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SHORT COMMUNICATION

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Further analysis for consensus of hybrid multiagent systems: A unified framework

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Abstract
Hybrid multiagent systems exist widely in real world and have been applied in many engineering fields. In this article, we consider the consensus problem for the hybrid multiagent system consisting of continuous-time and discrete-time dynamic agents. Different from the previous works, in this article, the interactions between different agents no longer only occur at the sampling time, but continuous-time dynamic agents can interact with their continuous-time dynamic neighbors in real time. By using graph theory and differential mean value theorem of matrix function, a consensus criterion is obtained for the hybrid multiagent system. A unified framework is also established for the consensus of continuous-time and discrete-time multiagent systems. Finally, a simulation example is given to illustrate the validity of our protocol.

KEYWORDS
consensus, continuous-time, discrete-time, hybrid multiagent system

1 | INTRODUCTION

Since the emergence of multiagent systems in the 1970s, it has been developed rapidly, and has become a method and tool for complex systems analysis and simulation. The goal of multiagent systems is to achieve complex intelligence through cooperation among a mass of agents which have simple intelligence but are easier to manage and control. Up to now, the research of multiagent systems mainly focuses on consensus,[1] flocking,[2,3] tracking,[4,5] game,[6,7] containment control,[8] formation control,[9] coverage control,[10] winner-take-all,[11] and so on.

As a fundamental problem of distributed coordination control, consensus has attracted extensive interest. It refers to that a great many agents in multiagent systems adjust and update their behavior under local mutual communication and cooperation, and finally all agents can agree on certain quantity of their interest. In the field of distributed decision, Tsitsiklis et al.[12] considered convergence and asymptotic agreement of multiagent system. In Reference 13, Vicsek et al. investigated a discrete-time system with n autonomous agents which can eventually move in the plane at the same speed and in the same direction. The observed behavior of Vicsek model was further explained theoretically in Reference 14. At the same time, continuous-time multiagent system has also attracted much attentions. In Reference 15, the authors studied the consensus of multiagent systems with fixed and switching topologies, they proposed two realistic and effective consensus protocols for communication networks with and without time-delays and provided the convergence analysis. Reference 16 extended the results of Reference 14 from undirected graphs to directed graphs, and gave some more relaxable conditions for solving the consensus of continuous-time multiagent systems.

In the real world, continuous-time and discrete-time dynamic agents often coexist in one system. Hence, Zheng et al.[17] considered a multiagent system which keeps switching between continuous-time and discrete-time subsystems, and gave some consensus criteria for the switched multiagent system under arbitrary switching. More generally, different dynamic agents in the system can interact and cooperate with each other. For instance, Halloy et al.[18] showed that collective decision-making of mixed groups of socially integrated autonomous robots and cockroaches leads to the choice of shared shelter, where the collective decision is generated by nonlinear feedbacks based on local interactions. This result is of great theoretical and practical significance, because it demonstrates the possibility of using intelligent autonomous devices to study and control self-organized behavioral patterns of social animals. In consequence, many researchers began to study the hybrid multiagent systems which are composed of continuous-time and discrete-time dynamic agents. In Reference 19, the authors obtained some consensus criteria for the hybrid multiagent system under three kinds of protocols, respectively. Next to, the second-order consensus of hybrid multiagent system is also considered in Reference 20. In Reference 21, the authors also investigated the consensus problem for the hybrid multiagent systems with heterogeneous dynamic. Reference 22 designed some pulse-modulated protocols for solving the consensus of hybrid multiagent system. In Reference 23, Su et al. proposed an event-triggered method to solve the consensus problem for the second-order hybrid multiagent system. In Reference 24, the author proposed a hybrid censoring strategy in order to reach resilient consensus for cooperative agents of hybrid multiagent system with some Byzantine agents. Ma et al.[25] used a game-theoretic approach to model the interactions between continuous-time and discrete-time dynamic agents, and designed a suitable cost function to achieve consensus of the considered hybrid multiagent system.

