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Research article

Synchronous and asynchronous resilient impulsive control for group consensus of second-order multi-agent systems with communication delays[☆]

Yunru Zhu [a,*], Qi Huang [a], Jiahui Wu [a], Yuanshi Zheng [a], Huaizhu Wang [b,*]

[a] Shaanxi Key Laboratory of Space Solar Power Station System, School of Mechano-electronic Engineering, Xidian University, Xi'an 710071, PR China
[b] School of Intelligent Engineering and Technology, Ningxia University (Zhongwei), Zhongwei, Ningxia, 755000, PR China

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ABSTRACT

This paper studies the resilient group consensus of continuous-time second-order multi-agent systems (MASs) with malicious agents. Adopting the idea that each normal agent ignores the most extreme values from neighbors, synchronous resilient impulsive algorithm based on sampled data is proposed for normal agents with bounded communication delays to achieve group consensus. Meanwhile, asynchronous resilient impulsive algorithm is also proposed for MASs where each agent has its own time clock. Sufficient topological conditions are obtained for solving resilient group consensus under synchronous and asynchronous settings, respectively. Numerical examples are provided to illustrate the effectiveness of the theoretical results.

1. Introduction

As an important branch of artificial intelligence, distributed multi-agent systems (MASs) have attracted extensive attention in light of their autonomy, scalability and coordination. As the most fundamental and important problem, consensus of MASs has evoked great enthusiasm of researchers. In the past few decades, there have been countless studies on consensus. The research on the consensus problem can be traced back to [1]. Jadbabaie et al. [2] use the graph theory to analyze the consensus for the first time. So far, the consensus problem of MASs also takes communication delays [3], quantization [4], event triggering control [5], hybrid dynamics [6], saturation [7] etc. into account.

The deep integration of information and physical systems in networked control systems brings a lot of advantages, but also makes the systems more susceptible to malicious attacks. False information injected by attackers might cause physical faults and even lead to the collapse of the whole system. However, it is challenging to identify the malicious agents in distributed MASs. Therefore, resilient consensus gains more and more attention, which aims to design distributed algorithms to guarantee the consensus of normal agents in the network with malicious agents.

To eliminate or mitigate the effects of malicious agents, ignoring the suspicious values is a most adopted idea. The largest and smallest f neighbors' values were removed for approximate resilient consensus in a complete graph with at most f malicious agents [8]. This algorithm inspired a family of algorithms, called the Mean-Subsequence Reduced (MSR) algorithms [9]. To extend the work in [8] to partially connected networks, a modified MSR algorithm, named W-MSR algorithm, was presented [10]. Moreover, the notion of network robustness was introduced to characterize the necessary network topology. To solve resilient consensus of second-order MASs, an adapted form of W-MSR algorithm, called DP-MSR algorithm, was proposed [11,12]. The results in [10] was extended to systems with locally bounded [11] and globally bounded [12] malicious agents, respectively. Yan et al. [13] proposed the resilient impulsive algorithm for second-order MASs. Sufficient topological conditions were derived by using the property of Sarymsakov matrices. Saldana et al. [14] introduced SW-MSR algorithm for MASs with time-varying graphs. Adopting the SW-MSR algorithm, resilient leader-follower consensus was solved [15]. To relax the constraints on the topology, the trusted agents were utilized in SW-MSR [16]. Mustafa and Modares [17] proposed an adaptive resilient algorithm to mitigate attacks on sensors and actuators. Zhao et al. [18] developed a distributed attack isolation algorithm for MASs with general higher order dynamics. In addition to the above works, some researchers also considered the resilient consensus under different contexts, such as event-trigged communication [19,20], quantization [21], time-delays [22], switched MASs [23], heterogeneous MASs [24], etc.

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* Corresponding authors.
E-mail addresses: zhuuruu@163.com (Y. Zhu), wanghz78@qq.com (H. Wang).

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Due to the changes of environments or cooperative tasks, the agents may not agree on the same value. Thus, a lot of works devoted to group consensus from various perspectives [25–35]. By using double-tree-form transformation, some sufficient conditions were presented for first-order MASs to reach group consensus under switching topology [25]. A novel protocol for group consensus was designed for heterogeneous MASs [26]. Based on the state prediction scheme, An et al. [27] investigated group consensus for MASs with communication delays. Ren et al. [30] obtained sufficient condition to solve H_∞ group consensus. Altafini [32] introduced the concept of bipartite consensus, which can be seen as a special group consensus. MASs working in an open and harsh environment are vulnerable to attack. Due to malicious attacks, some agents may become non-cooperative. For example, if a multi-army vehicle is attacked on the battlefield, it may become uncontrolled. This vehicle can be seen as a malicious agent. Recently, some researchers have paid attention to resilient group consensus. Oksuz and Akar [33] studied resilient group consensus of first-order discrete-time MASs, where the network topology is structured into multiple layers. Sufficient topological conditions were presented under the proposed MSR-like algorithms. Shang investigated resilient group consensus problem for first-order hybrid [34] and switched [35] MASs, respectively.

Considering that many real-world systems, e.g., harmonic oscillators [36] and vehicles are modeled by second-order dynamics, we study the resilient group consensus problem for second-order MASs under the synchronous and asynchronous settings, respectively. The presence of malicious agents, communication delays, asynchronous clock and control make the convergence analysis of the MASs very challenging. With the merits of fast transient and smaller control effort, impulsive control scheme has been widely used in (group) consensus of MASs [37–41]. In light of W-MSR algorithm, resilient impulsive algorithms are proposed to mitigate the effects of malicious agents. The main contributions of this paper are as follows: (1) Compared with the related work in [33–35], where the first-order MASs are studied, this paper considers MASs with second-order agents and communication delays, which can describe more complicated applications. (2) The resilient impulsive algorithms for consensus in [13] is modified for resilient group consensus, which cover consensus as a special case. Compared with the work in [13], this paper also considers the communication delays and asynchronous setting, where each agent has its own clock. (3) Under the proposed algorithm, some sufficient conditions related to the network topology are established for solving resilient group consensus with communication delays.

This paper is organized as follows. Problem setup is given in Section 2. Section 3 is devoted to resilient group consensus under the synchronous setting. Section 4 focuses on resilient group consensus under the asynchronous setting. Section 5 gives the simulation examples. Section 6 is the conclusion.

Notation: Let $\mathbb{R}$ be the set of real numbers. $\mathbb{N}$ and $\mathbb{N}^+$ are used to denote the sets of non-negative integers and positive integers, respectively. $co\{x_1, x_2, \dots, x_n\}$ represents the set $\{x|x = \sum_{i=1}^n \mu_i x_i, \sum_{i=1}^n \mu_i = 1, \mu_i > 0, i = 1, \dots, n\}$. $\mathcal{I}_d$ and $\mathcal{I}_d^+$ denote the sets $\{0, 1, 2, \dots, d\}$ and $\{1, 2, \dots, d\}$, respectively.

2. Problem setup

2.1. Graph theory

Given a digraph $G = (V, E)$ with the node set $V = \mathcal{I}_n^+$, and edge set $E \subset V \times V$. For node i, the set of neighbors is denoted as $N_i = \{j \in V : (j, i) \in E\}$. The adjacency matrix $A = [a_{ij}]_{n\times n}$ is defined as $a_{ij} > 0$ if $(j, i) \in E$, and otherwise $a_{ij} = 0$ with $\sum_{j=1}^n a_{ij} \le 1$. The path from node i_1 to i_p is a sequence $\{i_1, i_2, \dots, i_p\}$ with $(i_j, i_{j+1}) \in V, j \in \mathcal{I}_{p-1}^+$. The graph has a spanning tree if there exists a node, called root, from which there is a path to every other node in the graph. A graph $G_1 = (V_1, E_1)$ is said to be the subgraph of $G = (V, E)$ if $V_1 \subseteq V$ and $E_1 \subseteq E$ contains directed edges within V_1. The adjacency matrix A_1 associated with G_1 inherits the adjacency matrix A of G.

