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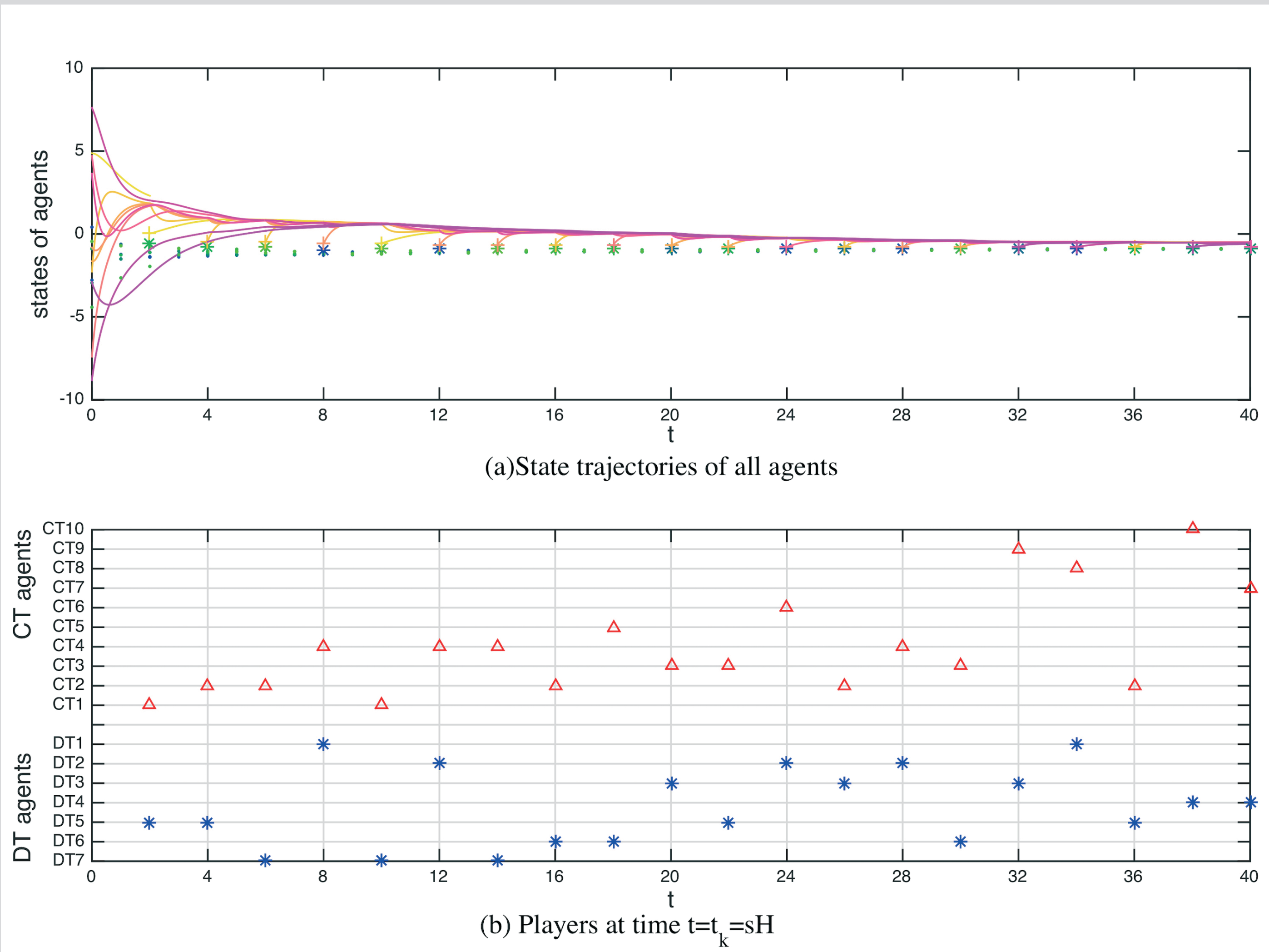

(a)State trajectories of all agents

(b) Players at time $t=t_k=sH$

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RESEARCH ARTICLE

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Consensus analysis of hybrid multiagent systems: A game-theoretic approach

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Summary

This paper considers a consensus problem for hybrid multiagent systems, which comprise two groups of agents: a group of continuous-time dynamic agents and a group of discrete-time dynamic agents. Firstly, a game-theoretic approach is adopted to model the interactions between the two groups of agents. To achieve consensus for the considered hybrid multiagent systems, the cost functions are designed. Moreover, it is shown that the designed game admits a unique Nash equilibrium. Secondly, sufficient/necessary conditions of solving consensus are established. Thirdly, we find that the convergence speed of the system depends on the game. By the mechanism design of the game, the convergence speed is increased. Finally, simulation examples are given to validate the effectiveness of the theoretical results.

