Received: 1 July 2021 | Revised: 20 August 2021 | Accepted: 26 October 2021

DOI: 10.1002/rnc.5883

RESEARCH ARTICLE

WILEY

Containment control of hybrid multi-agent systems

Yarui Chen[1,2] | Qi Zhao[1] | Yuanshi Zheng[1] | Yunru Zhu[1]

[1]Shaanxi Key Laboratory of Space Solar Power Station System, School of Mechano-electronic Engineering, Xidian University, Shaanxi, China

[2]Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai, China

Correspondence
Yuanshi Zheng, Shaanxi Key Laboratory of Space Solar Power Station System, School of Mechano-electronic Engineering, Xidian University, Xi'an 710071, China.
Email: zhengyuanshi2005@163.com

Abstract

In this article, the containment control problem is studied for hybrid multi-agent systems (MASs), which is comprised of continuous-time and discrete-time dynamic agents. For first-order hybrid MASs, two effective distributed protocols are designed when followers have continuous-time dynamics, and one distributed protocol is designed when followers have discrete-time dynamics. Meanwhile, for second-order hybrid MASs, we also propose three kinds of distributed protocols to solve the containment control. By utilizing stability theory and system transformation method, some criteria are derived for solving the containment control problem of hybrid MASs. Simulation examples are provided to show the efficiency of the theoretical results.

KEYWORDS

containment control, first-order, hybrid multi-agent systems, second-order

1 | INTRODUCTION

In the past few decades, coordinated control of multi-agent systems (MASs) has received extensive attention in the control science community. That is mainly due to its wide application in consensus,[1,2] formation control,[3] flocking,[4] rendezvous,[5] constraint control,[6] and so forth. The purpose of coordinated control is to complete some complex tasks by mutual communication and cooperation between a group of agents, so as to improve the overall performance of the system. Among these issues, consensus is an important and basic one to coordinated control. It means that a group of autonomous agents reach an agreement on certain quantities of interest via local interaction. The leaderless consensus and consensus tracking problems have been investigated, see References 7-9, just to name a few.

In some realistic scenarios, such as environmental monitoring, transportation of supplies, search and rescue, removal or transfer of hazardous materials, multiple leaders may be required. These leaders usually have outstanding abilities in sensing, calculation, decision-making, and so forth. In general, this kind of consensus issue is called containment control. Its main objective is to drive followers into the convex hull spanned by leaders by designing some appropriate distributed protocols. Dimarogonas et al.[10] proposed a discontinuous time-varying feedback control strategy based on the unicycle model. Inspired by the swarming phenomenon in silkworm moths, Notarstefano et al.[11] modeled containment control of continuous-time (CT) MASs with interval communication in undirected topologies. Liu et al.[12] designed some distributed protocols based on CT and sampled-data to solve the containment control problem of first- and second-order MASs under directed networks. It is worth mentioning that they proved an important lemma related to Laplacian matrix and network topology and derived the final convergence states of followers. The containment control problem of CT MASs with time delays and communication noises were also considered in References 13 and 14, respectively. For discrete-time (DT) MASs, Cao and Ren[15] obtained some necessary and sufficient conditions to achieve containment under fixed and switching directed networks, in light of a Lyapunov-based approach. In Reference 16, the authors transformed the asynchronous containment control problem into a stability problem of synchronous error system equivalently. Then, by using graph theory and nonnegative matrix theory, the asynchronous containment control problem of second-order DT MASs with time-varying delays was solved. Zhu et al.[17] investigated the containment control problem of first- and second-order MASs with DT-CT switching behaviors, and then the case of general second-order dynamics was also considered in Reference 18. Xu et al.[19] studied the containment control problem for high-order DT MASs with input saturation.

Note that the aforementioned works merely focus on the case that all agents have identical dynamics in homogeneous MASs. However, in reality, a collection of agents with different dynamics might be required to cooperate with each other for achieving the prespecified goals. For example, cockroaches and socially integrated autonomous robots with different dynamics showed collective decision-making,[20] which indicates that the necessity of studying heterogeneous MASs. Consensus of heterogeneous MASs composed of first- and second-order integrator agents was considered in References 21 and 22. By virtue of Lyapunov direct method, Zheng et al.[21] studied the consensus of heterogeneous MASs based on undirected connected graphs and leader-following networks. In Reference 22, the authors further analyzed the consensus problem based on fixed and switched directed topologies by using non-negative matrix theory. The containment control problem for CT heterogeneous MASs has been studied extensively. In Reference 23, Zheng and Wang proposed a linear protocol and a nonlinear protocol to address the containment control problem of heterogeneous MASs. And Asgari and Atrianfar[24] extended the results of Reference 23 to the case that DT dynamic agents have fixed time delay. Recently, the cases in which followers have different dynamics, time delays and state dimensions have also been taken into account.[25-27]

Hybrid MASs are a class of systems in which CT and DT dynamic agents coexist. An early attempt on consensus of hybrid MASs was made in Reference 28, and the concept of hybrid consensus was put forward. They devised three kinds of hybrid consensus protocols in accordance with the interaction modes between agents. Moreover, necessary and sufficient conditions involving the sampling period and the degree matrix of the system was established to guarantee the consensus. Zhao et al.[29] considered a new interaction mode among agents and established a unified framework for the consensus of CT and DT MASs. In Reference 30, the second-order consensus of hybrid MASs was further analyzed, and two kinds of consensus protocols with absolute velocity information were devised. The analysis tools developed in References 28-30 mainly include algebraic graph theory, system transformation method, matrix theory and differential mean value theorem of matrix function. A game-theoretic approach was also proposed in Reference 31 to analyze the consensus of hybrid MASs. In addition, the hybrid consensus based on event-triggered[32,33] and malicious nodes[34] has also been reported in details. The containment control problem that agents have CT dynamics,[12-14] DT dynamics[15,16] and CT-DT switching behaviors[17,18] has been considered in the existing papers. By contrast, the containment control problem in which CT agents and DT agents coexist is still a novel topic. The development of computer science and robotics provides conditions for the interaction between CT individuals and DT individuals. Therefore, the containment control of hybrid MASs composed of CT agents and DT agents may be applied to our real life, such as robotic dogs instead of shepherd dogs to shepherd the sheep, robotic fish instead of otters to drive fish, a team of unmanned aerial vehicles to encircle the enemy, and so forth. All of these have aroused our interest in studying containment control of hybrid MASs.

Inspired by all the above analyses, our attention is focused on the containment control of first- and second-order hybrid MASs, which are composed of CT followers/leaders and DT leaders/followers. There are two main difficulties in solving this kind of containment control problem. One is the design of distributed protocols. Obviously, the design of those general containment control protocols is not feasible here. Compared with the containment control problem of homogeneous MASs, it is more difficult for hybrid MASs to understand the interaction modes between agents. Motivated by References 28-30, this article analyzes three new types of interaction modes: (a) each CT follower can observe its own states in real time and interact with all its neighbors at the sampling time; (b) each CT follower can observe its own states in real time, interact with the leader neighbors at the sampling time, and interact with the follower neighbors in real time; (c) each DT follower can observe its own states and interact with all its neighbors only at the sampling time. Meanwhile, three novel distributed protocols are designed for first- and second-order hybrid MASs, respectively. The other is system analysis. For the containment control of homogeneous MASs, the system matrix is relatively easy to obtain. However, for hybrid MASs, the interaction between CT agents and DT agents undoubtedly increases the difficulty of system analysis. The main idea of this article is to construct the error function by using convex hull spanned by leaders, so that the containment control of hybrid MASs can be converted into the stability analysis problem of the error system. Then, by means of graph theory, matrix theory, stability theory, and system transformation method, some criteria to achieve containment are derived.

The rest of this article is organized as follows. In Section 2, we present some notions in graph theory and matrix theory, and list some key lemmas. The first- and second-order dynamic models of hybrid MASs are given in Section 3 and Section 4, respectively. Moreover, some distributed protocols are designed for first- and second-order hybrid MASs and four theorems are proposed. In Section 5, we give some simulations to illustrate the validity of the theoretical results. Lastly, the conclusion is drawn in Section 6.

Notations: Throughout this article, we use $\mathbb{R}$, $\mathbb{R}^N$, and $\mathbb{R}^{N\times N}$ for the set of real numbers, the N-dimensional real vector space, and $N \times N$ real matrix space, respectively. Let $\mathbb{N}$, $\mathbb{C}$, and $\mathbb{C}^{N\times N}$ be the set of natural numbers, the set of complex numbers, and $N \times N$ complex matrix space, respectively. I_N is the $N \times N$ identity matrix, and $\mathbf{0}$ is the all-zeros vector or matrix with compatible dimension. S^{T} is the transpose of vector S. $Re(\lambda)$ and $Im(\lambda)$ represent the real and imaginary parts of a complex number λ. $\overline{\lambda}$ and $|\lambda|$ are the conjugate complex number and the module of λ, respectively. Meanwhile, $diag(\cdot)$ denotes the diagonal matrix, $det(\cdot)$ denotes the determinant of the matrix, and $\|\cdot\|$ represents the Euclidian norm.

2 | PRELIMINARIES

2.1 | Graph theory

A weighted digraph of order N is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{s_1, s_2, \dots, s_N\}$ is the vertex set, $\mathcal{E} = \{e_{ij} = (s_i, s_j)\} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, and $\mathcal{A} = [a_{ij}]_{N\times N}$ is the weighted adjacency matrix. For the edge $(s_i, s_j) \in \mathcal{E}$, s_i is the beginning of the edge, s_j is the end of the edge, which indicates that s_j can receive information from s_i. If $(s_i, s_j) \in \mathcal{E}$ for $i \neq j$, then $a_{ij} > 0$, otherwise $a_{ij} = 0$. And we assume that $a_{ii} = 0$ for $i = 1, 2, \dots, N$. Let $\mathcal{N}_i$ denote the set of neighbors of s_i, $\mathcal{N}_i = \{s_j \in \mathcal{V} : e_{ji} = (s_j, s_i) \in \mathcal{E}, j \neq i\}$. A directed path between two distinct vertices s_j and s_i is denoted by a finite ordered sequence $(s_j, s_{j1}), (s_{j1}, s_{j2}), \dots, (s_{jk}, s_i)$. For a directed tree, there is a special vertex called the root whose in-degree is 0, and the in-degree of the remaining vertices is 1. A directed forest consists of one or more directed trees without common vertices. Then, a directed spanning forest is a directed forest composed of all vertices and some edges in a digraph $\mathcal{G}$. The degree matrix $\mathcal{D} = [d_{ij}]_{N\times N}$ is a diagonal matrix with $d_i = \sum_{j:s_j\in\mathcal{N}_i} a_{ij}$. The Laplacian matrix associated with digraph $\mathcal{G}$ is defined as $L = [l_{ij}]_{N\times N} = \mathcal{D} - \mathcal{A}$.

2.2 | Some useful lemmas and definitions

Lemma 1 (35). *Let $A = [a_{ij}] \in \mathbb{C}^{N\times N}$ and $r_i(A) = \sum_{j=1,j\neq i}^{N} |a_{ij}|,\ i = 1, 2, \dots, N$. Then all eigenvalues of A satisfy*

$$\bigcup_{i=1}^{N} \{z \in \mathbb{C} :\ |z - a_{ii}| \leq r_i(A)\}.$$

Definition 1 (36). The set $P \subset \mathbb{R}^N$ is said to be convex if there is $(1 - \eta)x + \eta y \in P$ for any $x \in P$, $y \in P$, and $\eta \in [0, 1]$. Let $Co\{x_1, x_2, \dots, x_N\} = \{\sum_{i=1}^{N} \xi_i x_i | \xi_i \in \mathbb{R},\ \xi_i \geq 0,\ \sum_{i=1}^{N} \xi_i = 1\}$ denote the convex hull for a finite set of points.