Definition 1 (*Primary Layer Subgraphs [29]*)**.** Consider a digraph $G = (V, E)$. There exist l_1 disjoint subsets $V_{1,m} \subseteq V$, $m \in \mathcal{I}_{l_1}^+$ such that each $V_{1,m}$ is the largest possible subset that has a spanning tree for its subgraph $G_{1,m}$ and for all $i \in V_{1,m}$ and $j \notin V_{1,m}$, it has $(j, i) \notin E$. The subgraph $G_{1,m}$ is called the primary layer subgraph of G.

For graph $G = (V, E)$ with l_1 primary layer subgraphs, $\bar{V}$ is used to denote set of node which are not in the primary layer subgraphs, that is $\bar{V} = V \setminus \bigcup_{m=1}^{l_1} V_{1,m}$.

Definition 2 (*Secondary Layer Subgraphs [29]*)**.** Consider a digraph $G = (V, E)$. There exist l_2 disjoint subsets $V_{2,m} \subseteq \bar{V}$, $m \in \mathcal{I}_{l_2}^+$ such that each $V_{2,m}$ has a spanning tree for its subgraph $G_{2,m}$ and there exists exactly a node $i \in V_{2,m}$, which is the root of the spanning tree of $G_{2,m}$, satisfying that there exist at least two nodes p and q outside $V_{2,m}$ in two different subgraphs (either primary or secondary layer) such that $(p, i) \in E$ and $(q, i) \in E$ and for all $j \in V_{2,m} \setminus i$ and $r \in V \setminus V_{2,m}$ it has $(r, j) \notin E$. The subgraph $G_{2,m}$ is called the secondary layer subgraph of G.

Definition 3 (*[10]*)**.** Given a digraph $G = (V, E)$ and $r \in \mathbb{N}$, if for any pair of nonempty and disjoint subsets $V_1, V_2 \subseteq V$, at least one of them $V_i, i \in \{1, 2\}$ has a node j such that $|N_j \setminus V_i| \ge r$, then we say that G is r-robust.

2.2. System model

Consider a MAS consist of n agents with communication graph $G = (V, E)$. The agents in V are partitioned into two disjoint sets V_N and V_M, with $V = V_N \cup V_M$, where V_N and V_M denote the set of normal and malicious agents, respectively. The dynamics of agents is described by:

$$\begin{cases} \dot{x}_i(t) = v_i(t), \dot{v}_i(t) = u_i(t), & i \in V_N \\ x_i(t) : \text{arbitary}, & i \in V_M \end{cases} \tag{1}$$

where $x_i(t), v_i(t) \in \mathbb{R}$ are the position and velocity of agent i, respectively, and $u_i(t) \in \mathbb{R}$ is the predefined control input.

Definition 4. The MAS (1) with communication graph $G = (V, E)$ is said to reach resilient group consensus if the following conditions are satisfied:

- Group Agreement: There exist m disjoint nonempty subsets $V_s, s \in \mathcal{I}_m^+$ satisfying $\bigcup_{s=1}^m V_s = V$ such that for any initial states, it has $\lim_{t\to\infty} x_i(t) = c_s$, where $c_s \in \mathbb{R}$ is a constant and $i \in V_s \cap V_N$.
- Validity: There exists an interval H_s such that $x_i(t) \in H_s$ for all $i \in V_s \cap V_N$.

3. Resilient group consensus under the synchronous setting

Assume that there exists l_1 primary layers subgraphs $G_{1,1} = (V_{1,1}, E_{1,1})$, ..., $G_{1,l_1} = (V_{1,l_1}, E_{1,l_1})$ and l_2 secondary layers subgraphs $G_{2,1} = (V_{2,1}, E_{2,1}), \dots, G_{2,l_2} = (V_{2,l_2}, E_{2,l_2})$ in graph $G = (V, E)$. Let $N_i^{s,m} = N_i \bigcap V_{s,m}$, $s \in \mathcal{I}_2^+$, $m \in \mathcal{I}_{l_s}^+$ be the set of agent i's neighbors within subgraph $G_{s,m}$. In each subgraph $G_{s,m}$, assume that there are $n_{s,m}$ normal agents and at most $f_{s,m}$ malicious agents in $N_i^{s,m}$.

3.1. Synchronous resilient impulsive algorithm

In this section, it is assumed that each agent can update and send its state values at identical sampling instants. $\{t_k\}|_{k=0}^{\infty}$ is the sequence of sampling time and satisfies that $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. In consideration of communication delays, at time t_k, the information received by agent i from j is $x_j(t_k - \tau_{ij}(t_k))$, where $\tau_{ij}(t_k)$ is the communication delay in the edge (j, i). Since there are malicious agents, in light of the W-MSR algorithm, the following synchronous resilient impulsive algorithm is designed for solving resilient group consensus:

Step 1: At $t = t_k$, each normal agent i, $i \in V_{s,m}$ sorts the most recently received neighbors' state values $x_j(t_k - \tau_{ij}(t_k))$, $j \in N_i$ in non-increasing order.

Step 2: If agent i is the root of $G_{2,m}$, it performs a two-round removal procedure. In the first round, agent i removes $f_{s,m}$ largest values in $N_i^{s,m}$ that are higher than $x_i(t_k)$. If there are less than $f_{s,m}$ values that are higher than $x_i(t_k)$ in $N_i^{s,m}$, then remove all of them. The similar process is adopted for the values that are lower than $x_i(t_k)$. In the second round, let $\Theta_i(t_k)$ represent the set of remaining values in $N_i^{s,m}$ after the first-round of removal. Remove all remaining values that are higher than the largest value or lower than the smallest value in the set $\{\Theta_i(t_k) \cup x_i(t_k)\}$.

Otherwise, agent i remove $f_{s,m}$ largest values in N_i that are higher than $x_i(t_k)$. If there are less than $f_{s,m}$ values higher than $x_i(t_k)$, remove all the values. The similar process is applied to the values that are lower than $x_i(t_k)$.

Step 3: Let $R_i(t_k)$ denote the set of agents removed in the above procedure at t_k. The input $u_i(t)$ is given by:

$$u_i(t) = \left(k_1 \sum_{j\in N_i\backslash R_i(t_k)} a_{ij}(x_j(t_k - \tau_{ij}(t_k)) - x_i(t_k)) - k_2(x_i(t_k) - x_i(t_{k-1})) \right) \delta(t - t_k) \tag{2}$$

where $k_1, k_2 > 0$ are the control gains, $\delta(\cdot)$ represents the Dirac function. $h_k = t_k - t_{k-1}$ is the sampling interval. We assume $\underline{h} \le h_k \le \bar{h}$. Since each agent only measures and sends its information at the discrete instance t_k, the neighbor's position state value $x_j(t_k - \tau_{ij}(t_k))$ received by agent i can be rewritten as $x_j(t_{k-d_{ij}(k)})$, where $d_{ij}(k) \in \mathbb{N}$. Assume the communication delays are bounded, that is $0 \le d_{ij}(k) \le d$. Let $\bar{d} = max\{1, d\}$.

Remark 1. In step1 and step 2, normal agent i will sort and remove the extreme values received from neighbors, which can be realized by using the standard procedures such as Quicksort with a time complexity of $O(n^2)$. In step 3, the impulsive input $u_i(t)$ using the remaining values is applied on agent i. In impulsive control scheme, only sampled data are utilized, but different from the traditional sample-and-hold case in [11,12], there is no requirement on the zero-order holder and $u_i(t)$ is only implemented at t_k by impulsive actuator, which will cause the velocity state of second-order agent to jump at t_k.