KEYWORDS

consensus, convergence speed, hybrid multiagent systems, Nash equilibrium

1 | INTRODUCTION

Multiagent systems (MASs) (eg, sensor networks,[1] multirobot systems[2]) have attracted much attention from both the industry and academia in the past two decades due to their wide applications.[3] Among the various research topics for MASs (see, eg, containment control,[4,5] formation,[6,7] flocking,[8] controllability,[9,10] and coverage control[11]), consensus is one of the main lines of research.[12,13]

Consensus means that a group of agents reach an agreement upon certain quantities of interest. DeGroot[14] proposed a model to describe how a group of individuals reach consensus on estimating some unknown parameters. Vicsek et al[12] investigated a discrete-time model of n agents all moving in the plane with the same speed. It appears that, based on local interaction rules, all agents eventually move in the same direction. By virtue of graph theory, Jadbabaie et al[15] gave a theoretical explanation for the consensus behavior of the Vicsek model. Olfati-Saber and Murray[16] established some necessary and/or sufficient conditions for achieving the average consensus for multiagent systems under switching topologies and time delays. For multiagent systems under directed communication topologies, Ren and Beard[17] proved that consensus is achievable if and only if the directed communication graph has a spanning tree. The consensus of multiagent systems with double-integrator agents was also investigated.[18,19] Zheng et al considered distributed coordination of heterogeneous MASs which consist of both single- and double-integrators. They studied consensus problem and containment control of heterogeneous MASs,[20-22] respectively. Moreover, fast consensus was considered in other works[16,23-25] and optimal consensus was investigated in the works of Cao and Ren[26] and Ma et al.[27] In particular, Olfati-Saber and Murray[16] found that the convergence rate of the consensus algorithm can be quantified by the algebraic connectivity of the communication graph (ie, the second smallest eigenvalue of the Laplacian matrix). Moreover, the algebraic connectivity of the graph can be increased by designing weights based on semi-definite convex programming.[23] In the works of Cao and Ren[26] and Ma et al,[27] LQR-based optimal communication graph for solving consensus problems were considered.

The agents considered in the aforementioned works are either discrete-time agents or continuous-time agents. However, different types of agents might work together in multiagent systems. For example, autonomous robots and natural critters can interact as a group,[28] and different types of mobile autonomous robots (eg, unmanned ground vehicles and unmanned air vehicles) need to cooperate.[29] Actually, many practical systems are hybrid systems, which contain two distinct types of interacting components: subsystems with continuous dynamics and subsystems with discrete dynamics.[30] Applications of such systems arise in various fields including cyber-physical systems,[31] multicell wireless data networks,[32] power grid,[33] among others. Along with this fact, consensus problems of hybrid multiagent systems, in which continuous-time dynamic agents (CT-agents) and discrete-time agents (DT-agents) coexist, were explored in the works of Zheng et al.[34,35] Inspired by Zheng et al,[34,35] this paper intends to propose a new consensus protocol for hybrid multiagent systems, which contain two groups of agents: a group of CT-agents and a group of DT-agents. Different from the works of Zheng et al,[34,35] a game-theoretic model is adopted in this paper to describe the interactions among the agents in distinct groups.

Game theoretical approaches have been leveraged for distributed control of multiagent systems. Cooperative game theory was used to solve optimal control of multiagent systems. To ensure consensus seeking, Semsar-Kazerooni and Khorasani[36] considered a combination of the individual cost as the team cost, and the minimization of this cost function results in a set of Pareto-efficient solutions. In the work of Vamvoudakis et al,[37] a cooperative multiplayer game was formulated for solving tracking problem. Noncooperation game theory is utilized to depict the competitive interactions in multiagent systems. In the works of Altafini[38] and Zhu et al,[39] signed graphs were employed to model the social networks, where negative edges demonstrate antagonistic relationships, and bipartite consensus problems were investigated. Qin et al[40] investigated group synchronization problems for multiagent systems with competitive interactions. Ma et al studied the competition of the leaders.[41,42] They[43] considered noncooperation phenomena between two competitive groups. Mei and Bullo[44] considered competitive propagation models over a social network. Motivated by the aforementioned works, we design a game to depict the interacting behaviors among the agents and propose a new consensus protocol based on the designed game. Compared with the existing works, the main contributions of this paper are summarized as follows.

1. The cost functions for the players in the interaction game between the two groups are designed based on a tradeoff between the intergroup consensus and the intragroup consensus. Based on the designed costs, the uniqueness of the Nash equilibrium is proven.
2. Based on the designed game, sufficient/necessary conditions to ensure the consensus of the hybrid multiagent system are provided.
3. The linkage between the convergence speed and the mechanism of the game is established.

The rest of this paper is organized as follows. Section 2 presents the preliminaries on the graphs and the system model. Section 3 shows our main results. Numerical simulations are given in Section 4 to illustrate the effectiveness of theoretical results. Some conclusions are drawn in Section 5.

Throughout this paper, the following notations will be used. Let $\mathbb{N}$, $\mathbb{N}^+$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of nonnegative integral numbers, positive integral numbers, real numbers, and complex numbers, respectively. For $x \in \mathbb{C}$, $|x|$ is magnitude of x. $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices. The column vector with all entries equal to one (to zero) is denoted as $\mathbf{1}_n$ (or $\mathbf{0}_n$). Moreover, $\mathbf{e}_r$ is the canonical vector with a 1 in the rth entry and 0's elsewhere. I_n denotes an n-dimensional identity matrix. $\mathcal{I}_n = \{1, \dots, n\}$ is an index set. A matrix is said to be nonnegative if all its entries are nonnegative. Nonnegative matrix $S \in \mathbb{R}^{n\times n}$ is said to be a stochastic matrix if its all row sums are 1. A stochastic matrix S is called indecomposable and aperiodic (SIA) if $\lim_{k\to\infty} S^k = \mathbf{1}\mathbf{p}^T$, where $\mathbf{p}$ is a column vector.