For the analysis of consensus problem for hybrid multiagent systems, the difficulty mainly lies in how to design the interactive modes of different dynamic agents to reach consensus. In Reference 19, Zheng et al. presented three kinds of protocols to solve the consensus problem for hybrid multiagent system. In the system of Reference 19, the information interactions between agents and their neighbors either all happen at sampling time, or only continuous-time dynamic agents can use their own states to update their behavior in real time. On account of it does not cover the situation that the continuous-time dynamics agents can interact with their neighbors with continuous-time dynamics in real time, Reference 19 does not establish a unified framework of the consensus problem for continuous-time and discrete-time multiagent systems. In order to enrich the interactive modes between different dynamic agents in hybrid multiagent systems, we propose a novel consensus protocol to extend the aforementioned results. Since more position-like information is used in this consensus protocol than the previous works, our consensus analysis is more difficult than that in Reference 19. Specifically, the mathematical tools used in Reference 19 no longer meet our need to prove the consensus problem of this hybrid multiagent system. Therefore, the use of the differential mean value theorem of matrix function is added on the basis of the original mathematical tools used in Reference 19. The main contributions of this article mainly concentrate on three aspects. Firstly, we propose a novel consensus protocol for the hybrid multiagent system, and enrich the interactive modes between different dynamic agents in hybrid multiagent systems. Secondly, a criterion for solving the consensus of the system is obtained by using graph theory and differential mean value theorem of matrix function. Thirdly, we establish a unified framework of the consensus problem for continuous-time and discrete-time multiagent systems under the proposed protocol.

The rest of this article is organized as follows. In Section 2, we introduce some related theories and give the hybrid multiagent system with its consensus protocol and definition. In Section 3, we propose the main theoretical results of this article. In Section 4, a numerical simulation is given to illustrate the effectiveness of our results. Finally, some conclusions are provided in Section 5.

Notation: Throughout this article, $\mathbb{R}$ represents the set of real number, $\mathbb{R}^n$ denotes the n-dimensional real vector space. $\mathbb{N}$ represents the set of natural number. Suppose that $\mathcal{I}_m = \{1, 2, \dots, m\}$, $\mathcal{I}_n \setminus \mathcal{I}_m = \{m+1, m+2, \dots, n\}$. For a given matrix or vector X, X^T denotes its transpose, $\|X\|$ denotes the Euclidean norm of a vector X. A matrix is nonnegative if all its elements are nonnegative. Denote by $\mathbf{1}_n$ (or $\mathbf{0}_n$) the n-dimensional column vector with all entries equal to one (or all zeros). I_n is an n-dimensional identity matrix. Matrix $A > B$ means all elements in matrix A are greater than the corresponding elements in matrix B, in other words, $A_{ij} > B_{ij}$ always holds.

2 | PRELIMINARIES

2.1 | Algebraic graph theory and matrix theory

A weighted directed graph $\mathcal{G}(\mathcal{A}) = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ of order n consists of a vertex set $\mathcal{V}$, an edge set $\mathcal{E}$ and a nonnegative matrix $\mathcal{A} = [a_{ij}]_{n\times n}$. The agent i's neighbor set is $\mathcal{N}_i = \{j : a_{ij} > 0\}$. A directed tree is a directed graph, where every node, except the root, has exactly one parent. A directed spanning tree is a directed tree, which is composed of all the nodes and some edges in graph $\mathscr{G}$. The degree matrix $\mathscr{D} = [d_{ij}]_{n\times n}$ is a diagonal matrix with $d_{ii} = \sum_{j:v_j\in\mathscr{N}_i} a_{ij}$ and the Laplacian matrix associated with graph $\mathscr{G}$ is defined as $\mathscr{L} = [l_{ij}]_{n\times n} = \mathscr{D} - \mathscr{A}$. It is easy to know that $\mathscr{L}\mathbf{1}_n = 0$.

A nonnegative matrix is called to be a (row) stochastic matrix if all row sums of it are 1. A stochastic matrix $P = [p_{ij}]_{n\times n}$ is called indecomposable and aperiodic (SIA) if $\lim_{k\to\infty} P^k = \mathbf{1_n}y^T$, where y is some column vector. Next, we will introduce some theories about the relationship between a stochastic matrix and its associated graph.

Lemma 1 (16). *A stochastic matrix has algebraic multiplicity equal to 1 for its eigenvalue $\lambda = 1$, if and only if the graph associated with the matrix has a spanning tree. In addition, a stochastic matrix with positive diagonal elements has the following properties: every eigenvalue not equal to 1 satisfies $|\lambda| < 1$.*

Lemma 2 (16). *Let $A = [a_{ij}]_{n\times n}$. If A has an eigenvalue $\lambda = 1$ with algebraic multiplicity equal to 1, and the remaining eigenvalues satisfy $|\lambda| < 1$, then A is SIA, that is to say, $\lim_{m\to\infty} A^m = \mathbf{1}_n v^T$,where v satisfies $\mathbf{1}_n^T v = 1$ and $A^T v = v$. Furthermore, all elements of v are nonnegative.*

Remark 1. Based on Lemmas 1 and 2, we can easily get the following result. Let $A = [a_{ij}]_{n\times n}$ is a stochastic matrix with positive diagonal elements. If the graph corresponding to the matrix A has a spanning tree, then A is SIA. It means that $\lim_{m\to\infty} A^m = \mathbf{1}_n v^T$, where v satisfies $\mathbf{1}_n^T v = 1$ and $A^T v = v$, and all elements of v are nonnegative.