3.2. Convergence analysis

From (1) and (2), the closed-loop dynamics of agent i can be written as:

$$\begin{cases} \dot{x}_i(t) = v_i(t), \dot{v}_i(t) = 0, & t \in (t_k, t_{k+1}], \\ \Delta v_i(t_k) = k_1 \sum\limits_{j\in N_i\backslash R_i(t_k)} a_{ij}(x_j(t_k - \tau_{ij}(t_k)) - x_i(t_k)) \\ \quad - k_2(x_i(t_k) - x_i(t_{k-1})), \end{cases} \tag{3}$$

where $\Delta v_i(t_k) = v_i(t_k^+) - v_i(t_k)$, $v_i(t_k^+) = \lim_{t\to t_k^+} v_i(t)$. Thus, it has

$$x_i(t_{k+1}) = x_i(t_k) + h_{k+1} v_i(t_k^+), \tag{4}$$

and

$$\begin{aligned} v_i(t_k^+) &= v_i(t_k) + k_1 \sum_{j\in N_i\backslash R_i(t_k)} a_{ij}(x_j(t_{k-d_{ij}(k)}) - x_i(t_k)) \\ &\quad - k_2(x_i(t_k) - x_i(t_{k-1})) \\ &= k_1 \sum_{j\in N_i\backslash R_i(t_k)} a_{ij}(x_j(t_{k-d_{ij}(k)}) - x_i(t_k)) \\ &\quad + (\frac{1}{h_k} - k_2)(x_i(t_k) - x_i(t_{k-1})). \end{aligned} \tag{5}$$

Substituting (5) into (4) gives:

$$\begin{aligned} x_i(t_{k+1}) =& (1 + \frac{h_{k+1}}{h_k} - k_2 h_{k+1} - k_1 h_{k+1} \sum_{j\in N_i\backslash R_i(t_k)} a_{ij}) x_i(t_k) \\ &+ h_{k+1} k_1 \sum_{j\in N_i\backslash R_i(t_k)} a_{ij} x_j(t_{k-d_{ij}(k)}) \\ &+ (k_2 h_{k+1} - \frac{h_{k+1}}{h_k}) x_i(t_{k-1}) \end{aligned} \tag{6}$$

Theorem 1. *Suppose $\frac{1}{\underline{h}} < k_2 < \frac{2}{\bar{h}} - k_1$, then the second-order MAS (1) under the synchronous resilient impulsive algorithm can reach resilient group consensus if each subgraph $G_{s,m}$, $s \in \mathcal{I}_2^+$, $m \in \mathcal{I}_{l_s}^+$ is $(2f_{s,m} + 1)$-robust.*

Proof. For $k \ge 0$, define $\underline{x}_{s,m}(t_k) = \min_{j\in V_{s,m}\cap V_N, \tau\in\mathcal{I}_{\bar{d}}} x_j(t_{k-\tau})$, $\bar{x}_{s,m}(t_k) = \max_{j\in V_{s,m}\cap V_N, \tau\in\mathcal{I}_{\bar{d}}} x_j(t_{k-\tau})$ and $H_{s,m}(t_k) = [\underline{x}_{s,m}(t_k), \bar{x}_{s,m}(t_k)]$. With the assumption $\frac{1}{\underline{h}} < k_2 < \frac{2}{\bar{h}} - k_1$, the coefficients of system (6) are non-negative and less than 1. Let γ denote the smallest value of all non-zero coefficients. Thus, for $i \in V_N$, $x_i(t_{k+1}) \in co\{x_i(t_k), x_i(t_{k-1}), x_j(t_{k-d_{ij}(k)}), j \in N_i\backslash R_i(t_k)\}$.

If normal agent $i \in V_{2,m} \bigcap V_N$ is the root, it has $x_j(t_{k-d_{ij}(k)}) \in H_{2,m}(t_k), j \in N_i^{2,m}\backslash R_i(t_k)$, since i's any malicious neighbors in $G_{2,m}$ with values outside the interval $H_{2,m}(t_k)$ have been ignored in the first round of removal. It follows that the values in $\Theta_i(t_k)$ are in the interval $H_{2,m}(t_k)$. Then, we have $x_j(t_{k-d_{ij}(k)}) \in H_{2,m}(t_k), j \in N_i\backslash R_i(t_k)$ because of the second round of removal. Therefore, we obtain that $x_i(t_{k+1}) \in H_{2,m}(t_k)$. If $i \notin V_{2,s}$ or $i \in V_{2,s}$ is not the root, it has $N_i = N_i^{s,m}$. Thus, it has $x_j(t_{k-d_{ij}(k)}) \in H_{s,m}(t_k), j \in N_i\backslash R_i(t_k)$ since the $f_{s,m}$ largest and smallest values at time t_k have been removed. It follows that $x_i(t_{k+1}) \in H_{s,m}(t_k)$. Therefore, it can be concluded that $\bar{x}_{s,m}(t_{k+1}) \le \bar{x}_{s,m}(t_k)$, $\underline{x}_{s,m}(t_{k+1}) \ge \underline{x}_{s,m}(t_k)$, and each normal agent $x_i(t_k) \in H_{s,m}(t_0)$, $k \ge 0$.

Since monotone functions $\bar{x}_{s,m}(t_k)$ and $\underline{x}_{s,m}(t_k)$ are bounded, the limits of these two functions exist. We define that $\bar{x}_{s,m}^* := \lim_{k\to+\infty} \bar{x}_{s,m}(t_k) \le \underline{x}_{s,m}^* := \lim_{k\to+\infty} \underline{x}_{s,m}(t_k)$. Next, we will prove that the resilient group consensus by proving that $\bar{x}_{s,m}^* = \underline{x}_{s,m}^*$.

By contradiction, suppose that $\bar{x}_{s,m}^* > \underline{x}_{s,m}^*$. Choose $\epsilon_0 > 0$ and $\epsilon > 0$ to satisfy that $\underline{x}_{s,m}^* + \epsilon_0 < \bar{x}_{s,m}^* - \epsilon_0$ and $\epsilon < \frac{\gamma^{(\bar{d}+1)n_{s,m}}\epsilon_0}{1-\gamma^{(\bar{d}+1)n_{s,m}}}$. Define the sequence $\{\epsilon_l\}$ via $\epsilon_{l+1} = \gamma\epsilon_l - (1-\gamma)\epsilon$, $l \in \mathcal{I}_{(\bar{d}+1)n_{s,m}-1}$. Obviously, $\epsilon_{l+1} \le \epsilon_l$ for all l. Moreover, $\epsilon_{(\bar{d}+1)n_{s,m}} = \gamma^{(\bar{d}+1)n_{s,m}}\epsilon_0 - (1-\gamma^{(\bar{d}+1)n_{s,m}})\epsilon > 0$. Take k_ϵ such that $\bar{x}_{s,m}(t_k) < \bar{x}_{s,m}^* + \epsilon$ and $\underline{x}_{s,m}(t_k) > \underline{x}_{s,m}^* - \epsilon$ for $k \ge k_\epsilon$.