Lemma 2 (37). *For a given block matrix*

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where B_{11}, B_{12}, B_{21}, $B_{22} \in \mathbb{R}^{N\times N}$. If $B_{21}B_{22} = B_{22}B_{21}$ then $det(B) = det(B_{11}B_{22} - B_{12}B_{21})$ and if $B_{11}B_{21} = B_{21}B_{11}$ then $det(B) = det(B_{11}B_{22} - B_{21}B_{12})$.

Lemma 3 (38). *A quadratic complex coefficient polynomial is given as*

$$H(s) = s^2 + \omega_1 s + \omega_0,$$

where ω_1 and ω_0 are complex numbers. Then, $h(s)$ is Hurwitz stable if and only if $Re(\omega_1) > 0$ and $Re(\omega_1)Im(\omega_1)Im(\omega_0) + Re^2(\omega_1)Re(\omega_0) - Im^2(\omega_0) > 0$.

Lemma 4. *Let $g_1(t) = \frac{e^{\alpha(t-t_k)}-1}{\alpha}$, $g_2(t) = e^{\alpha(t-t_k)}$, and $g_3(t) = \alpha e^{\alpha(t-t_k)}$ for $t \in (t_k, t_{k+1}]$. Here, $t_k = kh$, $k \in \mathbb{N}$ and $h \in \mathbb{R}^+$. Then, $g_1(t)$, $g_2(t)$, and $g_3(t)$ are monotone bounded functions for the constant $\alpha < 0$.*

Proof. First, it is easy to get

$$\dot{g_1}(t) = e^{\alpha(t-t_k)} > 0,\ \dot{g_2}(t) = \alpha e^{\alpha(t-t_k)} < 0,\ \dot{g_3}(t) = \alpha^2 e^{\alpha(t-t_k)} > 0,$$

hence $g_1(t)$ and $g_3(t)$ are monotone increasing functions, $g_2(t)$ is a monotone decreasing function.

When $t_k = kh$, $k \in \mathbb{N}$, we have $0 < t - t_k \le h$. Thence, for the constant $\alpha < 0$, we can also get

$$0 < g_1(t) \le \frac{e^{h\alpha} - 1}{\alpha},\ e^{h\alpha} \le g_2(t) \le 1,\ \alpha < g_3(t) \le \alpha e^{h\alpha},$$

which indicates that $g_1(t)$, $g_2(t)$, and $g_3(t)$ are bounded functions.

In conclusion, $g_1(t)$, $g_2(t)$, and $g_3(t)$ are monotone bounded functions for the constant $\alpha < 0$. ∎

Definition 2 (12). For an MAS, an agent is called a leader if it has no neighbors, and a follower if it has at least one neighbor. In this article, we assume that an n-agent system has M $(M < N)$ followers and $N - M$ leaders. Let $\mathcal{I}_F = \{1, 2, \dots, M\}$ denote the set of followers and $\mathcal{I}_R = \{M+1, M+2, \dots, N\}$ denote the set of leaders. Then the Laplacian matrix L associated the digraph $\mathcal{G}$ can be partitioned to be

$$L = \begin{bmatrix} L_F & L_{FR} \\ 0_{(N-M)\times M} & 0_{(N-M)\times(N-M)} \end{bmatrix},$$

where $L_F \in \mathbb{R}^{M\times M}$ and $L_{FR} \in \mathbb{R}^{M\times(N-M)}$.

Lemma 5 (12). *L_F is invertible if and only if the communication digraph $\mathcal{G}$ has a directed spanning forest. Moreover, all eigenvalues of L_F have positive real parts, each element of $-L_F^{-1}L_{FR}$ is non-negative, and the sum of each row of $-L_F^{-1}$ L_{FR} is 1.*

Definition 3. For MASs with first- or second-order dynamics, the containment control problem can be solved if the states of all followers ultimately converge to the convex hull spanned by those of leaders under any initial conditions.

Lemma 6 (12). *Consider the first-order MAS*

$$\begin{cases} \dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(x_j(t_k) - x_i(t_k)), & t \in (t_k, t_{k+1}], \quad t_k = kh, \quad k \in \mathbb{N}, \quad i \in \mathcal{I}_F, \\ \dot{x}_i(t) = 0, & i \in \mathcal{I}_R, \end{cases}$$

with sampled-data based protocol under the fixed digraph $\mathcal{G}$. Here, $h > 0$ is the sampling period and $\mathcal{A} = [a_{ij}]_{N\times N}$ is the weighted adjacency matrix associated with the digraph $\mathcal{G}$. When $i \in \mathcal{I}_R$, we can easily get $x_i(t) = x_i(0)$. That is to say, the position of each leader does not change with time t and is always the initial state. Therefore, these leaders are stationary and the convex hull spanned by these leaders is also stationary. Then all followers will converge to the stationary convex hull spanned by the stationary leaders for any initial conditions if and only if the digraph $\mathcal{G}$ contains a directed spanning forest and the sampling period satisfies $h < \min_{i\in\mathcal{I}_F}\left\{\frac{2Re(\lambda_i)}{|\lambda_i|}\right\}$, where λ_i are the eigenvalues of L_F.

3 | CONTAINMENT CONTROL OF FIRST-ORDER HYBRID MULTI-AGENT SYSTEMS

In this section, the containment control problem of hybrid MASs with first-order integrator agents is investigated. Then, we consider the following two cases: (a) followers have CT dynamics and leaders have DT dynamics; (b) followers have DT dynamics and leaders have CT dynamics.

3.1 | Followers are agents with first-order CT dynamics

Consider the containment control of the first-order hybrid MAS, which is comprised of CT followers and DT leaders. The dynamics of followers and leaders are given by

$$\begin{cases} \dot{x}_i(t) = u_i(t), & t \in (t_k, t_{k+1}], & i \in \mathcal{I}_F, \\ x_i(t_{k+1}) = x_i(t_k) + hu_i(t_k), & t_k = kh, \quad k \in \mathbb{N}, & i \in \mathcal{I}_R, \end{cases} \tag{1}$$

where $h > 0$ is the sampling period, $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the position and the control input of agent i, respectively. Let $x_F(\cdot) = [x_1(\cdot), x_2(\cdot), \dots, x_M(\cdot)]^T$, $x_R(\cdot) = [x_{M+1}(\cdot), x_{M+2}(\cdot), \dots, x_N(\cdot)]^T$, $x_F(0) = [x_1(0), x_2(0), \dots, x_M(0)]^T$, and $x_R(0) = [x_{M+1}(0), x_{M+2}(0), \dots, x_N(0)]^T$, where $x_i(0)$ is the initial position of agent i.

When followers have CT dynamics and leaders have DT dynamics, it is evident that each follower can observe its own states in real time and interact with leaders at the sampling time. As far as all followers are concerned, they can interact with each other not only at the sampling time but also in real time. In this regard, we will consider the next two kinds of distributed protocols.

3.1.1 | Case 1

In this case, we assume that each follower interacts with all its neighbors at the sampling time t_k and observes its own states in real time. Thus, a distributed protocol for the hybrid MAS (1) is devised as

$$\begin{cases} u_i(t) = \sum_{j=1}^{N} a_{ij}(x_j(t_k) - x_i(t)), \quad t \in (t_k, t_{k+1}], & i \in \mathcal{I}_F, \\ u_i(t_k) = 0, & i \in \mathcal{I}_R, \end{cases} \tag{2}$$

where $\mathcal{A} = [a_{ij}]_{N\times N}$ is the weighted adjacency matrix associated with the digraph $\mathcal{G}$.

Theorem 1. *Consider a communication digraph $\mathcal{G}$. Then, the hybrid MAS (1) with protocol (2) can solve the containment control problem if and only if the digraph $\mathcal{G}$ contains a directed spanning forest.*

Proof. (*Sufficiency*): The proof of sufficiency can be divided into the following two steps.

Step 1: For the hybrid MAS (1) with protocol (2), we can get the position of each agent as

$$\begin{cases} x_i(t) = x_i(t_k) + \frac{1-e^{-\sum_{j=1}^{N} a_{ij}(t-t_k)}}{\sum_{j=1}^{N} a_{ij}} \sum_{j=1}^{N} a_{ij}(x_j(t_k) - x_i(t_k)), & i \in \mathcal{I}_F, \\ x_i(t_{k+1}) = x_i(t_k), & i \in \mathcal{I}_R. \end{cases} \tag{3}$$

When $t = t_{k+1}$, for $i \in \mathcal{I}_F$, we have

$$x_i(t_{k+1}) = x_i(t_k) + \frac{1 - e^{-\sum_{j=1}^{N} a_{ij} h}}{\sum_{j=1}^{N} a_{ij}} \sum_{j=1}^{N} a_{ij}(x_j(t_k) - x_i(t_k)). \tag{4}$$

From (3), we get $x_R(t_k) = x_R(0)$, hence (4) can be written in a compact form as

$$x_F(t_{k+1}) = (I_M - H_M L_F)x_F(t_k) - H_M L_{FR} x_R(0), \tag{5}$$

where

$$H_M = diag\left\{ \frac{1 - e^{-\sum_{j=1}^{N} a_{1j} h}}{\sum_{j=1}^{N} a_{1j}}, \frac{1 - e^{-\sum_{j=1}^{N} a_{2j} h}}{\sum_{j=1}^{N} a_{2j}}, \dots, \frac{1 - e^{-\sum_{j=1}^{N} a_{Mj} h}}{\sum_{j=1}^{N} a_{Mj}} \right\}.$$

Since the digraph $\mathcal{G}$ contains a directed spanning forest, the matrix L_F is invertible in line with Lemma 5. Then, we construct an error function based on the states of followers. Let $\delta_F(t_k) = x_F(t_k) + L_F^{-1} L_{FR} x_R(0)$. When $t_k \to \infty$, $\delta_F(t_k) \to \mathbf{0}$ and the final states of followers can be obtained. Then, from Equation (5), we can get a DT error system as

$$\delta_F(t_{k+1}) = P\delta_F(t_k), \tag{6}$$

where $P = I_M - H_M L_F$. For convenience, we denote the diagonal elements of matrix P with P_i that $P_i = e^{-d_i h}$, $i \in \mathcal{I}_F$. By Lemma 1, we have

$$|z - P_i| \le \sum_{j=1, j\neq i}^{M} \left| \frac{1 - e^{-d_i h}}{d_i} a_{ij} \right|$$

$$
\begin{aligned}
&= \frac{1-e^{-d_i h}}{d_i} \sum_{j=1, j\neq i}^{N} a_{ij} \\
&= \frac{1-e^{-d_i h}}{d_i} d_i \\
&= 1-e^{-d_i h},
\end{aligned}
$$

where $h > 0$. In addition, the digraph $\mathcal{G}$ contains a directed spanning forest, which implies that each follower has at least one neighbor. Hence $d_i > 0$, $i \in \mathcal{I}_F$. Furthermore, we have $0 < e^{-d_i h} < 1$. All eigenvalues of matrix P are in the union of all Gerschgorin circles with $e^{-d_i h}$ as the center and $(1 - e^{-d_i h})$ as the radius. Since the centers and radii of all Gerschgorin circles of matrix P have the same form, only one Gerschgorin circle is shown as a representative in Figure 1.