2 | PRELIMINARIES

2.1 | Graph theory

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a weighted directed graph consisting of a vertex set $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. In this paper, we suppose that there is no self-loop in a graph. An edge of $\mathcal{G}$ is denoted by (v_j, v_i), where v_j is called the parent vertex. The in-neighbor set of vertex i is $\mathcal{N}_i = \{v_j \in \mathcal{V} \,|\, (v_j, v_i) \in \mathcal{E}\}$. A directed tree is a directed graph, where every vertex has exactly one parent, except that the root is without any parent. A spanning tree of $\mathcal{G}$ is a directed tree, which consists of all the vertices and a subset of edges in $\mathcal{G}$. For a graph $\mathcal{G}$ with spanning trees, the root of a spanning tree is called the root vertex. $A = [a_{ij}]_{n\times n}$ is the adjacency matrix of $\mathcal{G}$ with $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix of $\mathcal{G}$ is $\mathcal{L} = [l_{ij}]_{n\times n}$, where $l_{ii} = \sum_{j=1}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$ if $i \neq j$. It is well known that $e^{-\mathcal{L}t}$ is a (row) stochastic matrix.

Let S is an $n \times n$-dimensional nonnegative matrix. Graph $\mathcal{G}_S$ with n vertexes is called the graph associated with S, such that there is a directed edge in $\mathcal{G}_S$ from v_j to v_i if the (i,j)-entry of S is positive.

Lemma 1 (See the work of Ren and Beard[17]).
Let S be a stochastic matrix with positive diagonal entries, and $\mathcal{G}_S$ be the associated graph of S. $\mathcal{L}$ is the Laplacian matrix of $\mathcal{G}_S$. Then,

- *S is SIA, that is, $\lim_{m\to\infty} S^m = \mathbf{1}v^T$, if and only if the graph $\mathcal{G}_S$ has a spanning tree. Moreover, v is a nonnegative vector and $S^T v = v$ and $\mathbf{1}^T v = 1$.*
- *$e^{-\mathcal{L}t}(t > 0)$ is SIA if and only if $\mathcal{G}_S$ has a spanning tree.*

Lemma 2 (See the work of Ren and Beard[17]).
Let $S_1, S_2, \dots S_k$ be a finite set of SIA matrices with positive diagonal entries. Then, for each infinite sequence $S_{i_1}, S_{i_2}, \dots$, there exists a column vector y such that $\lim_{j\to\infty} S_{i_j} S_{i_{j-1}}, \dots S_{i_1} = \mathbf{1}y^T$.

Suppose that $\mathcal{G}_S$ has a spanning tree and all diagonal entries of S are positive. It is easy to know that the convergence speed of $\lim_{k\to\infty} S^k$ is dependent on $|\lambda_2|$, the second largest eigenvalue magnitude of S.[24] Therefore, we define $\rho(S) = |\lambda_2|$ as the convergence speed of the discrete system $x(k+1) = Sx(k), k \in \mathbb{N}$, where $x(k) \in \mathbb{R}^n$ is the state of the system. The smaller $\rho(S)$ means the faster asymptotic convergence.

2.2 | System model

Consider a multiagent system consisting of n CT-agents and m DT-agents. Let $\mathcal{V}_c = \{v_1^c, v_2^c, \dots, v_n^c\}$ and $\mathcal{V}_d = \{v_1^d, v_2^d, \dots, v_m^d\}$ be the group of CT-agents and the group of DT-agents, respectively.

2.2.1 | The dynamics of group $\mathcal{V}_c$

Agents of group $\mathcal{V}_c$ are continuous-time agents. The interaction among agents of $\mathcal{V}_c$ is modeled by a directed graph $\mathcal{G}_c = (\mathcal{V}_c, \mathcal{E}_c)$. Each agent of $\mathcal{V}_c$ represents a vertex in $\mathcal{G}_c$ and $(v_i^c, v_j^c) \in \mathcal{E}_c$ if and only if agent v_j^c can receive information of v_i^c. Denote $\mathcal{A}_c = [a_{ij}]_{n\times n}$ as the adjacent matrices of $\mathcal{G}_c$. Suppose that $x_i \in \mathbb{R}$ is the state of agent $v_i^c \in \mathcal{V}_c$. The dynamics of CT-agent v_i^c can be described as follows:

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i^c} a_{ij}(x_j(t) - x_i(t)), i \in \mathcal{I}_n,$$

where $\mathcal{N}_i^c$ is the neighbor set of v_i^c in $\mathcal{G}_c$. Let $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$. The evolution of group $\mathcal{V}_c$ can be described in vector form as

$$\dot{x}(t) = -\mathcal{L}_c x(t), \tag{1}$$

where $\mathcal{L}_c$ is the Laplacian matrix of $\mathcal{G}_c$.

2.2.2 | The dynamics of group $\mathcal{V}_d$

Agents of group $\mathcal{V}_d$ are discrete-time dynamic agents. Let directed graph $\mathcal{G}_d = (\mathcal{V}_d, \mathcal{E}_d)$ indicate interaction relationship among agents of $\mathcal{V}_d$, where each agent is a vertex in $\mathcal{G}_d$ and $(v_i^d, v_j^d) \in \mathcal{E}_d$ if and only if agent v_j^d can receive information of v_i^d. Denote by $y_i(t_k) \in \mathbb{R}$ the state of DT-agent $v_i^d \in \mathcal{V}_d$ at time $t_k = kh(k \in \mathbb{N}, h > 0)$, where h is the update period of