Define $\Phi(t_{k_\epsilon+l}, \epsilon_l) = \{j \in V_{s,m} \bigcap V_N : x_j(t_{k_\epsilon+l}) > \bar{x}_{s,m}^* - \epsilon_l\}$ and $\phi(t_{k_\epsilon+l}, \epsilon_l) = \{j \in V_{s,m} \bigcap V_N : x_j(t_{k_\epsilon+l}) < \underline{x}_{s,m}^* + \epsilon_l\}$. It is obvious that the sets $\Phi(t_{k_\epsilon+l}, \epsilon_l) \bigcap \phi(t_{k_\epsilon+l}, \epsilon_l) = \emptyset$. If the disjoint sets $\Phi(t_{k_\epsilon+l}, \epsilon_l)$ and $\phi(t_{k_\epsilon+l}, \epsilon_l)$ are nonempty, considering a normal agent $j \notin \Phi(t_{k_\epsilon+l}, \epsilon_l)$, that is $x_j(t_{k_\epsilon+l}) \le \bar{x}^*_{s,m} - \epsilon_l$, it has

$$\begin{aligned} x_j(t_{k_\epsilon+l+1}) &\le (1-\gamma)\bar{x}_{s,m}(t_{k_\epsilon+l}) + \gamma(\bar{x}^*_{s,m} - \epsilon_l) \\ &\le (1-\gamma)(\bar{x}^*_{s,m} + \epsilon) + \gamma(\bar{x}^*_{s,m} - \epsilon_l) \\ &= \bar{x}^*_{s,m} - \epsilon_{l+1}, \end{aligned} \tag{7}$$

that is $j \notin \Phi(t_{k_\epsilon+l+1}, \epsilon_{l+1})$. Similarly, we can show that if $j \notin \phi(t_{k_\epsilon+l}, \epsilon_l)$, then $j \notin \phi(t_{k_\epsilon+l+1}, \epsilon_{l+1})$.

By the $(2f_{s,m}+1)$-robustness of the subgraph $G_{s,m}$, there exists an agent in $\Phi(t_{k_\epsilon+l}, \epsilon_l)$ or $\phi(t_{k_\epsilon+l}, \epsilon_l)$, which has at least $2f_{s,m}+1$ neighbors outside of its own set. Suppose the normal agent j, $j \in \Phi(t_{k_\epsilon+l}, \epsilon_l)$ has this property. Due to at most $f_{s,m}$ agents in $N_j^{s,m}$ are malicious, there are at least $f_{s,m}+1$ normal agents in $N_j^{s,m}$ are outside $\Phi(t_{k_\epsilon+l}, \epsilon_l)$. By the argument above, these normal agents will not be in $\Phi(t_{k_\epsilon+l+m}, \epsilon_{l+m})$ for $0 \le m \le \bar{d}$. According to resilient impulsive algorithm, at least one of these normal neighbors' values at current or previous time steps, upper bounded by $\bar{x}^*_{s,m} - \epsilon_{l+\bar{d}}$, will be used by agent j at time $t_{k_\epsilon} + l + \bar{d}$. Hence, we have

$$\begin{aligned} x_j(t_{k_\epsilon+l+\bar{d}+1}) &\le (1-\gamma)(\bar{x}^*_{s,m} + \epsilon) + \gamma(\bar{x}^*_{s,m} - \epsilon_{l+\bar{d}}) \\ &= \bar{x}^*_{s,m} - \epsilon_{l+\bar{d}+1}, \end{aligned} \tag{8}$$

that is $j \notin \Phi(t_{k_\epsilon+l+\bar{d}+1}, \epsilon_{l+\bar{d}+1})$. Similarly, if agent $j \in \phi(t_{k_\epsilon+l}, \epsilon_l)$ has at least $f_{s,m}+1$ normal neighbors in $V_{s,m}$ outside $\phi(t_{k_\epsilon+l}, \epsilon_l)$, it has $j \notin \phi(t_{k_\epsilon+l+\bar{d}+1}, \epsilon_{l+\bar{d}+1})$.

Follow the step above, since there are $n_{s,m}$ normal agents in $G_{s,m}$, there exists a finite-time $T_{s,m} \le (\bar{d}+1)(n_{s,m}-1)$ such that $\Phi(t_{k_\epsilon+T_{s,m}}, \epsilon_{T_{s,m}}) = \emptyset$ or $\phi(t_{k_\epsilon+T_{s,m}}, \epsilon_{T_{s,m}}) = \emptyset$. Assume that $\Phi(t_{k_\epsilon+T_{s,m}}, \epsilon_{T_{s,m}}) = \emptyset$. According to the analysis above, it has $\Phi(t_{k_\epsilon+(\bar{d}+1)(n_{s,m}-1)+l}, \epsilon_{(\bar{d}+1)(n_{s,m}-1)+l}) = \emptyset$, $l \in \mathcal{I}_{\bar{d}+1}$. It follows that $\bar{x}_{s,m}(t_{k_\epsilon+(\bar{d}+1)n_{s,m}}) \le \bar{x}^*_{s,m} - \epsilon_{(\bar{d}+1)n_{s,m}} < \bar{x}^*_{s,m}$. However, $\bar{x}_{s,m}(t_{k_\epsilon+(\bar{d}+1)n_{s,m}}) \ge \bar{x}^*_{s,m}$ due to the nonincreasing function $\bar{x}_{s,m}(t_k)$ with a limit $\bar{x}^*_{s,m}$. It is a contradiction. Thus, we have $\bar{x}^*_{s,m} = \underline{x}^*_{s,m}$. The proof is complete. □

4. Resilient group consensus under the asynchronous setting

In the synchronous algorithm in Section 3, all agents update their states at the same discrete times, which requires clock synchronization. As we known, clock synchronization requires a lot of communication and computing resources. Thus, this section will consider the resilient group consensus in an asynchronous setting.

4.1. Asynchronous resilient impulsive algorithm

We assume each agent has its own time clock. For agent i, $\Pi_i = \{t_k^i\}|_{k=1}^{\infty}$ denotes the sequence of sampling times and satisfies $0 = t_0^i < t_1^i < t_2^i < \cdots < t_k^i < t_{k+1}^i < \cdots$ and $\lim_{k\to\infty} t_k^i = \infty$. At sampling time t_k^i, normal agent i measures and sends its state $x_i(t_k^i)$ to neighbors, and updates its velocity using the information available. Let $\xi^j(t) = \max\{k | k \in \mathbb{N}, t_k^j \le t\}$ and $t^j_{\xi^j(t)}$ denote the latest sampling time of agent j at time t. The asynchronous resilient impulsive algorithm is as follows:

Step 1: At $t = t_k^i$, normal agent i, $i \in V_{s,m}$ sorts the most recently received neighbors' state values $x_j(t^j_{\xi^j(t_k^i)})$, $j \in N_i$ in non-increasing order.

Step 2: Same with the step 2 in synchronous algorithm.

Step 3: Let $R_i(t_k^i)$ denote the set of agents removed in the above procedure at t_k^i. The input $u_i(t)$ is given by:

$$\begin{aligned} u_i(t) = &\left(k_1 \sum_{j \in N_i \setminus R_i(t_k^i)} a_{ij}(x_j(t^j_{\xi^j(t_k^i)}) - x_i(t_k^i)) - k_2(x_i(t_k^i) - x_i(t_{k-1}^i)) \right) \\ &\times \delta(t - t_k^i). \end{aligned} \tag{9}$$

The sampling intervals $h_k^i = t_k^i - t_{k-1}^i, k \in \mathbb{N}^+$, is assumed to satisfy that $0 < \underline{h} \le h_k^i \le \bar{h}$, where $\underline{h}$ and $\bar{h}$ are real numbers.

4.2. Convergence analysis

Substituting (9) into (1), one has:

$$\begin{aligned} x_i(t_{k+1}^i) = &\left(1 + \frac{h_{k+1}^i}{h_k^i} - k_2 h_{k+1}^i - k_1 h_{k+1}^i \sum_{j \in N_i \setminus R_i(t_k^i)} a_{ij}\right) x_i(t_k^i) \\ &+ k_1 h_{k+1}^i \sum_{j \in N_i \setminus R_i(t_k^i)} a_{ij} x_j(t^j_{\xi^j(t_k^i)}) \\ &+ \left(k_2 h_{k+1}^i - \frac{h_{k+1}^i}{h_k^i}\right) x_i(t_{k-1}^i). \end{aligned} \tag{10}$$

For the convenience of analysis, define the set $\{t_k\}|_{k=0}^{\infty} = \{t_k^i \mid i \in V_N, k \in \mathbb{N}\}$, which satisfies that $t_k < t_{k+1}$.