Use μ_i to denote the eigenvalues of the matrix P and $|\mu_i|$ to denote the modules of μ_i. Take any point C in the circle shown in Figure 1 and connect OC and CD. Then, we have $|DC| = |DE|$ and $|OD| + |DC| = |OD| + |DE|$. Based on a property of triangle, we get $|OC| \leq |OD| + |DC| = |OE| = 1$. Owing to $|OC| = |\mu_i|$, it is clear that $|\mu_i| \leq 1$. And then, we will apply the contradiction method to prove $|\mu_i| \neq 1$.

Suppose that there is $|\mu_i| = 1$, hence we get $\mu_i = 1$ by Figure 1. Then, we denote the eigenvalues of $H_M L_F$ by λ_i'. The characteristic equation of the matrix P in (6) is

$$
\begin{aligned}
det(\mu I_M - P) &= det(\mu I_M - (I_M - H_M L_F)) \\
&= det((\mu - 1) I_M + H_M L_F) \\
&= \prod_{i=1}^{M} (\mu - 1 + \lambda_i') \\
&= 0.
\end{aligned}
$$

If $\mu_i = 1$, then $\lambda_i' = 0$. It means that matrix $H_M L_F$ has one or more eigenvalues of zero. Then, $det(H_M L_F) = 0$. By a property of determinant, we have $det(H_M L_F) = (\prod_{i=1}^{M} \frac{1-e^{-d_i h}}{d_i}) det(L_F)$, where $\frac{1-e^{-d_i h}}{d_i} > 0$. The matrix L_F is invertible, thus we get $det(L_F) \neq 0$, and further $det(H_M L_F) \neq 0$. Obviously, that contradicts the result above. That is to say, $|\mu_i| \neq 1$. And then $|\mu_i| < 1$ is confirmed.

Therefore, all eigenvalues of the matrix P are within the unit circle, which indicates that the error system (6) is Schur stable. Moreover, we can obtain $x_F(t_k) \to -L_F^{-1} L_{FR} x_R(0)$ as $t_k \to \infty$. From Lemma 5 and Definition 1, we know that all followers will asymptotically converge to the convex hull spanned by those of leaders. Thence, by Definition 3, the distributed protocol (2) can solve the containment control problem of the hybrid MAS (1) when $t_k \to \infty$.

Step 2: In this step, we will prove that all followers almost not depart from the convex hull spanned by those of leaders when $t \in (t_k, t_{k+1})$, $t \to \infty$. From the system (3), we have

$$
x_F(t) - x_F(t_k) = -H_M(t)(L_F x_F(t_k) + L_{FR} x_R(t_k)) = -H_M(t) L_F (x_F(t_k) + L_F^{-1} L_{FR} x_R(0)), \tag{7}
$$

where $H_M(t) = diag\left\{ \frac{1-e^{-\sum_{j=1}^{N} a_{1j}(t-t_k)}}{\sum_{j=1}^{N} a_{1j}}, \frac{1-e^{-\sum_{j=1}^{N} a_{2j}(t-t_k)}}{\sum_{j=1}^{N} a_{2j}}, \ldots, \frac{1-e^{-\sum_{j=1}^{N} a_{Mj}(t-t_k)}}{\sum_{j=1}^{N} a_{Mj}} \right\}$, $i \in \mathcal{I}_F$. By Lemma 4, we can easily get that $\frac{1-e^{-\sum_{j=1}^{N} a_{ij}(t-t_k)}}{\sum_{j=1}^{N} a_{ij}}$ is a bounded function about t. That is to say, every element of the matrix $H_M(t)$ is bounded. Moreover, when $t \to \infty$, we have $t_k \to \infty$. In terms of *Step 1*, we know that $(x_F(t_k) + L_F^{-1} L_{FR} x_R(0)) \to \mathbf{0}$ as $t_k \to \infty$. Consequently, from (7), we have

$$
\lim_{t \to \infty} \| x_F(t) - x_F(t_k) \| = \lim_{t \to \infty} \left\| -H_M(t) L_F (x_F(t_k) + L_F^{-1} L_{FR} x_R(0)) \right\| = 0,
$$

which shows that $x_F(t) \to x_F(t_k) \to -L_F^{-1} L_{FR} x_R(0)$ when $t \to \infty$.

Combining *Step 1* and *Step 2*, the containment control for the hybrid MAS (1) with protocol (2) can be achieved if the communication digraph $\mathcal{G}$ contains a directed spanning forest.

Necessity: Suppose that the communication digraph $\mathcal{G}$ has no directed spanning forest. Then there exists at least one follower that is incapable of receiving information from any leader, which implies that the position of this follower is

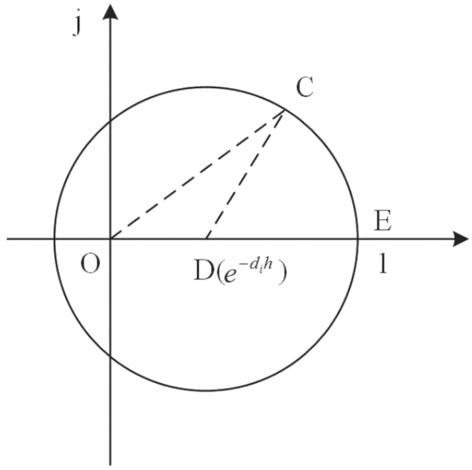

FIGURE 1 Any Gerschgorin circle of matrix P

uncorrelated with the positions of those leaders. Accordingly, the containment control of the hybrid MAS (1) cannot be achieved in this case. ■

3.1.2 | Case 2

In this case, we still assume that followers interact with all leaders at the sampling time t_k and observe their own positions in real time. However, unlike Case 1, the interaction between all followers is presumed to occur in real time. Thus, a distributed protocol for the hybrid MAS (1) is devised as

$$\begin{cases} u_i(t) = \sum_{j=1}^{M} a_{ij}(x_j(t) - x_i(t)) + \sum_{j=M+1}^{N} a_{ij}(x_j(t_k) - x_i(t)), \quad t \in (t_k, t_{k+1}], & i \in \mathcal{I}_F, \\ u_i(t_k) = 0, & i \in \mathcal{I}_R, \end{cases} \tag{8}$$

where $\mathcal{A} = [a_{ij}]_{N\times N}$ is the weighted adjacency matrix associated with the digraph $\mathcal{G}$.

Theorem 2. *Consider a communication digraph $\mathcal{G}$. Then, the hybrid MAS (1) with protocol (8) can solve the containment control problem if and only if the digraph $\mathcal{G}$ contains a directed spanning forest.*

Proof. (*Sufficiency*): For the hybrid MAS (1) with protocol (8), we have

$$\begin{cases} \dot{x}_i(t) = \sum_{j=1}^{M} a_{ij}(x_j(t) - x_i(t)) + \sum_{j=M+1}^{N} a_{ij}(x_j(t_k) - x_i(t)), & i \in \mathcal{I}_F, \\ x_i(t_{k+1}) = x_i(t_k), & i \in \mathcal{I}_R. \end{cases} \tag{9}$$

From the system (9), we have $x_R(t_k) = x_R(0)$. By Lemma 5, the matrix L_F is invertible. Thus, when $i \in \mathcal{I}_F$, the states of all followers can be written in a compact form as

$$\dot{x}_F(t) = -L_F x_F(t) - L_{FR} x_R(t_k) = -L_F(x_F(t) + L_F^{-1} L_{FR} x_R(0)). \tag{10}$$

Similar to Case 1, we devise an error function $\delta_F(t) = x_F(t) + L_F^{-1} L_{FR} x_R(0)$ with regard to t. By deriving the error function and using the result of Equation (10), one has

$$\dot{\delta}_F(t) = \dot{x}_F(t) = -L_F(x_F(t) + L_F^{-1} L_{FR} x_R(0)) = -L_F \delta_F(t). \tag{11}$$

By Lemma 5, all eigenvalues of $-L_F$ have negative real parts. In the light of Lyapunov stability criterion, we know that the error system (11) is asymptotically stable when $\delta_F(t) \to 0$. That is $x_F(t) \to -L_F^{-1} L_{FR} x_R(0)$ as $t \to \infty$. Therefore, from

Lemma 5 and Definition 1, we know that all followers will asymptotically converge to the convex hull spanned by those of leaders. Moreover, by Definition 3, the distributed protocol (8) can solve the containment control problem of the hybrid MAS (1).

Necessity: The proof of necessity is similar to Section 3.1.1, which is omitted here. ∎

Remark 1. Compared to the containment control of the first-order CT MAS with sampled-data based protocol, the containment control of hybrid MASs with CT followers involves more real time information. It brings the advantage of relaxing the reachable condition of the containment control. No matter Case 1 or Case 2, the sampling period h has no upper limit.

3.2 | Followers are agents with first-order DT dynamics

Consider the containment control of the first-order hybrid MAS, which is comprised of DT followers and CT leaders. The dynamics of leaders and followers are given by

$$\begin{cases} x_i(t_{k+1}) = x_i(t_k) + hu_i(t_k), & t_k = kh, \quad k \in \mathbb{N}, & i \in \mathcal{I}_F, \\ \dot{x}_i(t) = u_i(t), & t \in (t_k, t_{k+1}], & i \in \mathcal{I}_R, \end{cases} \tag{12}$$

where $h > 0$ is the sampling period, $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the position and the control input of agent i, respectively.

In view of the definition of leaders, we know that each leader only sends but does not receive any information. Thus, their control inputs are all zero. Those followers are agents with DT dynamics, hence they can only observe their own states and communicate with all its neighbors at the sampling time. Although leaders are agents with CT dynamics, followers only possess the ability to interact with leaders at the sampling time. Therefore, a distributed protocol for the hybrid MAS (12) is designed as

$$\begin{cases} u_i(t_k) = \sum_{j=1}^{N} a_{ij}(x_j(t_k) - x_i(t_k)), & i \in \mathcal{I}_F, \\ u_i(t) = 0, & i \in \mathcal{I}_R, \end{cases} \tag{13}$$

where $\mathcal{A} = [a_{ij}]_{N\times N}$ is the weighted adjacency matrix associated with the digraph $\mathcal{G}$.

Theorem 3. *Consider a communication digraph $\mathcal{G}$. Then, the hybrid MAS (12) with protocol (13) can solve the containment control problem if and only if the digraph $\mathcal{G}$ contains a directed spanning forest, and the sampling period satisfies*

$$h < min_{i\in\mathcal{I}_F} \left\{ \frac{2Re(\lambda_i)}{|\lambda_i|} \right\},$$

where λ_i are the eigenvalues of L_F.

Proof. For the hybrid MAS (12) with protocol (13), it can be noted that the proof is basically similar to that of Corollary 1 in Reference 12. Moreover, the range of sampling period h is the same after analysis. Thus the verification is omitted here. For more details, please refer to Reference 12. ∎

4 | CONTAINMENT CONTROL OF SECOND-ORDER HYBRID MULTI-AGENT SYSTEMS

In this section, we investigate the containment control problem of hybrid MASs with second-order integrator agents. Compared with the first-order system, the second-order system involves the velocity information, which increases the difficulty of solving the differential equations of the system. Similar to Section 3, we still discuss the following two situations: (a) followers are agents with CT dynamics and leaders are agents with DT dynamics; (b) followers are agents with DT dynamics and leaders are agents with CT dynamics.