DT-agents. The dynamics of agent v_i^d is

$$y_i(t_{k+1}) = \sum_{j=1}^{m} w_{ij} y_j(t_k), i \in \mathcal{I}_m,$$

where $\sum_{j=1}^{m} w_{ij} = 1$,

$$w_{ij} = \begin{cases} > 0 & i = j \,\text{or}\, v_j^d \in \mathcal{N}_i^d, \\ = 0 & \text{otherwise}, \end{cases}$$

and $\mathcal{N}_i^d$ is the neighbor set of v_i^d in $\mathcal{G}_d$. Let $\mathcal{W} = [w_{ij}]_{m\times m}$. It follows that $\mathcal{W}$ is a (row) stochastic matrix with positive diagonals. Denote $y(t_k) = [y_1(t_k), y_2(t_k), \dots, y_m(t_k)]^T$. The dynamics of group $\mathcal{V}_d$ can be written in vector form as

$$y(t_{k+1}) = \mathcal{W} y(t_k), \quad k \in \mathbb{N}. \tag{2}$$

2.2.3 | The interaction between groups $\mathcal{V}_c$ and $\mathcal{V}_d$

In what follows, we will model the interaction between $\mathcal{V}_c$ and $\mathcal{V}_d$ as a game. Let $H = rh$, where $r \in \mathbb{N}^+$. At time $t = t_k = sH, s \in \mathbb{N}^+$, a CT-agent $v_{p_1}^c$ and a DT-agent $v_{p_2}^d$ are chosen to exchange their states. Define $x_{p_1}((sH)^-)$ and $y_{p_2}((sH)^-)$ as the states of $v_{p_1}^c$ and $v_{p_2}^d$ before their interaction, respectively. After their communication, players will update their states independently. Let $x_{p_1}(sH)$ and $y_{p_2}(sH)$ be states of $v_{p_1}^c$ and $v_{p_2}^d$ after updating, respectively. The interaction between $v_{p_1}^c$ and $v_{p_2}^d$ can be modeled as the following game.

Definition 1. Game $\mathbb{G}$ is a two-person infinite game played by the CT-agent $v_{p_1}^c$ and the DT-agent $v_{p_2}^d$. The strategy of $v_{p_1}^c$ is $x_{p_1}(sH)$. Player $v_{p_1}^c$ decides $x_{p_1}(sH)$ to minimize its cost

$$P_c\left(x_{p_1}(sH), y_{p_2}(sH)\right) = \alpha_c\left[x_{p_1}(sH) - x_{p_1}((sH)^-)\right]^2 + \beta_c\left[x_{p_1}(sH) - y_{p_2}(sH)\right]^2, \tag{3}$$

where $0 < \beta_{\min} \le \beta_c \le \beta_{\max} < 1$, $\alpha_c = 1 - \beta_c$, $\beta_{\min}$ and $\beta_{\max}$ are two constants. The strategy of $v_{p_2}^d$ is $y_{p_2}(sH)$. Player $v_{p_2}^d$ updates its state $y_{p_2}(sH)$ to minimize its cost

$$P_d\left(x_{p_1}(sH), y_{p_2}(sH)\right) = \alpha_d\left[y_{p_2}(sH) - y_{p_2}((sH)^-)\right]^2 + \beta_d\left[y_{p_2}(sH) - x_{p_1}(sH)\right]^2, \tag{4}$$

where $0 < \beta_{\min} \le \beta_d \le \beta_{\max} < 1$ and $\alpha_d = 1 - \beta_d$. Moreover, $(x_{p_1}(sH), y_{p_2}(sH))$ is called the strategy pair of the game.

Definition 2. For players $v_{p_1}^c$ and $v_{p_2}^d$, a strategy pair $(x_{p_1}^*(sH), y_{p_2}^*(sH))$ is called the Nash equilibrium solution of the game if it satisfies

$$\begin{cases} P_c\left(x_{p_1}^*(sH), y_{p_2}^*(sH)\right) = \min\limits_{x_{p_1}(sH)} P_c\left(x_{p_1}(sH), y_{p_2}^*(sH)\right), \\ P_d\left(x_{p_1}^*(sH), y_{p_2}^*(sH)\right) = \min\limits_{y_{p_2}(sH)} P_d\left(x_{p_1}^*(sH), y_{p_2}(sH)\right). \end{cases} \tag{5}$$

Remark 1. We assume that the two players are self-interested and rational. On the one hand, achieving consensus would decrease their costs of disagreement, ie, $\beta_c[x_{p_1}(sH) - y_{p_2}(sH)]^2$ for $v_{p_1}^c$ and $\beta_d[y_{p_2}(sH) - x_{p_1}(sH)]^2$ for $v_{p_2}^d$, which indicates that cooperation is necessary. On the other hand, cooperation leads to the cost of changing their states, ie, $\alpha_c[x_{p_1}(sH) - x_{p_1}((sH)^-)]^2$ for $v_{p_1}^c$ and $\alpha_d[y_{p_2}(sH) - y_{p_2}((sH)^-)]^2$ for $v_{p_2}^d$. Therefore, they have different interests but are self-motivated to cooperate. To achieve consensus, each player has to compromise and make a trade-off between keeping the states and narrowing the gap of disagreement.

Remark 2. Actually, β_c and β_d are the weights for the disagreement costs between the two players. A higher β_c (β_d) means that the cost of disagreement is more important for the player. Moreover, $\beta_{\min}$ and $\beta_{\max}$ is the lower and the upper bound of β_c and β_d, respectively.

We have the following results.