Define $\tilde{x}_i(t_k) = x_i(t^i_{\xi^i(t_k)})$. Assume $t_s = t_k^i$, $t_{s+l_1} = t_{k+1}^i$ and $t_{s-l_2} = t_{k-1}^i$, where l_1 and l_2 are the functions of s. Then, it has

$$\begin{cases} x_i(t_{k-1}^i) = \tilde{x}_i(t_{s-l_2}) = \tilde{x}_i(t_{s-l_2+1}) = \cdots = \tilde{x}_i(t_{s-1}), \\ x_i(t_k^i) = \tilde{x}_i(t_s) = \tilde{x}_i(t_{s+1}) = \cdots = \tilde{x}_i(t_{s+l_1-1}). \end{cases} \tag{11}$$

According to (11), rewrite (10) as:

$$\begin{aligned} \tilde{x}_i(t_{s+l_1}) = &\left(1 + \frac{h^i_{\xi^i(t_{s+l_1})}}{h^i_{\xi^i(t_{s+l_1})-1}} - k_2 h^i_{\xi^i(t_{s+l_1})} - k_1 h^i_{\xi^i(t_{s+l_1})} \sum_{j \in N_i \setminus R_i(t_s)} a_{ij}\right) \\ &\times \tilde{x}_i(t_s) \\ &+ k_1 h^i_{\xi^i(t_{s+l_1})} \sum_{j \in N_i \setminus R_i(t_s)} a_{ij} \tilde{x}_j(t_s) + \left(k_2 h^i_{\xi^i(t_{s+l_1})} - \frac{h^i_{\xi^i(t_{s+l_1})}}{h^i_{\xi^i(t_{s+l_1})-1}}\right) \\ &\times \tilde{x}_i(t_{s-l_2}) \\ = &\left(1 + \frac{h^i_{\xi^i(t_{s+l_1})}}{h^i_{\xi^i(t_{s+l_1})-1}} - k_2 h^i_{\xi^i(t_{s+l_1})} - k_1 h^i_{\xi^i(t_{s+l_1})} \sum_{j \in N_i \setminus R_i(t_s)} a_{ij}\right) \\ &\times \tilde{x}_i(t_{s+l_1-1}) \\ &+ k_1 h^i_{\xi^i(t_{s+l_1})} \sum_{j \in N_i \setminus R_i(t_s)} a_{ij} \tilde{x}_j(t_s) + \left(k_2 h^i_{\xi^i(t_{s+l_1})} - \frac{h^i_{\xi^i(t_{s+l_1})}}{h^i_{\xi^i(t_{s+l_1})-1}}\right) \\ &\times \tilde{x}_i(t_{s-1}). \end{aligned} \tag{12}$$

For $i \in V_{s,m}$, let $\tilde{d}(s,m)$ be the upper bound for the number of elements in the set $\{t_j : t_j \in [t_k^i, t_{k+1}^i)\}$. Since $\underline{h} \le h_k^i \le \bar{h}$, any normal agent $j, j \in V_{s,m}, j \ne i$, updates its own state information at most $\lfloor \bar{h}/\underline{h} \rfloor$ times in time interval (t_k^i, t_{k+1}^i). Since there are at most $n_{s,m}-1$ possible j, we have $\tilde{d}(s,m) = (\lfloor \bar{h}/\underline{h} \rfloor + 1)(n_{s,m}-1)+1$. Let $\bar{d} = max\{\tilde{d}(s,m) : s \in \mathcal{I}_2^+, m \in \mathcal{I}_{l_s}^+\}$.

Let $t_{s+l_1} = t_{k+1}$. Based on (12), for $t_{k+1} \in \Pi_i$, it has

$$\begin{aligned} \tilde{x}_i(t_{k+1}) = &\left(1 + \frac{h^i_{\xi^i(t_{k+1})}}{h^i_{\xi^i(t_{k+1})-1}} - k_2 h^i_{\xi^i(t_{k+1})} \right. \\ &\left. - k_1 h^i_{\xi^i(t_{k+1})} \sum_{j \in N_i \setminus R_i(t_{k-l_1+1})} a_{ij}\right) \tilde{x}_i(t_k) \\ &+ k_1 h^i_{\xi^i(t_{k+1})} \sum_{j \in N_i \setminus R_i(t_{k-l_1+1})} a_{ij} \tilde{x}_j(t_{k-l_1+1}) \end{aligned}$$

$$
\begin{aligned}
&+\left(k_2 h^i_{\xi^i(t_{k+1})}-\frac{h^i_{\xi^i(t_{k+1})}}{h^i_{\xi^i(t_{k+1})-1}}\right)\tilde{x}_i(t_{k-l_1})\\
&=\left(1+\frac{h^i_{\xi^i(t_{k+1})}}{h^i_{\xi^i(t_{k+1})-1}}-k_2 h^i_{\xi^i(t_{k+1})}\right.\\
&\left.-k_1 h^i_{\xi^i(t_{k+1})}\sum_{j\in N_i\backslash R_i(t_{k-d_i(k)})} a_{ij}\right)\tilde{x}_i(t_k)\\
&+k_1 h^i_{\xi^i(t_{k+1})}\sum_{j\in N_i\backslash R_i(t_{k-d_i(k)})} a_{ij}\tilde{x}_j(t_{k-d_i(k)})\\
&+\left(k_2 h^i_{\xi^i(t_{k+1})}-\frac{h^i_{\xi^i(t_{k+1})}}{h^i_{\xi^i(t_{k+1})-1}}\right)\tilde{x}_i(t_{k-d_i(k)-1}),
\end{aligned}
\tag{13}
$$

where $0\le d_i(k)\le \bar{d}-1$. Otherwise,

$$
\tilde{x}_i(t_{k+1})=\tilde{x}_i(t_k). \tag{14}
$$

For normal agent $i\in V_{s,m}$, there exists $p\in\mathbb{N}$ such that $x_i(t^i_k)=\tilde{x}_i(t_p)$. Since $\{t^i_k\}|^{\infty}_{k=1}$ is a subsequence of $\{t_k\}|^{\infty}_{k=0}$, then $\lim_{k\to\infty}\tilde{x}_i(t_k)=c_s$ implies that $\lim_{k\to\infty}x_i(t^i_k)=c_s$. Hence, the normal agents with dynamics in (13) and (14) can reach resilient group consensus if the normal agents with dynamics (10) reach resilient group consensus.

Theorem 2. *Suppose $\frac{1}{\bar{h}}<k_2<\frac{2}{\bar{h}}-k_1$, then the second-order MAS (1) under the asynchronous resilient impulsive algorithm with input (9) can reach resilient group consensus if each subgraph $G_{s,m}$, $s\in\mathcal{I}^+_2$, $m\in\mathcal{I}^+_{l_s}$ is $(2f_{s,m}+1)$-robust.*

Proof. For $k\ge 0$, define $\underline{x}_{s,m}(t_k)=\min_{j\in V_{s,m}\cap V_N,\tau\in\mathcal{I}_{\bar{d}}}\tilde{x}_j(t_{k-\tau})$, $\bar{x}_{s,m}(t_k)=\max_{j\in V_{s,m}\cap V_N,\tau\in\mathcal{I}_{\bar{d}}}\tilde{x}_j(t_{k-\tau})$ and $H_{s,m}(t_k)=[\underline{x}_{s,m}(t_k),\bar{x}_{s,m}(t_k)]$. With the assumption $\frac{1}{\bar{h}}<k_2<\frac{2}{\bar{h}}-k_1$, the coefficients of (13) are non-negative and less than 1. Let γ denote the smallest value of all non-zero coefficients. Thus, for $i\in V_N$, if $t_{k+1}\in\Pi_i$, $\tilde{x}_i(t_{k+1})\in co\{\tilde{x}_i(t_k),\tilde{x}_i(t_{k-d_i(k)-1}),\tilde{x}_j(t_{k-d_i(k)}),j\in N_i\backslash R_i(t_{k-d_i(k)})\}$.