4.1 | Followers are agents with second-order CT dynamics

For the second-order hybrid MAS, the containment control problem with CT followers and DT leaders is considered in this section. Then, the dynamics of leaders and followers are given by

$$\begin{cases} \dot{x}_i(t) = v_i(t), & \dot{v}_i(t) = u_i(t), & & i \in \mathcal{I}_F, \\ x_i(t_{k+1}) = x_i(t_k) + hv_i(t_k), & v_i(t_{k+1}) = v_i(t_k) + hu_i(t_k), & t_k = kh, \quad k \in \mathbb{N}, & i \in \mathcal{I}_R, \end{cases} \tag{14}$$

where $h > 0$ is the sampling period, $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$, and $u_i \in \mathbb{R}$ are the position, the velocity, and the control input of agent i, respectively. Let $x_F(\cdot) = [x_1(\cdot), x_2(\cdot), \ldots, x_M(\cdot)]^{\mathrm{T}}$, $v_F(\cdot) = [v_1(\cdot), v_2(\cdot), \ldots, v_M(\cdot)]^{\mathrm{T}}$, $x_R(\cdot) = [x_{M+1}(\cdot), x_{M+2}(\cdot), \ldots, x_N(\cdot)]^{\mathrm{T}}$, $v_R(\cdot) = [v_{M+1}(\cdot), v_{M+2}(\cdot), \ldots, v_N(\cdot)]^{\mathrm{T}}$, $x_F(0) = [x_1(0), x_2(0), \ldots, x_M(0)]^{\mathrm{T}}$, $v_F(0) = [v_1(0), v_2(0), \ldots, v_M(0)]^{\mathrm{T}}$, $x_R(0) = [x_{M+1}(0), x_{M+2}(0), \ldots, x_N(0)]^{\mathrm{T}}$, and $v_R(0) = [v_{M+1}(0), v_{M+2}(0), \ldots, v_N(0)]^{\mathrm{T}}$, where $x_i(0)$ and $v_i(0)$ are the initial conditions of agent i.

Similar to the design of control protocols in Section 3.1 for the first-order hybrid MAS (1), the following two novel distributed protocols with absolute velocity information are proposed in terms of the interaction modes between followers.

4.1.1 | Case 1

In this case, we assume that those followers observe their own states in real time and interact with all their neighbors at the sampling time t_k. Thus, a distributed protocol with absolute velocity information for the hybrid MAS (14) is devised as

$$\begin{cases} u_i(t) = k_1 \sum_{j=1}^{N} a_{ij}(x_j(t_k) - x_i(t)) - k_2 v_i(t), \quad t \in (t_k, t_{k+1}], & i \in \mathcal{I}_F, \\ v_i(t_k) = 0, & i \in \mathcal{I}_R, \end{cases} \tag{15}$$

where $k_1 > 0, k_2 > 0$ are the feedback gains and $\mathcal{A} = [a_{ij}]_{N \times N}$ is the weighted adjacency matrix associated with the digraph $\mathcal{G}$.

Theorem 4. *Consider a communication digraph $\mathcal{G}$. Assume that the feedback gains satisfy $\frac{k_2^2}{k_1} > 4max_{i \in \mathcal{I}_F}\{d_i\}$. Then, the hybrid MAS (14) with protocol (15) can solve the containment control problem if and only if the digraph $\mathcal{G}$ contains a directed spanning forest.*

Proof. (*Sufficiency*): The proof of sufficiency can be divided into the following three steps.

Step 1: From (14) and (15), for $i \in \mathcal{I}_F$ and $t \in (t_k, t_{k+1}]$, we have

$$\ddot{x}_i(t) = \dot{v}_i(t) = k_1 \sum_{j=1}^{N} a_{ij}(x_j(t_k) - x_i(t)) - k_2 \dot{x}_i(t),$$

that is

$$\ddot{x}_i(t) + k_2 \dot{x}_i(t) + k_1 \sum_{j=1}^{N} a_{ij} x_i(t) = k_1 \sum_{j=1}^{N} a_{ij} x_j(t_k).$$

By solving the above second-order nonhomogeneous linear ordinary differential equation, we have

$$\begin{cases} x_i(t) = c_1 e^{p_1(t-t_k)} + c_2 e^{p_2(t-t_k)} + \frac{\sum_{j=1}^{N} a_{ij} x_j(t_k)}{\sum_{j=1}^{N} a_{ij}}, \\ v_i(t) = \dot{x}_i(t) = c_1 p_1 e^{p_1(t-t_k)} + c_2 p_2 e^{p_2(t-t_k)}, \end{cases} \tag{16}$$

where $p_1 = \frac{-k_2 + \sqrt{k_2^2 - 4k_1 \sum_{j=1}^{N} a_{ij}}}{2}$, $p_2 = \frac{-k_2 - \sqrt{k_2^2 - 4k_1 \sum_{j=1}^{N} a_{ij}}}{2}$, $c_1 = \frac{v_i(t_k) + \frac{p_2 \sum_{j=1}^{N} a_{ij}(x_j(t_k) - x_i(t_k))}{\sum_{j=1}^{N} a_{ij}}}{p_1 - p_2}$, $c_2 = \frac{v_i(t_k) + \frac{p_1 \sum_{j=1}^{N} a_{ij}(x_j(t_k) - x_i(t_k))}{\sum_{j=1}^{N} a_{ij}}}{p_2 - p_1}$. It is easy to know that $p_1 p_2 = k_1 \sum_{j=1}^{N} a_{ij}$, $p_1 + p_2 = -k_2$.

For (16), when $t = t_{k+1}$, we get

$$
\begin{aligned}
x_i(t_{k+1}) &= c_1 e^{p_1 h} + c_2 e^{p_2 h} + \frac{\sum_{j=1}^N a_{ij} x_j(t_k)}{\sum_{j=1}^N a_{ij}} \\
&= \frac{v_i(t_k) + \frac{p_2 \sum_{j=1}^N a_{ij}(x_j(t_k) - x_i(t_k))}{\sum_{j=1}^N a_{ij}}}{p_1 - p_2} e^{p_1 h} + \frac{v_i(t_k) + \frac{p_1 \sum_{j=1}^N a_{ij}(x_j(t_k) - x_i(t_k))}{\sum_{j=1}^N a_{ij}}}{p_2 - p_1} e^{p_2 h} + \frac{\sum_{j=1}^N a_{ij} x_j(t_k)}{\sum_{j=1}^N a_{ij}} \\
&= \frac{e^{p_1 h} - e^{p_2 h}}{p_1 - p_2} v_i(t_k) + \frac{p_2 e^{p_1 h} - p_1 e^{p_2 h}}{\sum_{j=1}^N a_{ij}(p_1 - p_2)} \sum_{j=1}^N a_{ij}(x_j(t_k) - x_i(t_k)) + \frac{\sum_{j=1}^N a_{ij} x_j(t_k)}{\sum_{j=1}^N a_{ij}} \\
&= x_i(t_k) + \frac{p_2 e^{p_1 h} - p_1 e^{p_2 h} + (p_1 - p_2)}{\sum_{j=1}^N a_{ij}(p_1 - p_2)} \sum_{j=1}^N a_{ij}(x_j(t_k) - x_i(t_k)) + \frac{e^{p_1 h} - e^{p_2 h}}{p_1 - p_2} v_i(t_k)
\end{aligned}
$$

and

$$
\begin{aligned}
v_i(t_{k+1}) &= c_1 p_1 e^{p_1 h} + c_2 p_2 e^{p_2 h} \\
&= \frac{p_1 v_i(t_k) + k_1 \sum_{j=1}^N a_{ij}(x_j(t_k) - x_i(t_k))}{p_1 - p_2} e^{p_1 h} + \frac{p_2 v_i(t_k) + k_1 \sum_{j=1}^N a_{ij}(x_j(t_k) - x_i(t_k))}{p_2 - p_1} e^{p_2 h} \\
&= v_i(t_k) + \frac{k_1(e^{p_1 h} - e^{p_2 h})}{p_1 - p_2} \sum_{j=1}^N a_{ij}(x_j(t_k) - x_i(t_k)) + \frac{p_1 e^{p_1 h} - p_2 e^{p_2 h} - (p_1 - p_2)}{p_1 - p_2} v_i(t_k).
\end{aligned}
$$

Then, we will convert the second-order system to the first-order system by way of variable substitution. Let $z_{i'}(t_k) = x_i(t_k) + k_3 v_i(t_k)$, and we get $v_i(t_k) = \frac{1}{k_3}(z_{i'}(t_k) - x_i(t_k))$. For $i, i' \in \mathcal{I}_F$, we have

$$
\begin{aligned}
x_i(t_{k+1}) &= x_i(t_k) + \frac{k_1\left(\frac{e^{p_1 h} - 1}{p_1} - \frac{e^{p_2 h} - 1}{p_2}\right)}{p_1 - p_2} \sum_{j=1}^N a_{ij}(x_j(t_k) - x_i(t_k)) + \frac{1}{k_3} \frac{e^{p_1 h} - e^{p_2 h}}{p_1 - p_2}(z_{i'}(t_k) - x_i(t_k)) \\
&= x_i(t_k) + k_1 k_4 \sum_{j=1}^N a_{ij}(x_j(t_k) - x_i(t_k)) + \frac{k_5}{k_3}(z_{i'}(t_k) - x_i(t_k))
\end{aligned}
$$

and

$$
\begin{aligned}
z_{i'}(t_{k+1}) &= x_i(t_{k+1}) + k_3 v_i(t_{k+1}) \\
&= x_i(t_k) + k_1 k_4 \sum_{j=1}^N a_{ij}(x_j(t_k) - x_i(t_k)) + \frac{k_5}{k_3}(z_{i'}(t_k) - x_i(t_k)) + k_3\left(v_i(t_k) + k_1 k_5 \sum_{j=1}^N a_{ij}(x_j(t_k) - x_i(t_k)) + k_6 v_i(t_k)\right) \\
&= y_{i'}(t_k) + \left(\frac{k_5}{k_3} + k_6\right)(z_{i'}(t_k) - x_i(t_k)) + k_1(k_4 + k_3 k_5)\left(\sum_{j=1}^N a_{ij}(x_j(t_k) - z_{i'}(t_k)) + \sum_{j=1}^N a_{ij}(z_{i'}(t_k) - x_i(t_k))\right) \\
&= z_{i'}(t_k) + k_1(k_4 + k_3 k_5) \sum_{j=1}^N a_{ij}(x_j(t_k) - z_{i'}(t_k)) + k_5\left(k_2 - \frac{1}{k_3} - k_1 k_3 d_i\right)\left(x_i(t_k) - z_{i'}(t_k)\right),
\end{aligned}
$$

where $k_4 = \frac{\frac{e^{p_1 h} - 1}{p_1} - \frac{e^{p_2 h} - 1}{p_2}}{p_1 - p_2}$, $k_5 = \frac{e^{p_1 h} - e^{p_2 h}}{p_1 - p_2}$, $k_6 = \frac{p_1 e^{p_1 h} - p_2 e^{p_2 h} - (p_1 - p_2)}{p_1 - p_2}$.