Lemma 3 (26). *Let $D^1 = [x_0, x_0 + \Delta x]$, $F(x) \in G^{r\times s}[x]$ defined on D^1. Then, for arbitrary $P \in R^r$ and $Q \in R^s$, there is at least one point $x_1 = x_0 + t_1\Delta x \in D^1$, $t_1 \in (0, 1)$, such that*

$$P^T \frac{dF}{dx}\Big|_{x=x_1} Q = P^T \frac{1}{\Delta x} G^T[F(x_0 + \Delta x) - F(x_0)]Q.$$

Specially, when $P = I_r$ and $Q = I_s$, one get

$$\frac{dF}{dx}\Big|_{x=x_1} = \frac{1}{\Delta x}[F(x_0 + \Delta x) - F(x_0)].$$

2.2 | Hybrid multiagent system

In this subsection, we investigate the consensus problem for hybrid multiagent system. The system consists of continuous-time and discrete-time dynamic agents. The number of agents is n, labeled 1 to n, where the number of continuous-time dynamic agents is m $(m \le n)$. Without loss of generality, we assume that agent 1 to agent m are continuous-time dynamic agents. Thus, the dynamics of the hybrid multiagent system are described as follows

$$\begin{cases} \dot{x}_i(t) = u_i(t), & i \in \mathcal{I}_m, \\ x_i(t_{k+1}) = x_i(t_k) + u_i(t_k), \quad t_k = kh, \quad k \in \mathbb{N}, & i \in \mathcal{I}_n/\mathcal{I}_m, \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are agent i's position-like and control input, respectively. $h = t_{k+1} - t_k > 0$ is the sampling period. The initial condition of agent i is $x_i(0)$. Let $X = [x_1, x_2, \dots, x_n]^T$.

We assume that all continuous-time dynamic agents can utilize their own states and communicate with their neighbors with continuous-time dynamics and update their control inputs in real time. Meanwhile, all discrete-time dynamic agents interact with their neighbors and update their control inputs at the sampling time t_k. Based on this, a novel consensus protocol for hybrid multiagent system (1) is given as

$$\begin{cases} u_i(t) = \sum_{j=1}^{m} a_{ij}(x_j(t) - x_i(t)) + \sum_{j=m+1}^{n} a_{ij}(x_j(t_k) - x_i(t)), \quad t \in (t_k, t_{k+1}], & i \in \mathcal{J}_m, \\ u_i(t_k) = h\sum_{j=1}^{n} a_{ij}(x_j(t_k) - x_i(t_k)), & i \in \mathcal{J}_n/\mathcal{J}_m, \end{cases} \tag{2}$$

where $\mathscr{A} = [a_{ij}]_{n\times n}$ is the aforementioned weighted adjacency matrix corresponding to graph $\mathscr{G}$.

Remark 2. In consensus protocol (2), the interactions between discrete-time dynamic agents and their neighbors still occurs at the sampling time t_k for $t \in (t_k, t_{k+1}]$, which is the same as that in Reference 19. Different from Reference 19, for continuous-time dynamic agents, except for their interactions with their discrete-time dynamic neighbors still take place at the sampling time, the rest of the information interactions all happen in real time. Therefore, a novel interaction mode is designed for the hybrid multiagent system to solve consensus problem in this article.

Definition 1 (19). Hybrid multiagent system (1) can be said to reach consensus if for any initial conditions, we have

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \quad for\ i,j \in \mathcal{I}_m \tag{3}$$

and

$$\lim_{t_k\to\infty} \|x_i(t_k) - x_j(t_k)\| = 0, \quad for\ i,j \in \mathcal{I}_n. \tag{4}$$

Remark 3. The problem studied in this article is similar to that in Reference 19, but the mathematical tools used are different. The main reason is that the interaction modes between agents are different. In the protocol of this article, continuous-time dynamic agents interact more information than continuous-time dynamic agents do in Reference 19. Therefore, the dynamic form of this system has changed, and the mathematical tools used in Reference 19 no longer meet our need to prove the consensus problem of this hybrid multiagent system. Hence, the use of the differential mean value theorem of matrix function (Lemma 3) is added on the basis of the original mathematical tools used in Reference 19.

3 | MAIN RESULTS

In this section, we give a criterion for the hybrid multiagent system (1) with protocol (2) to solve consensus problem. Moreover, the results are proved mathematically by using graph theory and matrix theory.