Similar to the analysis in Theorem 1, it has $\tilde{x}_i(t_{k+1})\in H_{s,m}(t_k)$ at $t_{k+1}\in\Pi_i$. Note that at $t_{k+1}\notin\Pi_i$, $\tilde{x}_i(t_{k+1})=\tilde{x}_i(t_k)\in H_{s,m}(t_k)$. Therefore, it can be concluded that $\bar{x}_{s,m}(t_{k+1})\le\bar{x}_{s,m}(t_k)$, $\underline{x}_{s,m}(t_{k+1})\ge\underline{x}_{s,m}(t_k)$, and each normal agent $\tilde{x}_i(t_k)\in H_{s,m}(t_0)$, $k\ge 0$.

Since monotone functions $\bar{x}_{s,m}(t_k)$ and $\underline{x}_{s,m}(t_k)$ are bounded, the limits of these two functions exist. We define that $\bar{x}^*_{s,m}:=\lim_{k\to+\infty}\bar{x}_{s,m}(t_k)\le\underline{x}^*_{s,m}:=\lim_{k\to+\infty}\underline{x}_{s,m}(t_k)$. Next, we will prove that the resilient group consensus by proving that $\bar{x}^*_{s,m}=\underline{x}^*_{s,m}$.

By contradiction, suppose that $\bar{x}^*_{s,m}>\underline{x}^*_{s,m}$. Choose $\epsilon_0>0$ and $\epsilon>0$ to satisfy that $\underline{x}^*_{s,m}+\epsilon_0<\bar{x}^*_{s,m}-\epsilon_0$ and $\epsilon<\frac{\gamma^{2\bar{d}n_{s,m}}\epsilon_0}{1-\gamma^{2\bar{d}n_{s,m}}}$. Define the sequence $\{\epsilon_l\}$ via $\epsilon_{l+1}=\gamma\epsilon_l-(1-\gamma)\epsilon$, $l\in\mathcal{I}_{2\bar{d}n_{s,m}-1}$. Obviously, $\epsilon_{l+1}\le\epsilon_l$ for all l. Moreover, $\epsilon_{2\bar{d}n_{s,m}}=\gamma^{\bar{d}n_{s,m}}\epsilon_0-(1-\gamma^{2\bar{d}n_{s,m}})\epsilon>0$. Take k_ϵ such that $\bar{x}_{s,m}(t_k)<\bar{x}^*_{s,m}+\epsilon$ and $\underline{x}_{s,m}(t_k)>\underline{x}^*_{s,m}-\epsilon$ for $k\ge k_\epsilon$.

Define $\Phi(t_{k_\epsilon+l},\epsilon_l)=\{j\in V_{s,m}\bigcap V_N:\tilde{x}_j(t_{k_\epsilon})>\bar{x}^*_{s,m}-\epsilon_l\}$ and $\phi(t_{k_\epsilon+l},\epsilon_l)=\{j\in V_{s,m}\bigcap V_N:\tilde{x}_j(t_{k_\epsilon+l})<\underline{x}^*_{s,m}+\epsilon_l\}$. It is obvious that the sets $\Phi(t_{k_\epsilon+l},\epsilon_l)\bigcap\phi(t_{k_\epsilon+l},\epsilon_l)=\emptyset$. If the disjoint sets $\Phi(t_{k_\epsilon+l},\epsilon_l)$ and $\phi(t_{k_\epsilon+l},\epsilon_l)$ are nonempty, considering a normal agent $j\notin\Phi(t_{k_\epsilon+l},\epsilon_l)$, that is $\tilde{x}_j(t_{k_\epsilon+l})\le\bar{x}^*_{s,m}-\epsilon_l$, for $t_{k_\epsilon+l+1}\in\prod_j$, according to (13), it has

$$
\begin{aligned}
\tilde{x}_j(t_{k_\epsilon+l+1})&\le(1-\gamma)\bar{x}_{s,m}(t_{k_\epsilon+l})+\gamma(\bar{x}^*_{s,m}-\epsilon_l)\\
&\le(1-\gamma)(\bar{x}^*_{s,m}+\epsilon)+\gamma(\bar{x}^*_{s,m}-\epsilon_l)\\
&=\bar{x}^*_{s,m}-\epsilon_{l+1}.
\end{aligned}
\tag{15}
$$

If $t_{k_\epsilon+l+1}\notin\prod_j$, it has $\tilde{x}_j(t_{k_\epsilon+l+1})=\tilde{x}_j(t_{k_\epsilon+l+1})\le\bar{x}^*_{s,m}-\epsilon_l$. From $\epsilon_{l+1}<\epsilon_l$, it has $\tilde{x}_j(t_{k_\epsilon+l+1})\le\bar{x}^*_{s,m}-\epsilon_{l+1}$. Thus, it can be concluded that if $j\notin\Phi(t_{k_\epsilon+l},\epsilon_l)$, then $j\notin\Phi(t_{k_\epsilon+l+1},\epsilon_{l+1})$. Similarly, we can show that if $j\notin\phi(t_{k_\epsilon+l},\epsilon_l)$, then $j\notin\phi(t_{k_\epsilon+l+1},\epsilon_{l+1})$.

By the $(2f_{s,m}+1)$-robustness of the subgraph $G_{s,m}$, there exists an agent in $\in\Phi(t_{k_\epsilon},\epsilon_0)$ or $\phi(t_{k_\epsilon},\epsilon_0)$, which has at least $2f_{s,m}+1$ neighbors outside of its own set. Suppose the normal agent j, $j\in\Phi(t_{k_\epsilon},\epsilon_0)$ has this property. Due to at most $f_{s,m}$ agents in $N^{s,m}_j$ are malicious, there are at least $f_{s,m}+1$ normal agents in $N^{s,m}_j$ are outside $\Phi(t_{k_\epsilon},\epsilon_0)$. Define the set $H_k=\{t_{k+1},t_{k+2},\ldots,t_{k+\bar{d}}\}$. Assume there are r times in $H_{k_\epsilon+\bar{d}}$ when agent i updates its velocity state. Since there are at most $\bar{d}$ elements in the set $\{t_j: t_j\in[t^i_k,t^i_{k+1})\}$, it has $r\ge 1$. Let $t_{k_\epsilon+\bar{d}+p^j_1}$ denote the updating time of agent j, where $1\le p^j_1\le\bar{d}$. Since there are at least $f_{s,m}+1$ normal agents in $N^{s,m}_j$ are outside $\Phi(t_{k_\epsilon},\epsilon_0)$, by the argument above, these normal agents will not be in $\Phi(t_{k_\epsilon+m},\epsilon_m)$ for $0\le m\le\bar{d}+p^j_1-1$. According to (13), at least one of these normal neighbors' position values at the current or previous time steps, upper bounded by $\bar{x}^*_{s,m}-\epsilon_{\bar{d}+p^j_1-1}$, will be used by agent j at time $t_{k_\epsilon+\bar{d}+p^j_1}$. Hence, one has

$$
\begin{aligned}
\tilde{x}_j(t_{k_\epsilon+\bar{d}+p^j_1})&\le(1-\gamma)(\bar{x}^*_{s,m}+\epsilon)+\gamma(\bar{x}^*_{s,m}-\epsilon_{\bar{d}+p^j_1-1})\\
&=\bar{x}^*_{s,m}-\epsilon_{\bar{d}+p^j_1},
\end{aligned}
\tag{16}
$$

that is $j\notin\Phi(t_{k_\epsilon+\bar{d}+p^j_1},\epsilon_{\bar{d}+p^j_1})$. By the argument above, it follows that $j\notin\Phi(t_{k_\epsilon+2\bar{d}},\epsilon_{2\bar{d}})$. Similarly, if agent $j\in\phi(t_{k_\epsilon},\epsilon_0)$ has at east $f_{s,m}+1$ normal neighbors in $V_{s,m}$ outside $\phi(t_{k_\epsilon},\epsilon_0)$, it has $j\notin\phi(t_{k_\epsilon+2\bar{d}},\epsilon_{2\bar{d}})$.