Let $\beta_i = k_2 - \frac{1}{k_3} - k_1 k_3 d_i$, $i \in \mathcal{I}_F$. Since the digraph $\mathcal{G}$ contains a directed spanning forest, we know that each follower has at least one neighbor, hence $d_i > 0$ for $i \in \mathcal{I}_F$. Let $\frac{k_2 - \sqrt{k_2^2 - 4k_1 d_i}}{2k_1 d_i} < k_3 < \frac{k_2 + \sqrt{k_2^2 - 4k_1 d_i}}{2k_1 d_i}$. Owing to $\frac{k_2^2}{k_1} > 4max_{i \in \mathcal{I}_F}\{d_i\}$, we get $\frac{k_2 - \sqrt{k_2^2 - 4k_1 d_i}}{2k_1 d_i} > 0$, $k_3 > 0$ and $\beta_i > 0$ for $i \in \mathcal{I}_F$. It is easy to know $p_2 < p_1 < 0$ according to $p_1 p_2 = k_1 \sum_{j=1}^N a_{ij}$ and $p_1 + p_2 = -k_2$, hence we have $k_4 > 0$ and $k_5 > 0$. Furthermore, we get $k_1 k_4 > 0$, $\frac{k_5}{k_3} > 0$ and $k_1(k_4 + k_3 k_5) > 0$.

Since those followers and leaders are agents with second-order dynamics and the velocity of leaders is zero, we can get a first-order MAS of $N+M$ agents with DT dynamics as follows:

$$\begin{cases} x_i(t_{k+1}) = x_i(t_k) + k_1k_4\sum_{j=1}^{N}a_{ij}(x_j(t_k)-x_i(t_k)) + \frac{k_5}{k_3}(z_{i'}(t_k)-x_i(t_k)), & i\in\mathcal{I}_F,\\ z_{i'}(t_{k+1}) = z_{i'}(t_k) + k_1(k_4+k_3k_5)\sum_{j=1}^{N}a_{ij}(x_j(t_k)-z_{i'}(t_k)) + k_5\left(k_2-\frac{1}{k_3}-k_1k_3d_i\right)(x_i(t_k)-z_{i'}(t_k)), & i\in\mathcal{I}_F,\\ x_i(t_{k+1}) = x_i(t_k), & i\in\mathcal{I}_R, \end{cases} \tag{17}$$

where $x_i\in\mathbb{R}$ and $z_{i'}\in\mathbb{R}$ are the states of the ith and i'th agents, respectively.

Step 2: Let $\mathcal{G}'=(\mathcal{V}',\mathcal{E}')$ be a communication digraph of the first-order MAS (17) with a vertex set $\mathcal{V}'=\mathcal{V}_1\cup\mathcal{V}_2\cup\mathcal{V}_3$, which $\mathcal{V}_1=\{s_1,s_2,\dots,s_M\}$, $\mathcal{V}_2=\{s_{1'},s_{2'},\dots,s_{M'}\}$ and $\mathcal{V}_3=\{s_{M+1},s_{M+2},\dots,s_N\}$. Suppose that the digraph $\mathcal{G}$ has a directed spanning forest $\mathcal{R}(\mathcal{G})$. For each edge $(s_j,s_i)\in\mathcal{R}(\mathcal{G})$, we have $(s_j,s_{i'})\in\mathcal{E}'$, $(s_{i'},s_i)\in\mathcal{E}'$ for $s_j\in\mathcal{V}_3$, $s_i\in\mathcal{V}_1$, and $(s_{j'},s_j)\in\mathcal{E}'$, $(s_j,s_{i'})\in\mathcal{E}'$, $(s_{i'},s_i)\in\mathcal{E}'$ for $s_j\in\mathcal{V}_1$, $s_i\in\mathcal{V}_1$. Adding these edges to $\mathcal{R}(\mathcal{G})$, we get a directed spanning forest $\mathcal{R}(\mathcal{G}')$ for $\mathcal{G}'$.

Combining *Step 1*, *Step 2*, and Lemma 6, we get that the first-order MAS (17) can solve the containment control problem. Moreover, when $t_k\to\infty$, we can get

$$x_F(t_k)\to -L_F^{-1}L_{FR}x_R(0),\ \ v_F(t_k)\to\mathbf{0}. \tag{18}$$

Step 3: In this step, we will prove that $x_F(t)\to x_F(t_k)$, $v_F(t)\to v_F(t_k)$ when $t\to\infty$. From (16), for $i\in\mathcal{I}_F$, we have

$$\begin{aligned} x_i(t)-x_i(t_k) &= c_1\left(e^{p_1(t-t_k)}-1\right)+c_2\left(e^{p_2(t-t_k)}-1\right)\\ &= \frac{v_i(t_k)+\frac{p_2\sum_{j=1}^{N}a_{ij}(x_j(t_k)-x_i(t_k))}{\sum_{j=1}^{N}a_{ij}}}{p_1-p_2}\left(e^{p_1(t-t_k)}-1\right)+\frac{v_i(t_k)+\frac{p_1\sum_{j=1}^{N}a_{ij}(x_j(t_k)-x_i(t_k))}{\sum_{j=1}^{N}a_{ij}}}{p_2-p_1}\left(e^{p_2(t-t_k)}-1\right)\\ &= k_1q_i^{(1)}(t)\sum_{j=1}^{N}a_{ij}(x_j(t_k)-x_i(t_k))+q_i^{(2)}(t)v_i(t_k), \end{aligned} \tag{19}$$

and

$$\begin{aligned} v_i(t)-v_i(t_k) &= c_1p_1\left(e^{p_1(t-t_k)}-1\right)+c_2p_2\left(e^{p_2(t-t_k)}-1\right)\\ &= \frac{p_1v_i(t_k)+k_1\sum_{j=1}^{N}a_{ij}(x_j(t_k)-x_i(t_k))}{p_1-p_2}\left(e^{p_1(t-t_k)}-1\right)+\frac{p_2v_i(t_k)+k_1\sum_{j=1}^{N}a_{ij}(x_j(t_k)-x_i(t_k))}{p_2-p_1}\left(e^{p_2(t-t_k)}-1\right)\\ &= k_1q_i^{(2)}(t)\sum_{j=1}^{N}a_{ij}(x_j(t_k)-x_i(t_k))+q_i^{(3)}(t)v_i(t_k), \end{aligned} \tag{20}$$

where $q_i^{(1)}(t)=\frac{\frac{e^{p_1(t-t_k)}-1}{p_1}-\frac{e^{p_2(t-t_k)}-1}{p_2}}{p_1-p_2}$, $q_i^{(2)}(t)=\frac{e^{p_1(t-t_k)}-e^{p_2(t-t_k)}}{p_1-p_2}$, $q_i^{(3)}(t)=\frac{p_1e^{p_1(t-t_k)}-p_2e^{p_2(t-t_k)}-(p_1-p_2)}{p_1-p_2}$, $i\in\mathcal{I}_F$.

From (17), we have $x_R(t_k)=x_R(0)$. Let $Q^{(1)}(t)=diag\{q_1^{(1)}(t),q_2^{(1)}(t),\dots,q_M^{(1)}(t)\}$, $Q^{(2)}(t)=diag\{q_1^{(2)}(t),q_2^{(2)}(t),\dots,q_M^{(2)}(t)\}$, $Q^{(3)}(t)=diag\{q_1^{(3)}(t),q_2^{(3)}(t),\dots,q_M^{(3)}(t)\}$, hence (19) and (20) can be written in the compact form as

$$x_F(t)-x_F(t_k)=-k_1Q^{(1)}(t)L_F(x_F(t_k)+L_F^{-1}L_{FR}x_R(0))+Q^{(2)}(t)v_F(t_k), \tag{21}$$

$$v_F(t)-v_F(t_k)=-k_1Q^{(2)}(t)L_F(x_F(t_k)+L_F^{-1}L_{FR}x_R(0))+Q^{(3)}(t)v_F(t_k). \tag{22}$$

By Lemma 4, we can obtain that $q_i^{(1)}(t)$, $q_i^{(2)}(t)$, and $q_i^{(3)}(t)$ are all bounded functions on t. Thus it is easy to discover that every element of matrix $Q^{(1)}(t)$, $Q^{(2)}(t)$ and $Q^{(3)}(t)$ is bounded. When $t\to\infty$, we have $t_k\to\infty$. Using (18) for (21)–(22), we get

$$\lim_{t\to\infty}\|x_F(t)-x_F(t_k)\|=\lim_{t\to\infty}\|v_F(t)-v_F(t_k)\|=0. \tag{23}$$

Furthermore, from (18) and (23), we can obtain $x_F(t)\to -L_F^{-1}L_{FR}x_R(0)$ and $v_F(t)\to 0$ when $t\to\infty$.

Suppose that the feedback gains satisfy $\frac{k_2^2}{k_1} > 4max_{i\in\mathcal{I}_F}\{d_i\}$, combining *Step 1*, *Step 2*, and *Step 3*, we prove that the hybrid MAS (14) with protocol (15) can solve the containment control problem if the digraph $\mathcal{G}$ contains a directed spanning forest.

Necessity: Suppose that the communication digraph $\mathcal{G}$ has no directed spanning forest. Then there exists at least one follower that is incapable of receiving information from any leader, which implies that the states of this follower are uncorrelated with the states of those leaders. Therefore, the containment control of the hybrid MAS (14) cannot be reached under the protocol (15). ■

4.1.2 | Case 2

In this case, we still assume that those followers interact with leaders at the sampling time and observe their own states in real time. However, unlike Case 1, all followers interact with each other in real time. Thus, a distributed protocol for the hybrid MAS (14) is devised as

$$\begin{cases} u_i(t) = k_1\left(\sum_{j=1}^{M} a_{ij}(x_j(t)-x_i(t)) + \sum_{j=M+1}^{N} a_{ij}(x_j(t_k)-x_i(t))\right) - k_2 v_i(t), \quad t\in(t_k,t_{k+1}], & i\in\mathcal{I}_F, \\ v_i(t_k)=0, & i\in\mathcal{I}_R, \end{cases} \tag{24}$$

where $k_1 > 0, k_2 > 0$ are the feedback gains and $\mathcal{A} = [a_{ij}]_{N\times N}$ is the weighted adjacency matrix associated with the digraph $\mathcal{G}$.

Remark 2. According to Section 4.1.1, we know that the premise of using the system transformation method is to solve the system differential equation. However, using protocol (24) for the hybrid MAS (14), the resulting differential equations are difficult to solve. Therefore, in this section, we will use stability theory to analyze the containment control problem.

Theorem 5. *Consider a communication digraph $\mathcal{G}$. Then, the hybrid MAS (14) with protocol (24) can solve the containment control problem if and only if the digraph $\mathcal{G}$ contains a directed spanning forest, and the feedback gains satisfy*

$$\frac{k_2^2}{k_1} > max_{i\in\mathcal{I}_F}\left\{\frac{Im^2(\lambda_i)}{Re(\lambda_i)}\right\},$$

where λ_i are the eigenvalues of L_F.

