-m} \end{pmatrix} \times \begin{pmatrix} x_F(0) \\ x_R(0) \end{pmatrix}. \end{aligned} \tag{15}$$

Thus,

$$\lim_{t\to\infty} x(t+1) = \begin{pmatrix} \lim_{t\to\infty}(\bar{B}_{11}^{-1}\bar{A}_{11})^{t+1} & -\sum_{n=0}^{\infty}(\bar{B}_{11}^{-1}\bar{A}_{11})^n\bar{B}_{11}^{-1}\bar{B}_{12} \\ \mathbf{0}_{(n-m)\times m} & I_{n-m} \end{pmatrix} \times \begin{pmatrix} x_F(0) \\ x_R(0) \end{pmatrix}. \tag{16}$$

We next show that $|\lambda(\bar{B}_{11}^{-1}\bar{A}_{11})| < 1$.

First, we have

$$\bar{A}_{11}^{-1}\bar{B}_{11} = \begin{pmatrix} \alpha_1^{-1} & -\frac{\beta_1}{\alpha_1 d_1}w'_{12} & \cdots & -\frac{\beta_1}{\alpha_1 d_1}w'_{1m} \\ -\frac{\beta_2}{\alpha_2 d_2}w'_{21} & \alpha_2^{-1} & \cdots & -\frac{\beta_2}{\alpha_2 d_2}w'_{2m} \\ \vdots & \vdots & & \vdots \\ -\frac{\beta_m}{\alpha_m d_m}w'_{m1} & -\frac{\beta_m}{\alpha_m d_m}w'_{m2} & \cdots & \alpha_m^{-1} \end{pmatrix}.$$

Assume $\lambda_1, \lambda_2, \ldots, \lambda_m$ are all eigenvalues of $\bar{A}_{11}^{-1}\bar{B}_{11}$. Then according to Lemma 2 in the Appendix, we have $\lambda_i \in \bigcup_{i=1}^{m} G_i, i \in F$, where

$$G_i = \{z \mid |z - \alpha_i^{-1}| \leq \sum_{j=1,j\neq i}^{m} |-\frac{\beta_i}{\alpha_i d_i}w'_{ij}|\}, = \{z \mid |z - \alpha_i^{-1}| \leq \frac{\beta_i}{\alpha_i d_i}\sum_{j=1,j\neq i}^{m} w'_{ij} \leq \frac{\beta_i}{\alpha_i d_i}\sum_{j=1,j\neq i}^{n} w'_{ij} \leq \frac{\beta_i}{\alpha_i}\}.$$

By the fact that $0 < \alpha_i < 1$ and $\alpha_i^{-1} - \frac{\beta_i}{\alpha_i} = 1$, we have $\lambda(\bar{A}_{11}^{-1}\bar{B}_{11}) \geq 1$.

Second, it is obvious that $I - \bar{A}_{11}^{-1}\bar{B}_{11} \triangleq H_1H_2$, where $H_1 = diag\{-\frac{\beta_1}{\alpha_1 d_1}, -\frac{\beta_2}{\alpha_2 d_2}, \ldots, -\frac{\beta_m}{\alpha_m d_m}\}$,

$$H_2 = \begin{pmatrix} d_1 & -w'_{12} & \cdots & -w'_{1m} \\ -w'_{21} & d_2 & \cdots & -w'_{2m} \\ \vdots & \vdots & & \vdots \\ -w'_{m1} & -w'_{m2} & \cdots & d_m \end{pmatrix}.$$

From Lemma 5 in the Appendix, we know that $|H_2| \neq 0$. According to $-\frac{\beta_i}{\alpha_i d_i} \neq 0, i \in F$, we can obtain that $|I - \bar{A}_{11}^{-1}\bar{B}_{11}| = |H_1| \cdot |H_2| \neq 0$, which means $\lambda(\bar{A}_{11}^{-1}\bar{B}_{11}) \neq 1$.