Theorem 1. *Game $\mathbb{G}$ has a unique Nash equilibrium solution given by*

$$((1-\lambda)x_{p_1}((sH)^-) + \lambda y_{p_2}((sH)^-), \mu x_{p_1}((sH)^-) + (1-\mu) y_{p_2}((sH)^-)), \tag{6}$$

where

$$\lambda = \frac{\alpha_d \beta_c}{\alpha_c + \alpha_d \beta_c} \, and \, \mu = \frac{\alpha_c \beta_d}{\alpha_d + \alpha_c \beta_d}. \tag{7}$$

Proof. For a fixed $y^*_{p_2}(sH))$, it follows from (3) that $P_c(x_{p_1}(sH), y^*_{p_2}(sH))$ is a quadratic function of $x_{p_1}(sH)$. Therefore, $P_c(x_{p_1}(sH), y^*_{p_2}(sH))$ has only one global minimum $x^*_{p_1}(sH)$, which satisfies

$$\begin{cases} \left.\frac{\partial P_c\left(x_{p_1}(sH), y_{p_2}(sH)\right)}{\partial x_{p_1}(sH)}\right|_{\left(x^*_{p_1}(sH), y^*_{p_2}(sH)\right)} = 0, \\ \left.\frac{\partial^2 P_c\left(x_{p_1}(sH), y_{p_2}(sH)\right)}{\partial x^2_{p_1}(sH)}\right|_{\left(x^*_{p_1}(sH), y^*_{p_2}(sH)\right)} > 0. \end{cases} \tag{8}$$

Likewise, $P_d(x^*_{p_1}(sH), y^*_{p_2}(sH))$ has only one global minimum $y^*_{p_2}(sH))$, which satisfies

$$\begin{cases} \left.\frac{\partial P_d\left(x_{p_1}(sH), y_{p_2}(sH)\right)}{\partial y_{p_2}(sH)}\right|_{\left(x^*_{p_1}(sH), y^*_{p_2}(sH)\right)} = 0, \\ \left.\frac{\partial^2 P_d\left(x_{p_1}(sH), y_{p_2}(sH)\right)}{\partial y^2_{p_2}(sH)}\right|_{\left(x^*_{p_1}(sH), y^*_{p_2}(sH)\right)} > 0. \end{cases} \tag{9}$$

According to Definitions 1 and 2, $(x^*_{p_1}(sH), y^*_{p_2}(sH))$ is the Nash equilibrium solution if and only if (5) holds. By (8) and (9), we can conclude that Game $\mathbb{G}$ has a unique Nash equilibrium solution. Moreover, $(x^*_{p_1}(sH), y^*_{p_2}(sH))$ is the Nash equilibrium solution if and only if

$$\begin{cases} \left.\frac{\partial P_c\left(x_{p_1}(sH), y_{p_2}(sH)\right)}{\partial x_{p_1}(sH)}\right|_{\left(x^*_{p_1}(sH), y^*_{p_2}(sH)\right)} = 0, \\ \left.\frac{\partial P_d\left(x_{p_1}(sH), y_{p_2}(sH)\right)}{\partial y_{p_2}(sH)}\right|_{\left(x^*_{p_1}(sH), y^*_{p_2}(sH)\right)} = 0. \end{cases} \tag{10}$$

By solving (10), we get

$$\begin{cases} x^*_{p_1}(sH) - \beta_c y^*_{p_2}(sH) = \alpha_c x_{p_1}((sH)^-), \\ -\beta_d x^*_{p_1}(sH) + y^*_{p_2}(sH) = \alpha_d y_{p_2}((sH)^-). \end{cases} \tag{11}$$

Since $1 - \beta_c \beta_d > 0$, it follows from (11) that

$$\begin{cases} x^*_{p_1}(sH) = \frac{\alpha_c}{1-\beta_c\beta_d} x_{p_1}((sH)^-) + \frac{\alpha_d\beta_c}{1-\beta_c\beta_d} y_{p_2}((sH)^-), \\ y^*_{p_2}(sH) = \frac{\alpha_c\beta_d}{1-\beta_c\beta_d} x_{p_1}((sH)^-) + \frac{\alpha_d}{1-\beta_c\beta_d} y_{p_2}((sH)^-). \end{cases}$$

Considering

$$1 - \beta_c\beta_d = (\alpha_c + \beta_c)(\alpha_d + \beta_d) - \beta_c\beta_d = \alpha_c + \alpha_d\beta_c = \alpha_d + \alpha_c\beta_d, \tag{12}$$

we have the game $\mathbb{G}$ has the unique Nash equilibrium solution (6). □

Theorem 2. *If α_d and β_d are fixed, then an increase of β_c would lead to an increase of λ and a decrease of μ. If α_c and β_c are fixed, then an increase of β_d would lead to an increase of μ and a decrease of λ.*

Proof. By (6) and (12), we have $\lambda = \frac{\alpha_d\beta_c}{1-\beta_c\beta_d}$ and $\mu = 1 - \frac{\alpha_d}{1-\beta_c\beta_d}$. Thus, it is easy to find that an increase of β_c leads to an increase of λ and a decrease of μ. Likewise, we can prove that an increase of β_d would lead to an increase of μ and a decrease of λ. □

Remark 3. Theorem 1 reveals that, in order to minimize its cost, each player would make a balance between its own state and other player's state. By Theorem 2, it can be concluded that an increase of λ and a decrease of μ mean a higher weight of $y_{p_2}((sH)^-)$ in both $x^*_{p_1}(sH)$ and $y^*_{p_2}(sH)$. This result can be reasoned as follows: a higher weight of disagreement punishment for $v^c_{p_1}$ would motivate $v^c_{p_1}$ to cooperate, and thereby to accept the state of player $v^d_{p_2}$. Moreover, since $v^c_{p_1}$ tends to cooperate, player $v^d_{p_2}$ would tend to keep its own state $y_{p_1}((sH)^-)$.