Follow the step above, since there are $n_{s,m}$ normal agents in $G_{s,m}$, there exists a finite-time $T_{s,m}\le 2\bar{d}(n_{s,m}-1)$ such that $\Phi(t_{k_\epsilon+T_{s,m}},\epsilon_{T_{s,m}})=\emptyset$ or $\phi(t_{k_\epsilon+T_{s,m}},\epsilon_{T_{s,m}})=\emptyset$. Assume that $\Phi(t_{k_\epsilon+T_{s,m}},\epsilon_{T_{s,m}})=\emptyset$. According to the analysis above, it has $\Phi(t_{k_\epsilon+2\bar{d}(n_{s,m}-1)+l},\epsilon_{2\bar{d}(n_{s,m}-1)+l})=\emptyset$, $l\in\mathcal{I}_{2\bar{d}}$. It follows that $\bar{x}_{s,m}(t_{k_\epsilon+2\bar{d}n_{s,m}})\le\bar{x}^*_{s,m}-\epsilon_{2\bar{d}n_{s,m}}<\bar{x}^*_{s,m}$. However, $\bar{x}_{s,m}(t_{k_\epsilon+2\bar{d}n_{s,m}})\ge\bar{x}^*_{s,m}$ due to the nonincreasing function $\bar{x}_{s,m}(t_k)$ with a limit $\bar{x}^*_{s,m}$. It is a contradiction. Thus, we have $\bar{x}^*_{s,m}=\underline{x}^*_{s,m}$. The proof is complete. □

Next, we will consider the asynchronous sampling with communication delays. For normal agent i at each time step t^i_k, the most recently received neighbor j'state value should be $x_j(t^j_{\xi^j(t^i_k)-d_{ij}(k)})$, where $d_{ij}(k)\in\mathbb{N}$ is the communication delay. Assume the communication delays are bounded, that is $0\le d_{ij}(k)\le d$. Hence, the input $u_i(t^i_k)$ in step 3 of asynchronous impulsive algorithm is designed as:

$$
\begin{aligned}
u_i(t^i_k)=&\left(k_1\sum_{j\in N_i\backslash R_i(t^i_k)}a_{ij}(x_j(t^j_{\xi^j(t^i_k)-d_{ij}(k)})-x_i(t^i_k))\right.\\
&\left.-k_2(x_i(t^i_k)-x_i(t^i_{k-1}))\right)\delta(t-t^i_k).
\end{aligned}
\tag{17}
$$

where $0\le d_{ij}(k)\le d$.

Substituting (17) into (1), one has:

$$
\begin{aligned}
x_i(t^i_{k+1})=&\left(1+\frac{h^i_{k+1}}{h^i_k}-k_2h^i_{k+1}-k_1h^i_{k+1}\sum_{j\in N_i\backslash R_i(t^i_k)}a_{ij}\right)x_i(t^i_k)\\
&+k_1h^i_{k+1}\sum_{j\in N_i\backslash R_i(t^i_k)}a_{ij}x_j(t^j_{\xi^j(t^i_k)-d_{ij}(k)})\\
&+\left(k_2h^i_{k+1}-\frac{h^i_{k+1}}{h^i_k}\right)x_i(t^i_{k-1}).
\end{aligned}
\tag{18}
$$

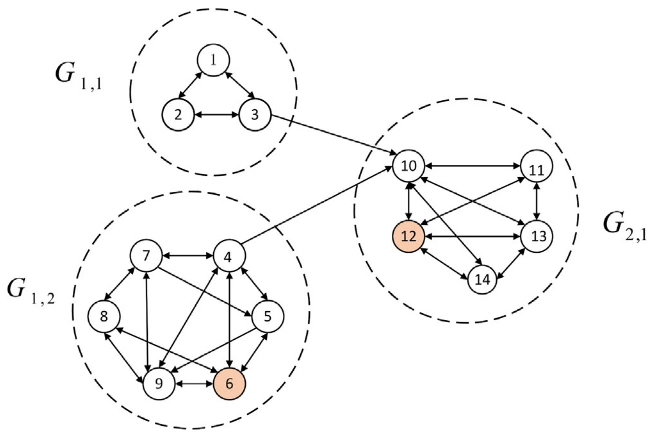

Fig. 1. A network with 3 subgroups.

Assume $t_s = t^i_k$, $t_{s+l_1} = t^i_{k+1}$ and $t_{s-l_{ij}(s)} = t^i_{k-d_{ij}(k)}$. Thus, we have

$$\begin{aligned}\tilde{x}_i(t_{s+l_1}) = {} & \left(1+\frac{h^i_{\xi^i(t_{s+l_1})}}{h^i_{\xi^i(t_{s+l_1})-1}} - k_2 h^i_{\xi^i(t_{s+l_1})} - k_1 h^i_{\xi^i(t_{s+l_1})}\sum_{j\in N_i\setminus R_i(t_s)} a_{ij}\right)\\ & \times \tilde{x}_i(t_{s+l_1-1})\\ & + k_1 h^i_{\xi^i(t_{s+l_1})}\sum_{j\in N_i\setminus R_i(t_s)} a_{ij}\tilde{x}_j(t_{s-l_{ij}(s)})\\ & + \left(k_2 h^i_{\xi^i(t_{s+l_1})} - \frac{h^i_{\xi^i(t_{s+l_1})}}{h^i_{\xi^i(t_{s+l_1})-1}}\right)\tilde{x}_i(t_{s-1}),\end{aligned} \tag{19}$$

where $0 \le l_{ij}(s) \le d\bar{d}$.

Let $\tilde{x}_i(t_{k+1}) = \tilde{x}_i(t_{s+l_1})$. According to (19), for $t_{k+1} \in \Pi_i$, we have

$$\begin{aligned}\tilde{x}_i(t_{k+1}) = {} & \left(1+\frac{h^i_{\xi^i(t_{k+1})}}{h^i_{\xi^i(t_{k+1})-1}} - k_2 h^i_{\xi^i(t_{k+1})}\right.\\ & \left. - k_1 h^i_{\xi^i(t_{k+1})}\sum_{j\in N_i\setminus R_i(t_{k-d_i(k)})} a_{ij}\right)\tilde{x}_i(t_k)\\ & + k_1 h^i_{\xi^i(t_{k+1})}\sum_{j\in N_i\setminus R_i(t_{k-d_i(k)})} a_{ij}\tilde{x}_j(t_{k-d_{ij}(k)})\\ & + \left(k_2 h^i_{\xi^i(t_{k+1})} - \frac{h^i_{\xi^i(t_{k+1})}}{h^i_{\xi^i(t_{k+1})-1}}\right)\tilde{x}_i(t_{k-d_i(k)-1}),\end{aligned} \tag{20}$$

where $0 \le d_{ij}(k) \le (d+1)\bar{d} - 1$ and $0 \le d_i(k) \le \bar{d} - 1$. Otherwise, $\tilde{x}_i(t_{k+1}) = \tilde{x}_i(t_k)$.