Following the analysis above, we can conclude that $\lambda(\bar{A}_{11}^{-1}\bar{B}_{11}) > 1$. Then, we have $|\lambda(\bar{B}_{11}^{-1}\bar{A}_{11})| < 1$. Therefore, $\lim_{t\to\infty}(\bar{B}_{11}^{-1}\bar{A}_{11})^{t+1} = 0$, $\sum_{n=0}^{\infty}(\bar{B}_{11}^{-1}\bar{A}_{11})^n = (I - \bar{A}_{11}^{-1}\bar{B}_{11})^{-1}$, which implies that

$$\begin{aligned}\lim_{t\to\infty} x(t+1) &= \begin{pmatrix} \mathbf{0}_{m\times m} & -(I - \bar{A}_{11}^{-1}\bar{B}_{11})^{-1}\bar{B}_{11}^{-1}\bar{B}_{12} \\ \mathbf{0}_{(n-m)\times m} & I_{n-m} \end{pmatrix} \begin{pmatrix} x_F(0) \\ x_R(0) \end{pmatrix} \\ &= \begin{pmatrix} -(I - \bar{A}_{11}^{-1}\bar{B}_{11})^{-1}\bar{B}_{11}^{-1}\bar{B}_{12}x_R(0) \\ x_R(0) \end{pmatrix}.\end{aligned}$$

Since

$$\begin{aligned}&(\bar{B}_{11} - \bar{A}_{11})\mathbf{1}_m + \bar{B}_{12}\mathbf{1}_{n-m} \\ &= \begin{pmatrix} 1 - \frac{\beta_1}{d_1}\sum_{j=2}^{m} w'_{1j} - \alpha_1 \\ \vdots \\ 1 - \frac{\beta_i}{d_i}\sum_{j=1,j\neq i}^{m} w'_{ij} - \alpha_i \\ \vdots \\ 1 - \frac{\beta_m}{d_m}\sum_{j=1}^{m-1} w'_{mj} - \alpha_m \end{pmatrix} + \begin{pmatrix} -\frac{\beta_1}{d_1}\sum_{j=m+1}^{n} w'_{1j} \\ \vdots \\ -\frac{\beta_i}{d_i}\sum_{j=m+1}^{n} w'_{ij} \\ \vdots \\ -\frac{\beta_m}{d_m}\sum_{j=m+1}^{n} w'_{mj} \end{pmatrix} \\ &= (1 - \frac{\beta_i}{d_i}\sum_{j=1,j\neq i}^{n} w'_{ij} - \alpha_i)_{m\times 1} \\ &= (1 - \beta_i - \alpha_i)_{m\times 1} \\ &= \mathbf{0}_{m\times 1}.\end{aligned} \quad (17)$$

i.e., $(\bar{B}_{11} - \bar{A}_{11})\mathbf{1}_m = -\bar{B}_{12}\mathbf{1}_{n-m}$. Multiplying both sides by $\bar{B}_{11}^{-1}$, we have $(I_m - \bar{B}_{11}^{-1}\bar{A}_{11})\mathbf{1}_m = -\bar{B}_{11}^{-1}\bar{B}_{12}\mathbf{1}_{n-m}$. Then, it is easy to be obtained that $-(I_m - \bar{B}_{11}^{-1}\bar{A}_{11})^{-1}\bar{B}_{11}^{-1}\bar{B}_{12}\mathbf{1}_n = \mathbf{1}_n$.

From Definition 4 in the Appendix, we know that the states of all followers converge to the convex hull spanned by those of leaders. This completes the proof of sufficiency.

(Necessity) If the network $\mathcal{G}'$ does not have a directed spanning forest, there exists at least one follower whose neighbors do not belong to the leader set. If the states of these followers are outside the convex hull spanned by those of leaders, then they would not enter the convex hull at any time. Therefore, the containment control will not be achieved. This completes the proof of necessity. □

6. Simulations

In order to show the validity of the mathematical results in Sections 4 and 5, some simulations will be provided in this section.

Example 1. Consider an MAS with six agents. The interaction topology $\mathcal{G}$ is shown in Fig. 1. It can be noted that $\mathcal{G}$ has a directed spanning tree. We model the interactions among the agents as a game $\mathbb{G}(\mathcal{V}, \Omega_i, C_i)$ which follows the definition in Section 3.

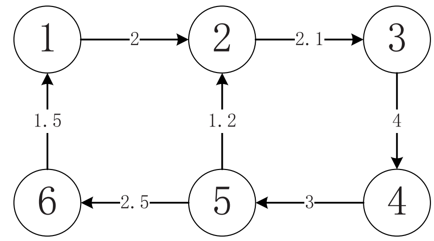

Fig. 1. A digraph $\mathcal{G}$.