Theorem 1. *Consider a directed communication graph* $\mathscr{G}$ *and suppose that* $0 < h < \frac{1}{\max_{i\in\mathcal{I}_n\setminus\mathcal{I}_m}\{d_{ii}\}}$. *Then, the hybrid multiagent system (1) with protocol (2) can solve consensus problem if graph* $\mathscr{G}$ *has a directed spanning tree.*

Proof. We divide the Laplacian matrix corresponding to the graph $\mathscr{G}$ into the following form

$$\mathscr{L} = \begin{pmatrix} \mathscr{L}_{cc} & \mathscr{L}_{cd} \\ \mathscr{L}_{dc} & \mathscr{L}_{dd} \end{pmatrix},$$

where $\mathscr{L}_{cc} \in \mathbb{R}^{m\times m}$, $\mathscr{L}_{cd} \in \mathbb{R}^{m\times(n-m)}$, $\mathscr{L}_{dc} \in \mathbb{R}^{(n-m)\times m}$, $\mathscr{L}_{dd} \in \mathbb{R}^{(n-m)\times(n-m)}$.

Under consensus protocol (2), we can express system (1) in matrix forms as follows

$$\begin{cases} \dot{X}_c(t) = -\mathscr{L}_{cc}X_c(t) - \mathscr{L}_{cd}X_d(t_k), & t \in (t_k, t_{k+1}], \\ X_d(t_{k+1}) = -h\mathscr{L}_{dc}X_c(t_k) + (I_{n-m} - h\mathscr{L}_{dd})X_d(t_k), \end{cases} \tag{5}$$

where $X_c(t) = (x_1(t), x_2(t), \dots, x_m(t))^T$, $X_d(t) = (x_{m+1}(t), x_{m+2}(t), \dots, x_n(t))^T$.

Next, we will prove the consensus of the hybrid multiagent system into two situations.

Situation 1: $\mathscr{L}_{cc}$ is an irreversible matrix, that is $\mathscr{L}_{cd} = 0$.

By combining formula (5) and $\mathscr{L}_{cd} = 0$, we can get

$$\begin{cases} X_c(t) = e^{-(t-t_k)\mathscr{L}_{cc}}X_c(t_k), & t \in (t_k, t_{k+1}], \\ X_d(t_{k+1}) = -h\mathscr{L}_{dc}X_c(t_k) + (I_{n-m} - h\mathscr{L}_{dd})X_d(t_k). \end{cases} \tag{6}$$

Let $t = t_{k+1}$. We can obtain the following discrete-time multiagent system

$$\begin{pmatrix} X_c(t_{k+1}) \\ X_d(t_{k+1}) \end{pmatrix} = B_1 \begin{pmatrix} X_c(t_k) \\ X_d(t_k) \end{pmatrix}, \tag{7}$$

where $B_1 = \begin{pmatrix} e^{-h\mathscr{L}_{cc}} & 0_{m\times(n-m)} \\ -h\mathscr{L}_{dc} & I_{n-m} - h\mathscr{L}_{dd} \end{pmatrix}$.

According to Remark 1, we will first prove matrix B_1 is a nonnegative matrix with positive diagonal elements. Let $\lambda = \max_{i\in\mathcal{I}_m}\{(h\mathscr{L}_{cc})_{ii}\}$. Then one get

$$e^{-h\mathscr{L}_{cc}} = e^{-\lambda I_m + (\lambda I_m - h\mathscr{L}_{cc})} = e^{-\lambda I_m} e^{\lambda I_m - h\mathscr{L}_{cc}},$$

where

$$e^{-\lambda I_m} = I_m - \lambda I_m + \frac{\lambda^2}{2!} I_m - \frac{\lambda^3}{3!} I_m + \ldots = e^{-\lambda} I_m$$

and

$$e^{\lambda I_m - h\mathscr{L}_{cc}} = I_m + (\lambda I_m - h\mathscr{L}_{cc}) + \frac{1}{2!}(\lambda I_m - h\mathscr{L}_{cc})^2 + \frac{1}{3!}(\lambda I_m - h\mathscr{L}_{cc})^3 + \ldots \ .$$

Obviously, $\lambda I_m - h\mathscr{L}_{cc}$ is a nonnegative matrix, in addition, all diagonal elements of the matrix $e^{\lambda I_m - h\mathscr{L}_{cc}}$ are positive and all its nondiagonal elements are nonnegative. Simultaneously, $e^{-\lambda} > 0$ always holds. Hence, $e^{-h\mathscr{L}_{cc}} = e^{-\lambda} \cdot e^{\lambda I_m - h\mathscr{L}_{cc}}$ is a nonnegative matrix and all its diagonal elements are positive. According to the properties of Laplacian matrix and $h > 0$, all elements of $-h\mathscr{L}_{dc}$ and all nondiagonal elements of $I_{n-m} - h\mathscr{L}_{dd}$ are obviously nonnegative. On account of $h < \frac{1}{\max_{i\in\mathcal{I}_n\setminus\mathcal{I}_m}\{d_{ii}\}}$, it is easy to know that all diagonal elements of $I_{n-m} - h\mathscr{L}_{dd}$ are positive.