Remark 4. At playing time sH, it follows from Theorem 1 that there exists information flow between groups $\mathcal{V}_c$ and $\mathcal{V}_d$. Thus, we denote $\mathcal{G}$ by the interaction graph of system. Easy to find that $\mathcal{G} = \{\mathcal{V}_c \cup \mathcal{V}_d, \mathcal{E}\}$, where $\mathcal{E} = \mathcal{E}_c \cup \mathcal{E}_d \cup (v^c_{p_1}, v^d_{p_2}) \cup (v^d_{p_2}, v^c_{p_1})$.

2.2.4 | The dynamics of the system

Let $s \in \mathbb{N}^+$. The dynamics of the system can be described as follows.

- For $t \in [(s-1)H, sH)$, agents of $\mathcal{V}_c$ update their states according to (1). We have

$$\begin{cases} x(t) = e^{-\mathcal{L}_c(t-(s-1)H)}x((s-1)H), \qquad t \in [(s-1)H, sH), \\ x((sH)^-) = \lim\limits_{t\to(sH)^-} x(t), \end{cases} \tag{13}$$

  where $x((sH)^-)$ denotes the state vector of group $\mathcal{V}_c$ at time sH before game $\mathbb{G}$.
- For $t_k \in \{(s-1)H, (s-1)H+h, \dots, sH-h\}$, agents of $\mathcal{V}_d$ update their states by (2). We get

$$\begin{cases} y(t_k) = \mathcal{W}^{k-(s-1)r}y((s-1)H), \qquad t_k \in \{(s-1)H, (s-1)H+h, \dots, sH-h\}, \\ y((sH)^-) = \mathcal{W}^r y((s-1)H), \end{cases} \tag{14}$$

  where $y((sH)^-)$ denotes the state vector of group $\mathcal{V}_d$ at time sH before the game $\mathbb{G}$.
- At time $t = t_k = sH$, CT-agent $v^c_{p_1}$ and DT-agent $v^d_{p_2}$ play game $\mathbb{G}$. They update their states according to Nash equilibrium solution (6). As a result, the state of $v^c_{p_1}$ has a jump from $x_{p_1}((sH)^-)$ to $x_{p_1}(sH)$. Meanwhile, the state of $v^d_{p_2}$ also has a jump from $y_{p_2}((sH)^-)$ to $y_{p_2}(sH)$. The process can be written as

$$\begin{cases} x_i(sH) = x_i((sH)^-), y_j(sH) = y_j((sH)^-), i \neq p_1, j \neq p_2, \\ x_{p_1}(sH) = (1-\lambda)x_{p_1}((sH)^-) + \lambda y_{p_2}((sH)^-), \\ y_{p_2}(sH) = \mu x_{p_1}((sH)^-) + (1-\mu)y_{p_2}((sH)^-). \end{cases} \tag{15}$$

  Let $E_{ij} = \mathbf{e}_i\mathbf{e}_j^T$. The matrix-form of (15) is

$$\begin{pmatrix} x(sH) \\ y(sH) \end{pmatrix} = \Phi \begin{pmatrix} x((sH)^-) \\ y((sH)^-) \end{pmatrix},$$

  where

$$\Phi = \begin{pmatrix} I_{n_r} - \lambda E_{p_1p_1} & \mathbf{0} & \lambda E_{p_1p_2} \\ \mathbf{0} & I_{n-n_r} & \mathbf{0} \\ \mu E^T_{p_1p_2} & \mathbf{0} & I_m - \mu E_{p_2p_2} \end{pmatrix}. \tag{16}$$

System (13)-(15) has a global task, reaching consensus. Therefore, we propose the definition of consensus for system (13)-(15).

Definition 3. System (13)-(15) is said to reach consensus if, for any initial conditions,

$$\lim_{t\to+\infty} x_i(t) = \lim_{t_k\to+\infty} y_j(t_k) \tag{17}$$

holds for all $i \in \mathcal{I}_m, j \in \mathcal{I}_n$.

3 | MAIN RESULTS

3.1 | Consensus of hybrid multiagent systems

In what follows, we will develop some conditions of solving consensus for system (13)-(15).

We assume the following.

A1 Players of game $\mathbb{G}$ are unchangeable at each time.
A2 $\mathcal{G}_c$ and $\mathcal{G}_d$ have a spanning tree.
A3 At least one player of game $\mathbb{G}$ is the root vertex of $\mathcal{G}_c$ or $\mathcal{G}_d$.

Let $\mathcal{R}_c$ and $\mathcal{R}_d$ be sets of the root vertexes in $\mathcal{G}_c$ and $\mathcal{G}_d$, respectively. Without loss of generality, let $\mathcal{R}_c = \{v_1^c, \dots, v_{n_r}^c\}$ and $\mathcal{R}_d = \{v_1^d, \dots, v_{m_r}^d\}$.