Proof. (*Sufficiency*): By Lemma 5, the matrix L_F is invertible. Then, by using protocol (24) for the hybrid MAS (14), it can be written in a compact form as

$$\begin{cases} \dot{x}_F(t) = v_F(t), \\ \dot{v}_F(t) = -k_1 L_F(x_F(t) + L_F^{-1}L_{FR}x_R(t_k)) - k_2 v_F(t), \\ x_R(t_{k+1}) = x_R(t_k) = x_R(0). \end{cases} \tag{25}$$

Define $\delta(t) = [\delta_x(t), \delta_v(t)]^{\mathrm{T}}$ with $\delta_x(t) = x_F(t) + L_F^{-1}L_{FR}x_R(0)$, $\delta_v(t) = v_F(t)$. Then, according to (25), the derivative of $\delta(t)$ can be obtained as follows:

$$\dot{\delta}(t) = \begin{bmatrix}\dot{\delta}_x(t)\\ \dot{\delta}_v(t)\end{bmatrix} = \begin{bmatrix}\dot{x}_F(t)\\ \dot{v}_F(t)\end{bmatrix} = \begin{bmatrix}\delta_v(t)\\ -k_1L_F\delta_x(t) - k_2\delta_v(t)\end{bmatrix} = \Gamma\delta(t), \tag{26}$$

where $\Gamma = \begin{bmatrix} 0 & I_M \\ -k_1 L_F & -k_2 I_M\end{bmatrix}$. By Lemma 2, the characteristic equation of Γ is written as

$$\begin{aligned} det(\mu I_{2M} - \Gamma) &= det\begin{pmatrix} \mu I_M & -I_M \\ k_1 L_F & \mu I_M + k_2 I_M \end{pmatrix} \\ &= det(\mu^2 I_M + k_2\mu I_M + k_1 L_F) \end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{M} g(\mu, \lambda_i) \\
&= 0,
\end{aligned}
$$

where $g(\mu, \lambda_i) = \mu^2 + k_2\mu + k_1\lambda_i$, λ_i are the eigenvalues of L_F, $i \in \mathcal{I}_F$. By Lemma 3, we get that $g(\mu, \lambda_i)$ is Hurwitz stable if and only if $k_2 > 0$ and $k_2^2 k_1 Re(\lambda_i) - k_1^2 Im^2(\lambda_i) > 0$ hold, that is, $\frac{k_2^2}{k_1} > max_{i\in\mathcal{I}_F}\left\{\frac{Im^2(\lambda_i)}{Re(\lambda_i)}\right\}$. When the above condition holds, we obtain that the CT error system (26) is asymptotically stable, which indicates that $x_F(t) \to -L_F^{-1}L_{FR}x_R(0)$ and $v_F(t) \to \mathbf{0}$ for $t \to \infty$. Thence, from Lemma 5 and Definition 1, we know that all followers will asymptotically converge to the convex hull spanned by those of leaders. And by Definition 3, the distributed protocol (24) can solve the containment control problem of the hybrid MAS (14).

Necessity: The proof of necessity is similar to Section 4.1.1, which is omitted here. ■

Remark 3. Consider a distributed protocol with relative velocity information described by

$$
\begin{cases}
u_i(t) = k_1\sum_{j=1}^{N} a_{ij}(x_j(t_k) - x_i(t)) + k_2\sum_{j=1}^{N} a_{ij}(v_j(t_k) - v_i(t)), \quad t \in (t_k, t_{k+1}], & i \in \mathcal{I}_F,\\
u_i(t_k) = 0, & i \in \mathcal{I}_R,
\end{cases}
\tag{27}
$$

where $k_1 > 0, k_2 > 0$ are the feedback gains and $\mathcal{A} = [a_{ij}]_{N\times N}$ is the weighted adjacency matrix associated with the digraph $\mathcal{G}$. Through simulation verification, we discover that the hybrid MAS (14) cannot reach the containment under the protocol (27). That is why we consider the above two distributed protocols with absolute velocity information.

4.2 | Followers are agents with second-order DT dynamics

For the second-order hybrid MAS, the containment control problem with DT followers and CT leaders is considered in this section. Then, the dynamics of leaders and followers are given by

$$
\begin{cases}
x_i(t_{k+1}) = x_i(t_k) + hv_i(t_k), \quad v_i(t_{k+1}) = v_i(t_k) + hu_i(t_k), \quad t_k = kh, \quad k \in \mathbb{N}, & i \in \mathcal{I}_F,\\
\dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = u_i(t), & i \in \mathcal{I}_R,
\end{cases}
\tag{28}
$$

where $h > 0$ is the sampling period, $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$, and $u_i \in \mathbb{R}$ are the position, the velocity, and the control input of agent i, respectively. Let $x_F(\cdot) = [x_1(\cdot), x_2(\cdot), \dots, x_M(\cdot)]^{\mathrm{T}}$, $v_F(\cdot) = [v_1(\cdot), v_2(\cdot), \dots, v_M(\cdot)]^{\mathrm{T}}$, $x_R(\cdot) = [x_{M+1}(\cdot), x_{M+2}(\cdot), \dots, x_N(\cdot)]^{\mathrm{T}}$, $v_R(\cdot) = [v_{M+1}(\cdot), v_{M+2}(\cdot), \dots, v_N(\cdot)]^{\mathrm{T}}$, $x_F(0) = [x_1(0), x_2(0), \dots, x_M(0)]^{\mathrm{T}}$, $v_F(0) = [v_1(0), v_2(0), \dots, v_M(0)]^{\mathrm{T}}$, $x_R(0) = [x_{M+1}(0), x_{M+2}(0), \dots, x_N(0)]^{\mathrm{T}}$, and $v_R(0) = [v_{M+1}(0), v_{M+2}(0), \dots, v_N(0)]^{\mathrm{T}}$, where $x_i(0)$ and $v_i(0)$ are the initial conditions of agent i.

This situation is similar to the analysis of the interaction mode between leaders and followers in Section 3.2. Those followers can only observe their own states and interact with all their neighbors at the sampling time. Thus, a distributed protocol with absolute velocity information for the hybrid MAS (28) is devised as

$$
\begin{cases}
u_i(t_k) = k_1\sum_{j=1}^{N} a_{ij}(x_j(t_k) - x_i(t_k)) - k_2 v_i(t_k), & i \in \mathcal{I}_F,\\
v_i(t) = 0, & i \in \mathcal{I}_R,
\end{cases}
\tag{29}
$$

where $k_1 > 0$, $k_2 > 0$, k_1, k_2 are the feedback gains and $\mathcal{A} = [a_{ij}]_{N\times N}$ is the weighted adjacency matrix associated with the digraph $\mathcal{G}$.

Theorem 6. *Consider a communication digraph $\mathcal{G}$. Then, the hybrid MAS (28) with protocol (29) can solve the containment control problem if and only if the digraph $\mathcal{G}$ contains a directed spanning forest, and the sampling period and the feedback gains satisfy*

$$
\begin{cases}
h < \frac{k_2 Re(\lambda_i)}{k_1|\lambda_i|^2},\\
k_1^3 h^4|\lambda_i|^4 + \theta_i|\lambda_i|^2 + \zeta_i > 0,
\end{cases}
\tag{30}
$$

where

$$\begin{cases} \theta_i = k_1 h^2 (4k_1(1-k_2h)Re(\lambda_i) + k_2^2), \\ \zeta_i = 2(k_2(2-k_2h)(k_2 - 2k_1hRe(\lambda_i))Re(\lambda_i) - 2k_1 Im^2(\lambda_i)), \end{cases}$$

and λ_i are the eigenvalues of L_F, $i \in \mathcal{I}_F$.

Proof. (*Sufficiency*): Using protocol (29) for the hybrid MAS (28), it can be written in a compact form as

$$\begin{cases} x_F(t_{k+1}) = x_F(t_k) + hv_F(t_k), \\ v_F(t_{k+1}) = (I_M - k_2hI_M)v_F(t_k) - k_1hL_F(x_F(t_k) + L_F^{-1}L_{FR}x_R(t_k)), \\ x_R(t) = x_R(0). \end{cases}$$

As far as protocol (29) is concerned, those followers only need to obtain the states of leaders at the sampling time. Then let $t = t_{k+1}$, we have $x_R(t_k) = x_R(0)$.

If the digraph $\mathcal{G}$ contains a directed spanning forest, then the matrix L_F is invertible in line with Lemma 5. Hence, we can define $\delta(t_k) = [\delta_x(t_k), \delta_v(t_k)]^{\mathrm{T}}$ with $\delta_x(t_k) = x_F(t_k) + L_F^{-1}L_{FR}x_R(t_k)$, $\delta_v(t_k) = v_F(t_k)$. Then, we have

$$\delta(t_{k+1}) = \begin{bmatrix} \delta_x(t_{k+1}) \\ \delta_v(t_{k+1}) \end{bmatrix} = \begin{bmatrix} \delta_x(t_k) + h\delta_v(t_k) \\ -k_1hL_F\delta_x(t_k) + (I_M - k_2hI_M)\delta_v(t_k) \end{bmatrix} = \Gamma'\delta(t_k), \tag{31}$$

where $\Gamma' = \begin{bmatrix} I_M & hI_M \\ -k_1hL_F & I_M - k_2hI_M \end{bmatrix}$. By Lemma 2, the characteristic equation of Γ' is written as

$$\begin{aligned} det(\mu I_{2M} - \Gamma') &= det\begin{pmatrix} (\mu-1)I_M & -hI_M \\ k_1hL_F & (\mu-1)I_M + k_2hI_M \end{pmatrix} \\ &= det(\mu^2 I_M + (k_2hI_M - 2I_M)\mu + k_1h^2L_F - k_2hI_M + I_M) \\ &= \prod_{i=1}^{M} g'(\mu, \lambda_i) \\ &= 0, \end{aligned}$$

where $g'(\mu, \lambda_i) = \mu^2 + (k_2h - 2)\mu + k_1h^2\lambda_i - k_2h + 1$, λ_i are the eigenvalues of L_F, $i \in \mathcal{I}_F$. Then, we need to analyze the Schur stability of the DT error system (31). Using the bilinear transformation $\mu = \frac{s+1}{s-1}$, we can obtain

$$r_i(s) = (s-1)^2 g'\left(\frac{s+1}{s-1}, \lambda_i\right) = k_1h^2\lambda_i s^2 + 2(k_2h - k_1h^2\lambda_i)s + k_1h^2\lambda_i - 2k_2h + 4. \tag{32}$$

Owing to $k_1h^2\lambda_i \neq 0$, the first coefficient of (32) can be reduced to 1. Then a new quadratic polynomial is written as

$$\hat{r}_i(s) = s^2 + \left(\frac{2k_2\overline{\lambda}_i}{k_1h|\lambda_i|^2} - 2\right)s + \frac{4\overline{\lambda}_i}{k_1h^2|\lambda_i|^2} - \frac{2k_2\overline{\lambda}_i}{k_1h|\lambda_i|^2} + 1 \triangleq s^2 + \omega_1 s + \omega_0, \tag{33}$$

where $\omega_1 = \frac{2k_2\overline{\lambda}_i}{k_1h|\lambda_i|^2} - 2$, $\omega_0 = \frac{4\overline{\lambda}_i}{k_1h^2|\lambda_i|^2} - \frac{2k_2\overline{\lambda}_i}{k_1h|\lambda_i|^2} + 1$.