Then, we will prove matrix B_1 is a stochastic matrix. Owing to $\mathscr{L}_{cd} = 0$, combined with the properties of Laplacian matrix, we can get $\mathscr{L}_{cc}\mathbf{1}_m = \mathbf{0}_m$ and $(\mathscr{L}_{dc} + \mathscr{L}_{dd})\mathbf{1}_n = \mathbf{0}_n$. For the first m rows of matrix B_1, the row sums are

$$e^{-h\mathscr{L}_{cc}}\mathbf{1}_m = \mathbf{1}_m - h\mathscr{L}_{cc}\mathbf{1}_m + \frac{h^2}{2!}(\mathscr{L}_{cc})^2\mathbf{1}_m - \frac{h^3}{3!}(\mathscr{L}_{cc})^3\mathbf{1}_m + \ldots = \mathbf{1}_m.$$

For the last $n - m$ rows of matrix B_1,

$$-h\mathscr{L}_{dc}\mathbf{1}_m + (I_{n-m} - h\mathscr{L}_{dd})\mathbf{1}_{n-m} = \mathbf{1}_{n-m} - h(\mathscr{L}_{dc} + \mathscr{L}_{dd})\mathbf{1}_n = \mathbf{1}_{n-m}.$$

We already know matrix B_1 is a nonnegative matrix, and therefore B_1 is a stochastic matrix.

We have proved that matrix B_1 is a stochastic matrix and all its diagonal elements are positive. Based on Remark 1, it is easy to know that B_1 is SIA if the communication graph $\mathscr{G}_1$ corresponding to matrix B_1 has a directed spanning tree. Hence, we need to prove that as long as graph $\mathscr{G}$ has a directed spanning tree, then graph $\mathscr{G}_1$ must have a directed spanning tree. That is to say, if $\mathscr{A}_{ij} \neq 0$ $(i \neq j)$, we need to have $(B_1)_{ij} \neq 0$ $(i \neq j)$. We already know that $e^{-h\mathscr{L}_{cc}} = e^{-\lambda} \cdot e^{\lambda I_m - h\mathscr{L}_{cc}}$ and $\lambda I_m - h\mathscr{L}_{cc}$ are nonnegative matrixes, where

$$e^{\lambda I_m - h\mathscr{L}_{cc}} = I_m + (\lambda I_m - h\mathscr{L}_{cc}) + \frac{1}{2!}(\lambda I_m - h\mathscr{L}_{cc})^2 + \frac{1}{3!}(\lambda I_m - h\mathscr{L}_{cc})^3 + \ldots \geq I_m + (\lambda I_m - h\mathscr{L}_{cc}).$$

When $i \neq j$, if $\mathscr{A}_{ij} \neq 0$, we have $(e^{\lambda I_m - h\mathscr{L}_{cc}})_{ij} \neq 0$. Together with $e^{-\lambda} > 0$ is always tenable, we get $(e^{-h\mathscr{L}_{cc}})_{ij} \neq 0$. In the case where $\mathscr{A}_{ij} \neq 0, i \neq j$, because $0 < h < \frac{1}{\max_{i\in\mathcal{I}_n\setminus\mathcal{I}_m}\{d_{ii}\}}$, we can easily get $(-h\mathscr{L}_{dc})_{ij} \neq 0$ and $(I_{n-m} - h\mathscr{L}_{dd})_{ij} \neq 0$. In conclusion, if $\mathscr{A}_{ij} \neq 0$ $(i \neq j)$, we certainly have $(B_1)_{ij} \neq 0$ $(i \neq j)$.

By the aforementioned analysis, when $0 < h < \frac{1}{\max_{i\in\mathcal{I}_n\setminus\mathcal{I}_m}\{d_{ii}\}}$, since graph $\mathscr{G}$ has a directed spanning tree, it is easy to get that $\lim_{t_k\to\infty} X(t_k) = \beta 1_n, \beta \in \mathbb{R}$. Hence, discrete-time multiagent system (7) can reach consensus, that is, $\lim_{t_k\to\infty} \|x_i(t_k) - x_j(t_k)\| = 0$, for $i, j \in \mathcal{I}_n$. Thus, we get that Equation (4) holds if graph $\mathscr{G}$ has a directed spanning tree.