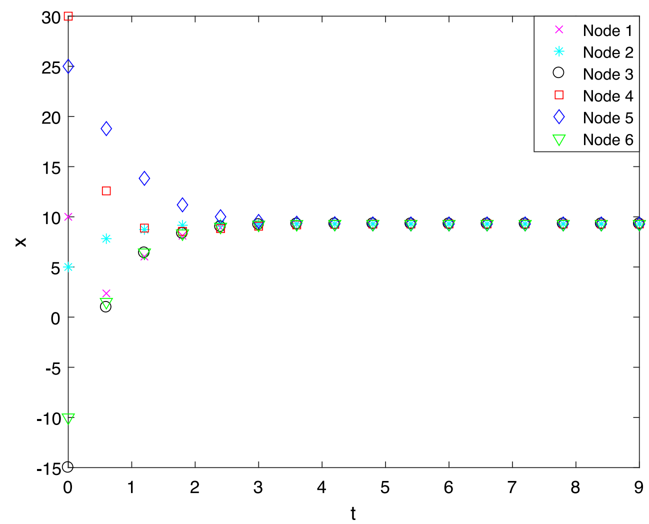

Fig. 2. State trajectories of all the agents with the digraph $\mathcal{G}$.

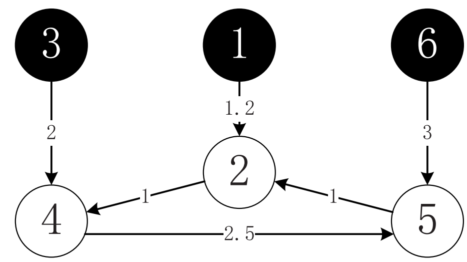

Fig. 3. A digraph $\mathcal{G}'$.

The cost functions of the agents are given in (2). Let $x(0) = [10, 5, -15, 20, 35, 30]^T$, $\alpha = [0.1, 0.2, 0.3, 0.4, 0.5, 0.6]^T$ and $\beta = [0.9, 0.8, 0.7, 0.6, 0.5, 0.4]^T$. The state trajectories of all the agents are shown in Fig. 2, which is consistent with Theorem 2.

Example 2. Consider an MAS with three leaders and three followers, 1, 3, 6 denote leaders and 2, 4, 5 denote followers. The interaction topology $\mathcal{G}'$ is shown in Fig. 3. Obviously, $\mathcal{G}'$ contains a directed spanning forest. We model the interactions among agents by a multi-player game $\mathbb{G}'(\mathcal{V}, \Omega_i, C_i)$. The definition of game $\mathbb{G}'(\mathcal{V}, \Omega_i, C_i)$ is similar to that in Section 3. The cost functions of leaders and followers are given in (2) and (3), respectively. Let $x(0) = [10, 24, 5, 19, -9, 0]^T$, $y(0) = [10, 30, 0, -9, -3, 10]^T$, $\alpha = [1, 0.6, 1, 0.5, 0.7, 1]^T$ and $\beta =$

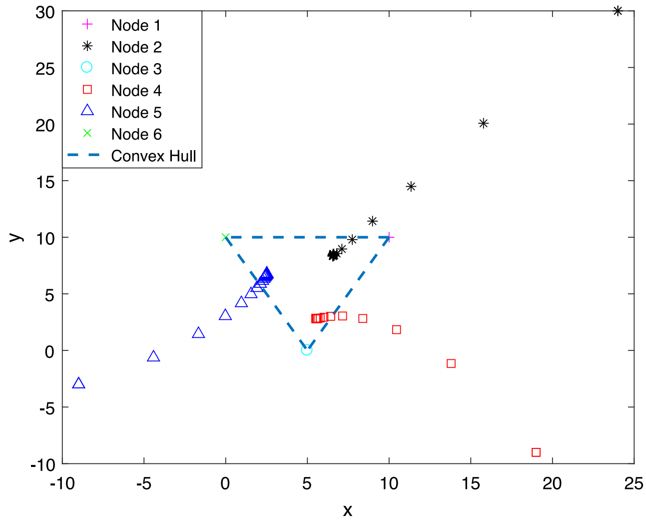

Fig. 4. State trajectories of all the agents with the digraph $\mathcal{G}'$.

$[0, 0.4, 0, 0.5, 0.3, 0]^T$. From Fig. 4, one can see that the trajectories of states of followers enter the convex hull spanned by those of leaders, which is consistent with Theorem 4.