The Schur stability of the DT error system (31) is equivalent to the Hurwitz stability of the quadratic polynomial (33). By Lemma 3, we know that $\hat{r}_i(s)$ is Hurwitz stable if and only if $Re(\omega_1) > 0$ and $Re(\omega_1)Im(\omega_1)Im(\omega_0) + Re^2(\omega_1)Re(\omega_0) - Im^2(\omega_0) > 0$. Here,

$$\begin{cases} Re(\omega_1) = \frac{2k_2Re(\lambda_i)}{k_1h|\lambda_i|^2} - 2, & Re(\omega_0) = \frac{4Re(\lambda_i)}{k_1h^2|\lambda_i|^2} - \frac{2k_2Re(\lambda_i)}{k_1h|\lambda_i|^2} + 1, \\ Im(\omega_1) = \frac{2k_2Im(\overline{\lambda}_i)}{k_1h|\lambda_i|^2}, & Im(\omega_0) = \frac{4Im(\overline{\lambda}_i)}{k_1h^2|\lambda_i|^2} - \frac{2k_2Im(\overline{\lambda}_i)}{k_1h|\lambda_i|^2}. \end{cases}$$

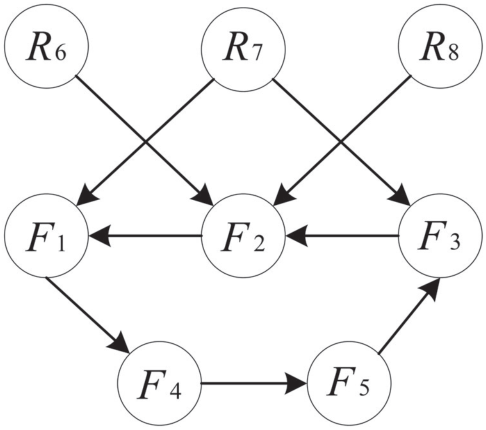

FIGURE 2 A communication digraph $\mathcal{G}$ of hybrid MASs

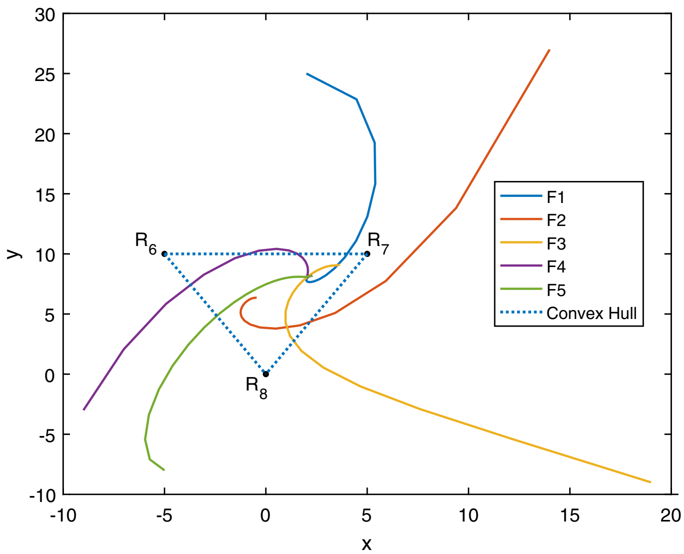

FIGURE 3 State trajectories of followers and leaders under the protocol (2)

Thus the DT error system (31) is asymptotically stable if and only if (30) holds. It shows that $x_F(t_k) \to -L_F^{-1}L_{FR}x_R(0)$ and $v_F(t_k) \to \mathbf{0}$ as $t_k \to \infty$. Thence, from Lemma 5 and Definition 1, we know that all followers will asymptotically converge to the convex hull spanned by those of leaders. And by Definition 3, the containment control for the hybrid MAS (28) with protocol (29) can be achieved.

Necessity: The proof of necessity is similar to Section 4.1.1, which is omitted here. ∎

5 | SIMULATIONS

In this section, we will give some simulations to demonstrate the validity of the theorems in Sections 3 and 4. Consider a communication digraph $\mathcal{G}$ depicted in Figure 2, in which followers are labeled $F_1 \sim F_5$ and leaders are labeled $R_6 \sim R_8$. Obviously, the digraph $\mathcal{G}$ has a directed spanning forest. For simplicity, we assume that the weight of each edge is 1. Furthermore, the eigenvalues of L_F are $\lambda_1 = 0.5567$, $\lambda_{2,3} = 1.3904 \pm 0.7072i$, and $\lambda_{4,5} = 2.8312 \pm 0.3224i$, respectively.

Example 1. Consider the hybrid MAS (1) under the digraph $\mathcal{G}$, in which $F_1 \sim F_5$ take the first-order CT dynamics, and $R_6 \sim R_8$ take the first-order DT dynamics. Suppose that the sampling period $h = 0.2$. Figures 3 and 4 display the position trajectories of followers and leaders using distributed protocol (2) and (8), respectively. In both cases, the CT follower

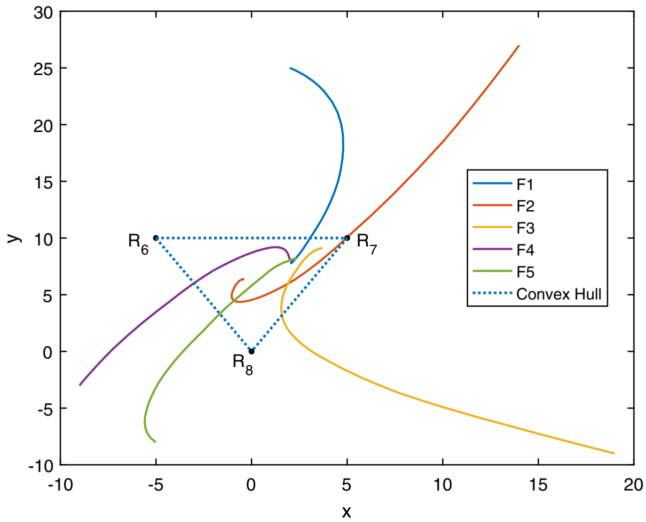

FIGURE 4 State trajectories of followers and leaders under the protocol (8)

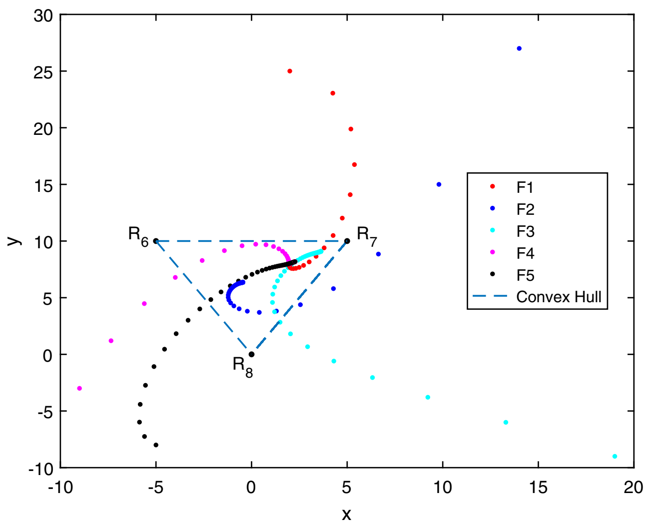

FIGURE 5 State trajectories of followers and leaders under the protocol (13)

$F_1 \sim F_5$ asymptotically converge to the convex hull spanned by the positions of the DT leader $R_6 \sim R_8$, which is consistent with the results of Theorems 1 and 2.

Example 2. Consider the hybrid MAS (12) under the digraph $\mathcal{G}$. Assume that the sampling period $h = 0.15$, which satisfies $h < min_{i\in\mathcal{I}_F}\left\{\frac{2Re(\lambda_i)}{|\lambda_i|}\right\} \approx 0.7064$. By using distributed protocol (13), the position trajectories of followers and leaders are depicted in Figure 5. The DT follower $F_1 \sim F_5$ asymptotically converge to the convex hull spanned by the positions of the CT leader $R_6 \sim R_8$, which is consistent with the results of Theorem 3.

Example 3. Consider the hybrid MAS (14) under the digraph $\mathcal{G}$, in which $F_1 \sim F_5$ take the second-order CT dynamics, and $R_6 \sim R_8$ take the second-order DT dynamics. Assume that the sampling period $h = 0.5$ and the feedback gains $k_1 = 2$, $k_2 = 5$. Then, we have $k_2^2/k_1 > 4max_{i\in\mathcal{I}_F}\{d_i\} = 12$ and $k_2^2/k_1 > max_{i\in\mathcal{I}_F}\left\{Im^2(\lambda_i)/Re(\lambda_i)\right\} \approx 0.3597$. By using distributed protocol (15) and (24), the state trajectories of followers and leaders are depicted in Figures 6 and 7, respectively. Figures 6A and 7A illustrate that the CT follower $F_1 \sim F_5$ asymptotically converge to the convex hull spanned by the positions of the DT leader $R_6 \sim R_8$. Figure 6B and 7B illustrate that the velocity of all agents eventually converge to zero. The result is consistent with Theorems 4 and 5.

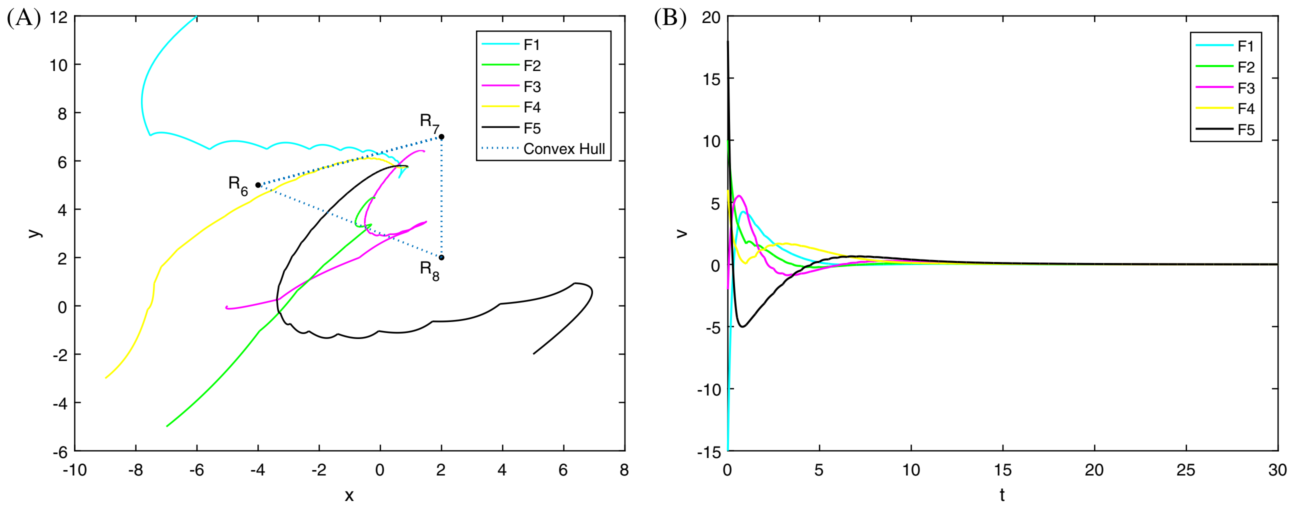

FIGURE 6 State trajectories of followers and leaders under the protocol (15). (A) Position. (B) Velocity