Next, we will prove (3) holds. From Equation (6), it can be easily get

$$X_c(t) - X_c(t_k) = (e^{-(t-t_k)\mathscr{L}_{cc}} - \mathcal{I}_m)X_c(t_k), \quad t \in (t_k, t_{k+1}] \,.$$

We assume that $e^{-\mathscr{L}_{cc}x} \in G^{m\times m}$ on $D^1 = [0, t - t_k]$, $t \in (t_k, t_{k+1}]$ and $0 < \alpha < 1$. According to Lemma 3, we have

$$e^{-(t-t_k)\mathscr{L}_{cc}} - \mathcal{I}_m = -(t - t_k)e^{-\alpha(t-t_k)\mathscr{L}_{cc}}\mathscr{L}_{cc}. \tag{8}$$

When $t \to \infty$, we have $t_k \to \infty$. Because the row sums of Laplacian matrix are 0, it can be obtained that $\mathscr{L}_{cc} \lim_{t_k\to\infty} X_c(t_k) = \mathbf{0}_m$. Therefore,

$$\lim_{t\to\infty}(X_c(t) - X_c(t_k)) = -\lim_{t\to\infty}(t - t_k)e^{-\alpha(t-t_k)\mathscr{L}_{cc}}\mathscr{L}_{cc}X_c(t_k) = \mathbf{0}_m,$$

which implies that

$$\lim_{t\to\infty} X_c(t) = \lim_{t_k\to\infty} X_c(t_k) = \beta 1_m,$$

that is to say,

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \quad for \;\; i,j \in \mathcal{I}_m.$$

Situation 2: $\mathscr{L}_{cd} \neq 0$, at this point, $\mathscr{L}_{cc}$ is an invertible matrix.

When $\mathscr{L}_{cd} \neq 0$, we can get from (5) that

$$\begin{cases} X_c(t) = e^{-(t-t_k)\mathscr{L}_{cc}}X_c(t_k) + (e^{-(t-t_k)\mathscr{L}_{cc}} - \mathcal{I}_m)\mathscr{L}_{cc}^{-1}\mathscr{L}_{cd}X_d(t_k), & t \in (t_k, t_{k+1}], \\ X_d(t_{k+1}) = -h\mathscr{L}_{dc}X_c(t_k) + (I_{n-m} - h\mathscr{L}_{dd})X_d(t_k). \end{cases} \tag{9}$$

When $t = t_{k+1}$, we can get a discrete-time multiagent system is shown below

$$\begin{pmatrix} X_c(t_{k+1}) \\ X_d(t_{k+1}) \end{pmatrix} = B_2 \begin{pmatrix} X_c(t_k) \\ X_d(t_k) \end{pmatrix}, \tag{10}$$

where $B_2 = \begin{pmatrix} e^{-h\mathscr{L}_{cc}} & (e^{-h\mathscr{L}_{cc}} - \mathcal{I}_m)\mathscr{L}_{cc}^{-1}\mathscr{L}_{cd} \\ -h\mathscr{L}_{dc} & I_{n-m} - h\mathscr{L}_{dd} \end{pmatrix}$.

By the analysis of Situation 1, we already know that $e^{-h\mathscr{L}_{cc}}$, $-h\mathscr{L}_{dc}$ and $I_{n-m} - h\mathscr{L}_{dd}$ are all nonnegative matrix, and all matrices B_2's diagonal elements are positive if $0 < h < \frac{1}{\max_{i\in\mathcal{I}_n\setminus\mathcal{I}_m}\{d_{ii}\}}$. From (8), it can be verified that

$$(e^{-h\mathscr{L}_{cc}} - \mathcal{I}_m)\mathscr{L}_{cc}^{-1}\mathscr{L}_{cd} = -he^{-\alpha h\mathscr{L}_{cc}}\mathscr{L}_{cd},$$

where $0 < \alpha < 1$. Given that $e^{-h\mathscr{L}_{cc}}$ is a nonnegative matrix, then $e^{-\alpha h\mathscr{L}_{cc}}$ is also a nonnegative matrix. According to the properties of Laplacian matrix, we know that $-\mathscr{L}_{cd}$ is a nonnegative matrix. Hence, $(e^{-h\mathscr{L}_{cc}} - \mathcal{I}_m)\mathscr{L}_{cc}^{-1}\mathscr{L}_{cd}$ is a nonnegative matrix. Therefore, matrix B_2 is a nonnegative matrix with positive diagonal elements.