Theorem 3. *Suppose $\frac{1}{h} < k_2 < \frac{2}{h} - k_1$, then the second-order MAS (1) under the asynchronous resilient impulsive algorithm with input (17) can reach resilient group consensus if each subgraph $G_{s,m}$, $s \in \mathcal{I}^+_2$, $m \in \mathcal{I}^+_{l_s}$ is $(2f_{s,m}+1)$-robust.*

Proof. Let $\hat{d} = (d+1)\bar{d} - 1$. For $k \ge 0$, define $\underline{x}_{s,m}(t_k) = \min_{j\in V_{s,m}\cap V_N, \tau\in\mathcal{I}_{\hat{d}}} \tilde{x}_j(t_{k-\tau})$, $\bar{x}_{s,m}(t_k) = \max_{j\in V_{s,m}\cap V_N, \tau\in\mathcal{I}_{\hat{d}}} \tilde{x}_j(t_{k-\tau})$ and $H_{s,m}(t_k) = [\underline{x}_{s,m}(t_k), \bar{x}_{s,m}(t_k)]$. The following proof is similar to that of Theorem 2. Hence, it is omitted to save space. □

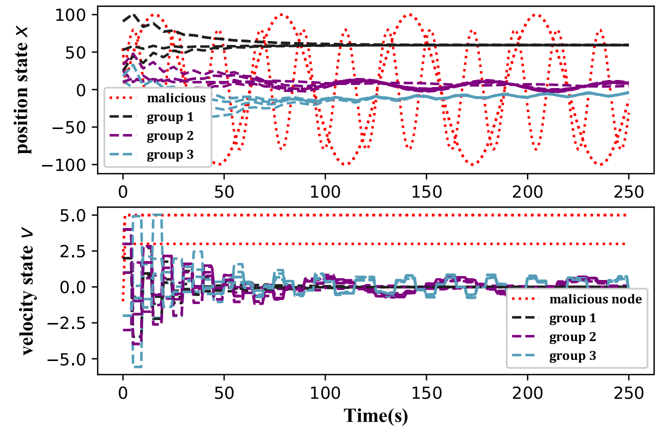

Fig. 2. State trajectories of agents under the impulsive algorithm (21).

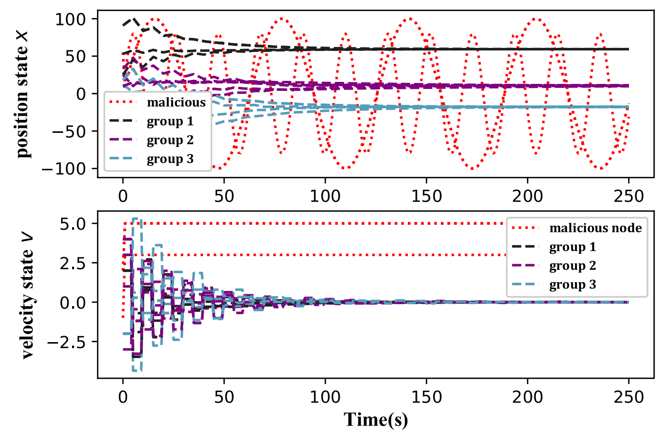

Fig. 3. State trajectories of agents under synchronous resilient impulsive algorithm.

5. Simulation example

Example 1. Consider a directed graph in Fig. 1, where agent 6 and 12 are malicious agents. $x_6(t) = 100sin(0.1t)$ and $x_{12}(t) = 80sin(0.3t)$. $G_{1,1}, G_{1,2}$ are primary layer subgraphs and $G_{2,1}$ is secondary layer subgraph. The subgraph $G_{1,1}$ with $f_{1,1} = 0$ contains a spanning tree. The subgraphs $G_{1,2}$ and $G_{2,1}$ with $f_{1,2} = f_{2,1} = 1$ are 3-robust. The nodes's initial state values are chosen as $x(0) = [53, 25, 91, 33, -30, 55, 25, 10, -45, 12, -81, -54, 18, -92]$ and $v(0) = [1, 4, 2, 3, 4, 2, -3, 1, 4, -2, 3, -1, 3, -2]$. The communication delays are assumed as $\tau = [2, 4, 1, 1, 3, 5, 4, 2, 3, 5, 1, 2, 3, 1]$. For simplification, we take $h_k \equiv 5$. Choose $k_1 = 0.05$, $k_2 = 0.34$ to satisfy the condition $\frac{1}{h} < k_2 < \frac{2}{h} - k_1$. Fig. 2 presents the state trajectories of agents in the adversarial environment under the following traditional impulsive algorithm:

$$\begin{aligned}u_i(t) = {} & \left(k_1\sum_{j\in N_i} a_{ij}(x_j(t_k - \tau_{ij}(t_k)) - x_i(t_k)) - k_2(x_i(t_k) - x_i(t_{k-1}))\right)\\ & \times \delta(t-t_k),\end{aligned} \tag{21}$$

where sorting and removal process are not required compared with the proposed resilient algorithm. As seen in Fig. 2, the normal agents cannot reach group consensus. As a comparison, the state trajectories of agents under the synchronous resilient impulsive algorithm are shown in Fig. 3. We can see that normal

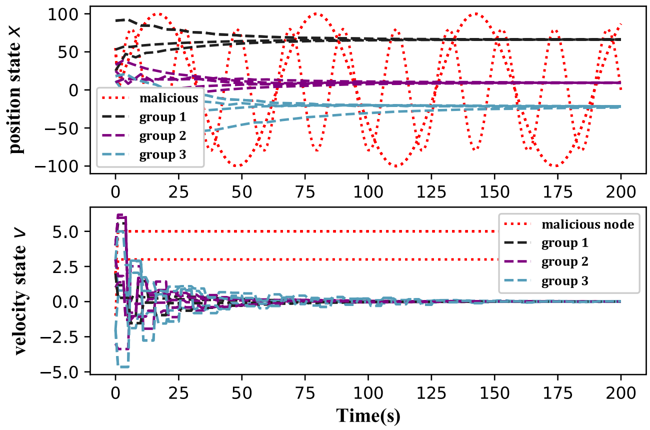

Fig. 4. State trajectories of agents under asynchronous resilient impulsive algorithm.

agents reach group consensus with 3 different equilibrium states as expected.

Example 2. Consider the MAS under the asynchronous resilient impulsive algorithm. Assume the sampling time $t_k^i = 3k$ for $i \in \{1, 4, 8, 11\}$, $t_k^i = 4k$ for $i \in \{2, 5, 9, 13\}$ and $t_k^i = 5k$ for $i \in \{3, 7, 10, 14\}$, $k \in N^+$. Keep all other parameters the same as those in Example 1. Fig. 4 shows that the group consensus is achieved with 3 different equilibrium states, which verify the correctness of Theorem 3.

6. Conclusions

This paper investigated the resilient group consensus problem for second-order agents with communication delays. Synchronous resilient impulsive algorithm was proposed for normal agents with bounded communication delays to achieve group consensus. It has been shown that the proposed algorithm guarantee group consensus if the subgraph in each primary and secondary layer is $(2f_{s,m} + 1)$-robust. Meanwhile, asynchronous resilient impulsive algorithm was also proposed and the same conclusion can be drawn for resilient group consensus. In order to decrease the communication load of agents, we will concentrate on resilient group consensus of MASs with communication delays under event-triggered impulsive algorithm, where the impulse occurs only when an event is triggered instead of time lapses. It is of interest to apply the proposed algorithm to practice. Future works will also pay attention to finite-time resilient group consensus because of practical requirements.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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