Theorem 3. *Suppose that* ***A1*** *holds. Then, system (13)-(15) can reach consensus if and only if* ***A2*** *and* ***A3*** *hold. Moreover, the consensus state is* $\mathbf{q}^T[x^T(0), y^T(0)]^T$*, where* $\mathbf{q}$ *satisfies* $\mathcal{A}^T\mathbf{q} = \mathbf{q}$ *and*

$$\mathcal{A} = \Phi \begin{pmatrix} e^{-\mathcal{L}_c H} & \mathbf{0} \\ \mathbf{0} & \mathcal{W}^r \end{pmatrix}.$$

Proof. (**Sufficiency**) For all $x(sH), y(sH)$, $s \in \mathbb{N}^+$, it follows from (13)-(15) that

$$\begin{pmatrix} x(sH) \\ y(sH) \end{pmatrix} = \mathcal{A} \begin{pmatrix} x((s-1)H) \\ y((s-1)H) \end{pmatrix}. \tag{18}$$

It follows from **A2** that $\mathcal{L}_c$ and $\mathcal{W}$ have the structure as

$$\mathcal{L}_c = \begin{pmatrix} L_{11} & \mathbf{0} \\ L_{21} & L_{22} \end{pmatrix} \quad \text{and} \quad \mathcal{W} = \begin{pmatrix} W_{11} & \mathbf{0} \\ W_{21} & W_{22} \end{pmatrix}.$$

Thus, we have

$$e^{-\mathcal{L}_c H} = \begin{pmatrix} P_{11} & \mathbf{0} \\ P_{21} & P_{22} \end{pmatrix} \quad \text{and} \quad \mathcal{W}^r = \begin{pmatrix} Q_{11} & \mathbf{0} \\ Q_{21} & Q_{22} \end{pmatrix},$$

where P_{11} is a $n_r \times n_r$-dimensional row stochastic matrix and Q_{11} is a $m_r \times m_r$-dimensional row stochastic matrix. Since **A3** holds, we have $v_{p_1}^c \in \mathcal{R}_c$ or $v_{p_2}^d \in \mathcal{R}_d$, ie, $p_1 \in \mathcal{I}_{n_r}$ or $p_2 \in \mathcal{I}_{m_r}$.

Firstly, we suppose $v_{p_1}^c \in \mathcal{R}_c$. By (16), we have

$$\mathcal{A} = \begin{pmatrix} (I_{n_r} - \lambda E_{p_1p_1})P_{11} & \mathbf{0} & \lambda E_{p_1p_2}\mathcal{W}^r \\ P_{21} & P_{22} & \mathbf{0} \\ \mu E_{p_1p_2}^T P_{11} & \mathbf{0} & (I_m - \mu E_{p_2p_2})\mathcal{W}^r \end{pmatrix}. \tag{19}$$

From Lemma 1, we have $e^{-\mathcal{L}_c H}$ and $\mathcal{W}^r$ are (row) stochastic matrixes with positive diagonals. Together with the fact that Φ is a (row) stochastic matrix with positive diagonals, we know that $\mathcal{A}$ is a (row) stochastic matrix with positive diagonals. Let $\mathcal{G}_u = (\mathcal{V}_u, \mathcal{E}_u)$ be a graph associated with $\mathcal{A}$, where $\mathcal{V}_u = \mathcal{V}_c \cup \mathcal{V}_d$. Clearly, $\mathcal{G}_c$ and $\mathcal{G}_d$ are two subgraphs of $\mathcal{G}_u$. Because $v_{p_1}^c$ is the root vertexes in $\mathcal{G}_c$, $\mathcal{G}_c$ has a spanning tree $\mathcal{T}_c$ with the root vertex $v_{p_1}^c$. It follows from **A2** that $\mathcal{G}_d$ has a spanning tree $\mathcal{T}_d$. Since the (p_1, p_2)-entry of $E_{p_1p_2}\mathcal{W}^r$ equals the (p_2, p_2)-entry of $\mathcal{W}^r$, $(p_1, n + p_2)$-entry of $\mathcal{A}$ is positive. Consequently, $(v_{p_2}^d, v_{p_1}^c)$ is an edge of $\mathcal{G}_u$. Easy to find that $(v_{p_2}^d, v_{p_1}^c) \cup \mathcal{T}_c \cup \mathcal{T}_d$ is a spanning tree of $\mathcal{G}_u$.

Likewise, for the case of $v_{p_2}^d \in \mathcal{R}_d$, we can also prove that $\mathcal{G}_u$ has a spanning tree. By Lemma 1, we get $\mathcal{A}$ is an SIA matrix, ie,

$$\lim_{s \to +\infty} \mathcal{A}^s = \mathbf{1}\mathbf{q}^T. \tag{20}$$

Then, it follows from (18) and (20) that

$$\lim_{s \to +\infty} x_i(sH) = \lim_{s \to +\infty} y_j(sH) = \mathbf{q}^T \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}. \tag{21}$$

Because $e^{-\mathcal{L}_c(t-(s-1)H)}$ and $\mathcal{W}$ are row stochastic matrixes, we have

$$\min_{j \in \mathcal{I}_n} x_j((s-1)H) \le x_i(t) \le \max_{j \in \mathcal{I}_n} x_j((s-1)H), i \in \mathcal{I}_n, t \in [(s-1)H, sH) \tag{22}$$

and

$$\min_{j \in \mathcal{I}_m} y_j((s-1)H) \le y_i(t_k) \le \max_{j \in \mathcal{I}_m} y_j((s-1)H), i \in \mathcal{I}_m, t_k \in \{(s-1)H, (s-1)H + h, \dots, sH - h\}. \tag{23}$$

Consequently, we get

$$\lim_{t \to +\infty} x_i(t) = \lim_{t_k \to +\infty} y_j(t_k) = \mathbf{q}^T \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}, \tag{24}$$

which means that system (13)-(15) achieves consensus.