Next, we will prove that matrix B_2 is a stochastic matrix. For the first m rows of matrix B_2, it is easy to know

$$e^{-h\mathscr{L}_{cc}} = I_m - h\mathscr{L}_{cc} + \frac{h^2}{2!}(\mathscr{L}_{cc})^2 - \frac{h^3}{3!}(\mathscr{L}_{cc})^3 + \ \dots$$

and

$$(e^{-h\mathscr{L}_{cc}} - \mathcal{I}_m)\mathscr{L}_{cc}^{-1}\mathscr{L}_{cd} = -h\mathscr{L}_{cd} + \frac{h^2}{2!}\mathscr{L}_{cc}\mathscr{L}_{cd} - \frac{h^3}{3!}\mathscr{L}_{cc}^2\mathscr{L}_{cd} + \ \dots\ .$$

Hence, it follows that

$$e^{-h\mathscr{L}_{cc}}\mathbf{1}_m + (e^{-h\mathscr{L}_{cc}} - \mathcal{I}_m)\mathscr{L}_{cc}^{-1}\mathscr{L}_{cd}\mathbf{1}_{n-m} = \mathbf{1}_m - h(\mathscr{L}_{cc} + \mathscr{L}_{cd})\mathbf{1}_n + \frac{h^2}{2!}\mathscr{L}_{cc}(\mathscr{L}_{cc} + \mathscr{L}_{cd})\mathbf{1}_n - \frac{h^3}{3!}\mathscr{L}_{cc}^2(\mathscr{L}_{cc} + \mathscr{L}_{cd})\mathbf{1}_n + \ \dots\ = \mathbf{1}_m.$$

In Situation 1, we have proved the last $n - m$ rows of matrix B_2 satisfy that the row sums are equal to 1. Thus, we can get that matrix B_2's all row sums are 1. On account of B_2 is a nonnegative matrix and all its row sums are 1, B_2 is a stochastic matrix.

On the basis of the above conditions, through Remark 1, we know that if graph $\mathscr{G}_2$ which corresponding to matrix B_2 has a directed spanning tree, then B_2 is SIA. We already know that the original directed communication graph $\mathscr{G}$

has a directed spanning tree. When $\mathscr{A}_{ij} \neq 0$ and $0 < h < \frac{1}{\max_{i\in\mathcal{I}_n\setminus\mathcal{I}_m}\{d_{ii}\}}$, we already have $(e^{-h\mathscr{L}_{cc}})_{ij} \neq 0$, $(-h\mathscr{L}_{dc})_{ij} \neq 0$ and $(I_{n-m} - h\mathscr{L}_{dd})_{ij} \neq 0$ for $i \neq j$. According to (8), it is easy to know

$$(e^{-h\mathscr{L}_{cc}} - I_m)\mathscr{L}_{cc}^{-1}\mathscr{L}_{cd} = -he^{-\alpha h\mathscr{L}_{cc}}\mathscr{L}_{cd},$$

where $0 < \alpha < 1$. We also can easily get that

$$e^{-\alpha h\mathscr{L}_{cc}} = e^{-\alpha\lambda I_m + (\alpha\lambda I_m - \alpha h\mathscr{L}_{cc})} = e^{-\alpha\lambda}e^{\alpha\lambda I_m - \alpha h\mathscr{L}_{cc}} \geq e^{-\alpha\lambda}I_m.$$

Combining the two conditions, it can be obtained that

$$(e^{-h\mathscr{L}_{cc}} - I_m)\mathscr{L}_{cc}^{-1}\mathscr{L}_{cd} \geq -he^{-\alpha\lambda}\mathscr{L}_{cd},$$

where $he^{-\alpha\lambda} > 0$ always holds. Therefore, if $\mathscr{A}_{ij} \neq 0$, one can get $((e^{-h\mathscr{L}_{cc}} - I_m)\mathscr{L}_{cc}^{-1}\mathscr{L}_{cd})_{ij} \neq 0$. To sum up, if $\mathscr{A}_{ij} \neq 0$ $(i \neq j)$, thus we must have $(B_2)_{ij} \neq 0$ $(i \neq j)$. In other words, the communication graph $\mathscr{G}_2$ has a directed spanning tree if graph $\mathscr{G}$ has a directed spanning tree.

Through the aforementioned analysis, if $0 < h < \frac{1}{\max_{i\in\mathcal{I}_n\setminus\mathcal{I}_m}\{d_{ii}\}}$ and graph $\mathscr{G}$ has a directed spanning tree, it is easy to know that $\lim_{t_k\to\infty} X(t_k) = \beta 1_n$, $\beta \in \mathbb{R}$. In other words, Equation (4) of Definition 1 holds.