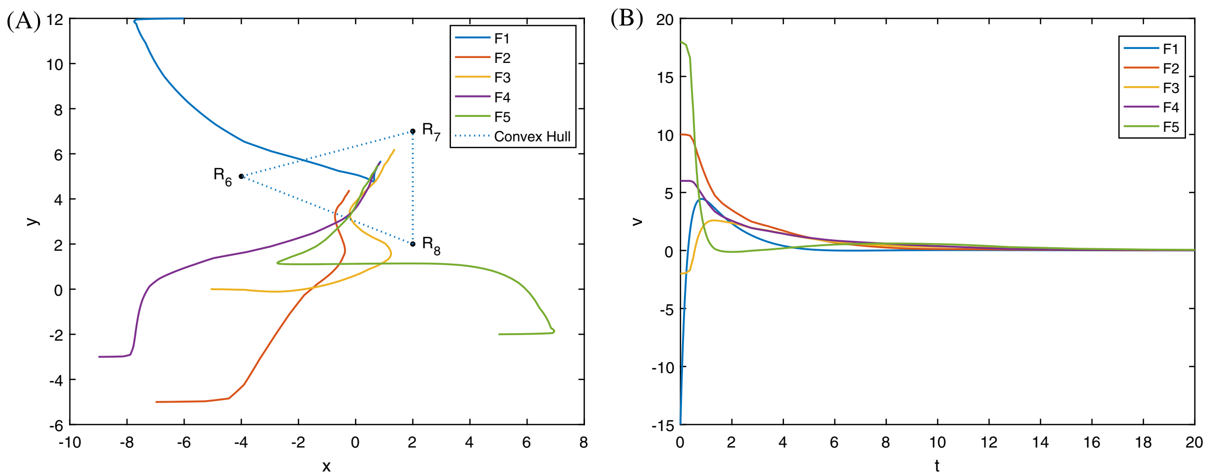

FIGURE 7 State trajectories of followers and leaders under the protocol (24). (A) Position. (B) Velocity

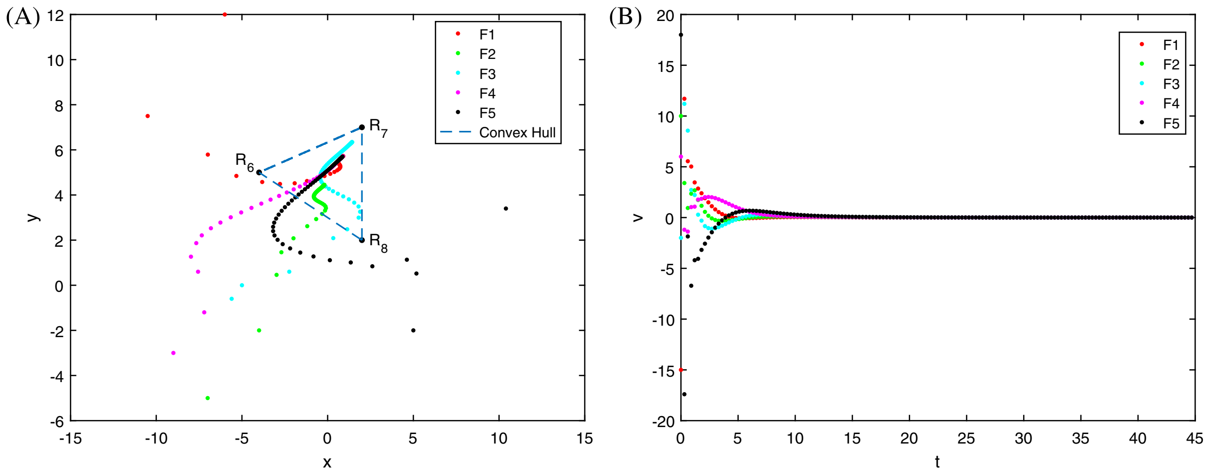

FIGURE 8 State trajectories of followers and leaders under the protocol (29). (A) Position. (B) Velocity

Example 4. Consider the hybrid MAS (28) under the digraph $\mathcal{G}$. Suppose that the sampling period $h = 0.3$ and the feedback gains $k_1 = 2$ and $k_2 = 5$, which satisfy the condition (30). By using distributed protocol (29), the state trajectories of followers and leaders are depicted in Figure 8. And we find that the DT follower $F_1 \sim F_5$ asymptotically converge to the convex hull spanned by the positions of the CT leader $R_6 \sim R_8$ and the velocity of all agents eventually converge to zero. That is consistent with the results of Theorem 6.

6 | CONCLUSIONS

This article investigated the containment control problem of hybrid MASs with the coexistence of CT followers/leaders and DT leaders/followers. In the case of multiple CT followers, two novel distributed protocols were designed for first- and second-order hybrid MASs, respectively, by analyzing the interaction modes between followers. Meanwhile, in the case of multiple DT followers, a distributed protocol was designed for first- and second-order hybrid MASs, respectively. For the containment control problem under different protocols, appropriate methods were adopted to achieve containment, and further some conditions were established to guarantee that CT/DT followers asymptotically converge to the convex hull spanned by DT/CT leaders. In particular, when followers have CT dynamics, we found that the achievement of containment control is not limited by the sampling period, no matter the first- or second-order hybrid MAS. Future work will focus on the containment control of hybrid MASs with time delays.

ACKNOWLEDGMENT

This work was supported in part by the National Natural Science Foundation of China under Grants 61773303 and 61962049.

CONFLICT OF INTEREST

The authors confirm that there is no conflict of interest in this article.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

ORCID

Yuanshi Zheng https://orcid.org/0000-0002-1143-2509

REFERENCES

1. Olfati-Saber R, Fax JA, Murray RM. Consensus and cooperation in networked multi-agent systems. *Proc IEEE*. 2007;95(1):215-233.
2. Chen S, Zhang Z, Zheng Y. H∞ scaled consensus for MASs with mixed time delays and disturbances via observer-based output feedback. *IEEE Trans Cybern*. 2020. doi:10.1109/TCYB.2020.3001643
3. Liu X, Xie Y, Li F, et al. Formation control of singular multiagent systems with switching topologies. *Int J Robust Nonlinear Control*. 2020;30(2):652-664.
4. Shao J, Zheng W, Shi L, et al. Leader-follower Cucker-Smale flocking with lossy links and general weight functions. *IEEE Trans Automat Contr*. 2020. doi:10.1109/TAC.2020.3046695
5. Lin J, Morse AS, Anderson BDO. The multi-agent rendezvous problem. Proceedings of the 42nd IEEE International Conference on Decision and Control; 2003:1508-1513; Maui, HI.
6. Liu L, Liu Y, Chen A, et al. Integral barrier Lyapunov function based adaptive control for switched nonlinear systems. *Sci China Inf Sci*. 2020;63(3):132203:1-132203:14.
7. Jadbabaie A, Lin J, Morse AS. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans Automat Contr*. 2003;48(6):988-1001.
8. Zhang Z, Chen S, Zheng Y. Fully distributed scaled consensus tracking of high-order multi-agent systems with time delays and disturbances. *IEEE Trans Ind Inform*. 2022;18(1):305-314.
9. Liu J, Ruan X, Zheng Y. Iterative learning control for discrete-time systems with full learnability. *IEEE Trans Neural Netw Learn Syst*. 2020. doi:10.1109/TNNLS.2020.3028388
10. Dimarogonas DV, Egerstedt M, Kyriakopoulos KJ. A leader-based containment control strategy for multiple unicycles. Proceedings of the 45th IEEE Conference on Decision and Control; 2006:5968-5973; San Diego, CA.
11. Notarstefano G, Egerstedt M, Haque M. Containment in leader–follower networks with switching communication topologies. *Automatica*. 2011;47(5):1035-1040.
12. Liu H, Xie G, Wang L. Necessary and sufficient conditions for containment control of networked multi-agent systems. *Automatica*. 2012;48(7):1415-1422.

13. Li B, Chen Z, Liu Z, et al. Containment control of multi-agent systems with fixed time-delays in fixed directed networks. *Neurocomputing*. 2016;173(3):2069-2075.
14. Zheng Y, Li T, Wang L. Containment control of multi-agent systems with measurement noises; 2014. arXiv preprint arXiv:1405.4681.
15. Cao Y, Ren W. Containment control with multiple stationary or dynamic leaders under a directed interaction graph. Proceedings of the 48th IEEE Conference on Decision, and 28th Chinese Control Conference; 2009:3014-3019; Shanghai, P.R. China.
16. Shi L, Shao J, Zheng W, et al. Asynchronous containment control for discrete-time second-order multi-agent systems with time-varying delays. *J Franklin Inst*. 2017;354(18):8552-8569.
17. Zhu Y, Zheng Y, Wang L. Containment control of switched multi-agent systems. *Int J Control*. 2015;88(12):2570-2577.
18. Wang F, Ni Y, Liu Z, et al. Containment control for general second-order multi-agent systems with switched dynamics. *IEEE Trans Cybern*. 2020;50(2):550-560.
19. Xu C, Li B, Yang L. Semi-global containment of discrete-time high-order multi-agent systems with input saturation via intermittent control. *IET Control Theory Appl*. 2020;14(16):2303-2309.
20. Halloy J, Sempo G, Caprari G, et al. Social integration of robots into groups of cockroaches to control self-organized choices. *Science*. 2007;318(5853):1155-1158.
21. Zheng Y, Zhu Y, Wang L. Consensus of heterogeneous multi-agent systems. *IET Control Theory Appl*. 2011;5(16):1881-1888.
22. Zheng Y, Wang L. Distributed consensus of heterogeneous multi-agent systems with fixed and switching topologies. *Int J Control*. 2012;85(12):1967-1976.
23. Zheng Y, Wang L. Containment control of heterogeneous multi-agent systems. *Int J Control*. 2014;87(1):1-8.
24. Asgari M, Atrianfar H. Necessary and sufficient conditions for containment control of heterogeneous linear multi-agent systems with fixed time delay. *IET Control Theory Appl*. 2019;13(13):2065-2074.
25. Xiao Q, Lewis FL, Zeng Z. Containment Control for multiagent systems under two intermittent control schemes. *IEEE Trans Automat Contr*. 2019;64(3):1236-1243.
26. Wang X, Shi L, Shao J. Containment control for high-order multi-agent systems with heterogeneous time delays. *IET Control Theory Appl*. 2018;12(9):1340-1348.
27. Zuo S, Song Y, Lewis FL, et al. Output containment control of linear heterogeneous multi-agent systems using internal model principle. *IEEE Trans Cybern*. 2017;47(8):2099-2109.
28. Zheng Y, Ma J, Wang L. Consensus of hybrid multi-agent systems. *IEEE Trans Neural Netw Learn Syst*. 2018;29(4):1359-1365.
29. Zhao Q, Zheng Y, Liu J, et al. Further analysis for consensus of hybrid multi-agent systems: a unified framework. *Int J Robust Nonlinear Control*. 2021. doi:10.1002/rnc.5721
30. Zheng Y, Zhao Q, Ma J, et al. Second-order consensus of hybrid multi-agent systems. *Syst Control Lett*. 2019;125:51-58.
31. Ma J, Ye M, Zheng Y, et al. Consensus analysis of hybrid multiagent systems: a game-theoretic approach. *Int J Robust Nonlinear Control*. 2019;29(6):1840-1853.
32. Wang X, Su H. Consensus of hybrid multi-agent systems by event-triggered/self-triggered strategy. *Appl Math Comput*. 2019;359:490-501.
33. Su H, Wang X, Zeng Z. Consensus of second-order hybrid multiagent systems by event-triggered strategy. *IEEE Trans Cybern*. 2020;50(11):4648-4657.
34. Shang Y. Consensus of hybrid multi-agent systems with malicious nodes. *IEEE Trans Circuits Syst II Express Briefs*. 2020;67(4):685-689.
35. Horn RA, Johnson CR. *Matrix Analysis*. Cambridge University Press; 1987.
36. Rockafellar RT. *Convex Analysis*. Princeton University Press; 1972.
37. Gantmakher FR. *The Theory of Matrices*. Chelsea; 1959.
38. Parks PC, Hahn V. *Stability Theory*. Prentice-Hall; 1993.

How to cite this article: Chen Y, Zhao Q, Zheng Y, Zhu Y. Containment control of hybrid multi-agent systems. *Int J Robust Nonlinear Control*. 2021;1-19. doi: 10.1002/rnc.5883