7. Conclusion

In this paper, we have modeled the interactions among agents of MASs as a multi-player game. The cost function of each agent in the game has been defined. Followers and leaders have been described by different forms of cost functions. We have studied the coordination control of two kinds of MASs. The first kind of MAS has no leaders. The second one is composed of leaders and followers. We have drawn a conclusion that agents need to interact with each other and find a balance between maintaining its own state unchanged and closing the gap with other players to achieve coordination. Besides, some criteria have been obtained for realizing coordination control of two kinds of MASs. The future work will focus on the case when there are errors between the next states of agents and the Nash equilibrium solution. In addition, we will also consider the coordination control based on games with other forms of cost functions.

CRediT authorship contribution statement

Liqi Zhou: Conceptualization, Software, Writing – original draft. **Yuanshi Zheng:** Conceptualization, Methodology, Writing – review & editing, Supervision. **Qi Zhao:** Methodology, Software, Writing – review & editing. **Feng Xiao:** Methodology, Writing – review & editing, Supervision. **Yuling Zhang:** Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

Lemma 1 ([35]). *$\mathcal{B} = (b_{ij}) \in R^{n\times n}$ is said to be strictly diagonally dominant (SDD) if $|b_{ii}| > \sum_{j=1,j\neq i}^{n} |b_{ij}|$, $i \in \mathcal{I}_n$. The SDD matrix is invertible, i.e.* $\det \mathcal{B} \neq 0$.

Lemma 2 (*Gersghorin Circle Theorem [18]*). *Let $\mathcal{C} = (c_{ij}) \in \mathbb{R}^{n\times n}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be all eigenvalues of $\mathcal{C}$. We have $\lambda_i \in \bigcup_{i=1}^{n} G_i$, $i \in \mathcal{I}_n$, where $G_i = \{z \mid |z - c_{ii}| \leq \sum_{j=1,j\neq i}^{N} |c_{ij}|\}$.*

Lemma 3 ([36]). *Let $A = [a_{ij}]_{n\times n}$ be a stochastic matrix with positive diagonal elements. If the graph $\mathcal{G}_A$ associated with the matrix A has a spanning tree, then A is SIA, that is, $\exists\, \upsilon$, s.t. $\lim_{m\to\infty} A^m = \mathbf{1}_n \upsilon^T$. Here, υ is a nonnegative vector and $A^T\upsilon = \upsilon$ and $\mathbf{1}_n^T \upsilon = 1$.*

Lemma 4 ([37]). *Let A be weighted adjacency matrix of a graph $\mathcal{G}$. Then $a_K(i,j)$, the ij entry in the matrix A^K, denotes the weight of paths of length K from v_j to v_i.*

Definition 4 ([38]). A set $P \subset \mathbb{R}^n$ is convex if $(1-\eta)x + \eta y \in P$ for any $x, y \in P$ and any $\eta \in [0, 1]$. The convex hull $co\{x_1, \ldots, x_n\} = \{\sum_{i=1}^{n} \alpha_i x_i \mid \alpha_i \in \mathbb{R}, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1\}$ is the minimal convex set which contains all points in $X = \{x_1, \ldots, x_n\}$.

Lemma 5 ([39]). *Let L represent the Laplacian matrix of the corresponding graph $\mathcal{G}$. Assume that there are $m (m < n)$ followers and $n - m$ leaders in the network. The set of followers is $\{v_1, v_2, \ldots, v_m\}$ and the set of leaders is $\{v_{m+1}, v_{m+2}, \ldots, v_n\}$. Let $F = \{1, 2, \ldots, m\}$ and $R = \{m+1, m+2, \ldots, n\}$. Then, L can be partitioned as*

$$\begin{pmatrix} L_{FF} & L_{FR} \\ \mathbf{0}_{(n-m)\times m} & \mathbf{0}_{(n-m)\times(n-m)} \end{pmatrix}, \quad (18)$$

where $L_{FF} \in \mathbb{R}^{m\times m}$ and $L_{FR} \in \mathbb{R}^{m\times(n-m)}$. Then, L_{FF} is invertible if and only if the directed graph $\mathcal{G}$ has a directed spanning forest.

Lemma 6 ([40]). *Let $A \in \mathbb{R}^{n\times n}$ be partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (19)$$

with $A_{ii} \in \mathbb{R}^{n_i\times n_i}$, $i = 1, 2$ and $n_1 + n_2 = n$. The correspondingly partitioned presentation of A^{-1} is

$$A^{-1} = \begin{pmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & A_{11}^{-1}A_{12}(A_{21}A_{11}^{-1}A_{12} - A_{22})^{-1} \\ (A_{21}A_{11}^{-1}A_{12} - A_{22})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{pmatrix}, \quad (20)$$

where all the relevant inverses exist.

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