Then, we will prove Equation (3) holds. From Equation (6), we can easily know

$$X_c(t) - X_c(t_k) = (e^{-(t-t_k)\mathscr{L}_{cc}} - \mathcal{I}_m)(X_c(t_k) + \mathscr{L}_{cc}^{-1}\mathscr{L}_{cd}X_d(t_k), \quad t \in (t_k, t_{k+1}].$$

On account of the row sums of Laplacian matrix are 0, it can be verified that

$$\mathscr{L}_{cc}\lim_{t_k\to\infty} X_c(t_k) + \mathscr{L}_{cd}\lim_{t_k\to\infty} X_d(t_k) = \mathbf{0}_m.$$

When $t \to \infty$, we have $t_k \to \infty$. Therefore,

$$\lim_{t\to\infty}(X_c(t) - X_c(t_k)) = -\lim_{t\to\infty}(t - t_k)e^{-\alpha(t-t_k)\mathscr{L}_{cc}}(\mathscr{L}_{cc}X_c(t_k) + \mathscr{L}_{cd}X_d(t_k)) = \mathbf{0}_m.$$

It implies that

$$\lim_{t\to\infty} X_c(t) = \lim_{t_k\to\infty} X_c(t_k) = \beta 1_m.$$

Hence, we obtain

$$\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0, \quad for \;\; i,j \in \mathcal{I}_m.$$

■

Remark 4. In this article, hybrid multiagent system contains both discrete-time and continuous-time dynamic agents. Different from Reference 19, the continuous-time dynamic agents in this article can interact with their continuous-time dynamic neighbors in real time. By using the consensus protocol (2), we establish a unified framework of the consensus problem for continuous-time and discrete-time multiagent systems in this article.

4 | SIMULATION

It is assumed that there are eight agents in the hybrid multiagent system. The continuous-time and discrete-time dynamic agents are denoted by 1–4 and 5–8, respectively. All agents' dynamics are described in (1) and (2). The communication graph $\mathscr{G}$ with 0–1 weights is shown in Figure 1. It is easy to see that $\mathscr{G}$ has a directed spanning tree and $\max_{i\in\mathcal{I}_8\setminus\mathcal{I}_4}\{d_{ii}\} = 2$. Let $x(0) = [-6, -5, 9, -11, 12, -9, 7, -3]^T$.

We choose the sampling period $h = 0.2$, which satisfies $0 < h < \frac{1}{\max_{i\in\mathcal{I}_8\setminus\mathcal{I}_4}\{d_{ii}\}}$. Under the control of consensus protocol (2), the state trajectories of all agents are shown in Figure 2, which is in agreement with Theorem 1.

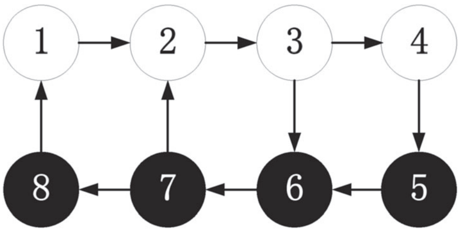

FIGURE 1 A directed graph $\mathscr{G}$

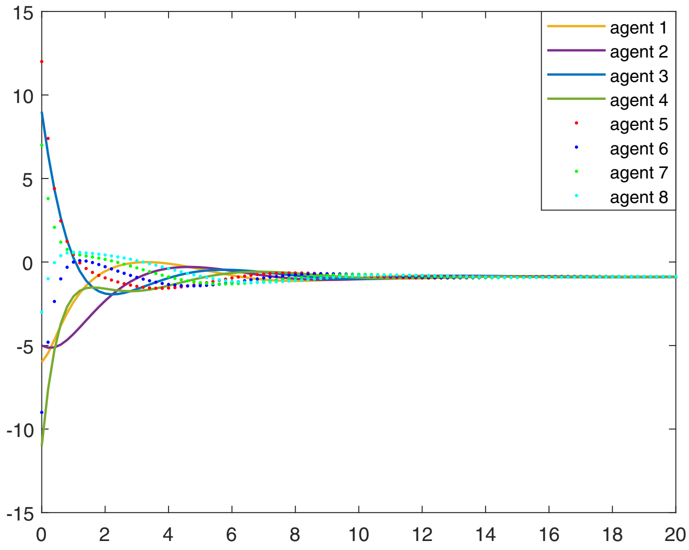

FIGURE 2 When $h = 0.2$, all agents' state trajectories with communication graph $\mathscr{G}$ and consensus protocol (2)

5 | CONCLUSION

In this article, we mainly investigated the consensus problem for the hybrid multiagent system which is composed of continuous-time and discrete-time dynamic agents. Crucially, we presented a novel protocol in which all continuous-time dynamic agents can obtain their own states, and can interact with their own neighbors with continuous-time dynamics in real time. The remaining information exchange takes place at the sampling time. Through mathematical tools such as graph theory and differential mean value theorems of matrix function, we obtained the consensus criterion for this hybrid multiagent system. In the future, the novel interactive mode will be applied to the second-order consensus problem for hybrid multiagent systems, containment control of hybrid multiagent systems, and so on.

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CONFLICT OF INTEREST

The authors confirm that there is no conflict of interest for this article.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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