(Necessity) Suppose that **A2** or **A3** does not hold. Then, $\mathcal{G}_u$ has not a spanning tree, which means that $\mathcal{A}$ is not SIA. Therefore, we have that (20) does not hold and system (13)-(15) cannot reach consensus. □

Theorem 3 gives a sufficient and necessary condition of achieving consensus for system (13)-(15). Now, we will give a sufficient condition for reaching consensus under the scenario where players of game $\mathbb{G}$ are not fixed. We assume the following.

A4 Players of game $\mathbb{G}$ are time-variant.

A5 At each time sH, at least one player is root vertexes of $\mathcal{G}_c$ or $\mathcal{G}_d$.

Theorem 4. *Suppose that **A4** holds. Then, system (13)-(15) can reach consensus if **A2** and **A5** hold.*

Proof. Since players of the game are time-variant, we let $v^c_{p_1(s)}$ and $v^d_{p_2(s)}$ be two players at time sH. Then, it follows that (18) can be written as

$$\begin{pmatrix} x(sH) \\ y(sH) \end{pmatrix} = \mathcal{A}(sH) \begin{pmatrix} x((s-1)H) \\ y((s-1)H) \end{pmatrix}, \tag{25}$$

where

$$\mathcal{A}(sH) = \Phi(sH) \begin{pmatrix} e^{-\mathcal{L}_c H} & \mathbf{0} \\ \mathbf{0} & \mathcal{W}^r \end{pmatrix},$$

$$\Phi(sH) = I_{n+m} - \lambda \mathbf{e}_{p_1(s)}(\mathbf{e}_{p_1(s)} - \mathbf{e}_{n+p_2(s)})^T - \mu \mathbf{e}_{n+p_2(s)}(\mathbf{e}_{n+p_2(s)} - \mathbf{e}_{p_1(s)})^T.$$

Then, by a similar proof for Theorem 3, we know that $\mathcal{A}(sH)$ is an SIA matrix with positive diagonal entries. Moreover, all possible $\mathcal{A}(sH)$ can compose of a finite set of SIA matrixes. By Lemma 2, we know that there exist a column vector $\mathbf{q}$ such that $\lim_{s\to\infty} \mathcal{A}(sH)\mathcal{A}((s-1)H)\dots\mathcal{A}(H) = \mathbf{1}\mathbf{q}^T$. Therefore, we have

$$\lim_{s\to+\infty} x_i(sH) = \lim_{s\to+\infty} y_j(sH) = \mathbf{q}^T \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

Then, similar with the proof of Theorem 3, we can prove that system (13)-(15) will solve consensus problem. □

Remark 5. According to Remark 4, the interaction graph $\mathcal{G}$ has a spanning tree when **A2** and **A3** hold. Theorems 3-4 indicate that the consensus of the agents can be achieved if $\mathcal{G}$ has a spanning tree.

Remark 6. By using the tool of matrix Kronecker products, the results of Theorems 3 and 4 can be extended into high-dimension space.

3.2 | Fast consensus by mechanism design

In this section, the convergence speed of system (13)-(15) will be investigated. By Theorem 3, the sates of the agents at time $t = t_k = sH$ is decided by (18). Moreover, the states at other time satisfy inequations (22) and (23). Therefore, the convergence speed of system (13)-(15) is decided by that of the system (18). Since update period of system (18) is H, $\rho(\mathcal{A})$ denotes the convergence efficiency at each period $[sH, (s+1)H]$. Considering that the update period of DT-agents equals $h = \frac{H}{r}$, we define $[\rho(\mathcal{A})]^{\frac{1}{r}}$ as the convergence speed of system (13)-(15), a smaller $[\rho(\mathcal{A})]^{\frac{1}{r}}$ indicates a faster convergence speed. Thus, in order to accelerate the convergence of system (13)-(15), we consider to minimize $[\rho(\mathcal{A})]^{\frac{1}{r}}$ by

$$\begin{aligned} &\min_{r,\beta_c,\beta_d} [\rho(\mathcal{A})]^{\frac{1}{r}} \\ &\text{s.t.} \quad r \in \mathbb{N}^+, \ \ \beta_{\min} \le \beta_c \le \beta_{\max}, \ \ \beta_{\min} \le \beta_d \le \beta_{\max}, \end{aligned} \tag{26}$$

or

$$\begin{aligned} &\min_{p_1,p_2} \rho(\mathcal{A}) \\ &\text{s.t.} \ v^c_{p_1} \in \mathcal{R}_c \quad \text{or} \quad v^d_{p_2} \in \mathcal{R}_d. \end{aligned} \tag{27}$$

Remark 7. Since $\rho(\mathcal{A})$ is the second largest eigenvalue magnitude of $\mathcal{A}$, it is the second largest solution magnitude of $|I_{n+m}z - \mathcal{A}| = 0$. By (7) and (19), we know that $|I_{n+m}z - \mathcal{A}| = z^{n+m} + \sum_{l=1}^{n+m} a_l(\beta_c, \beta_d) z^{n+m-l}$, where $a_l(\beta_c, \beta_d)$, is a nonlinear and nonconvex function of β_c and β_d.

Remark 8. According to (19), computing the eigenvalues of $(n \times m_r + n_r \times m - n_r \times m_r)$ matrixes is required to solve (27).

It follows from Remarks 7 and 8 that it is difficult to solve (26) and (27) efficiently, especially for large n and m. In the following context, we will analyze the convergence speed of system (13)-(15) from a new perspective.

The goal of system (13)-(15) is to make two groups reach consensus. In order to perform this global task, two groups $\mathcal{V}_c$ and $\mathcal{V}_d$ influence each other via game $\mathbb{G}$. Therefore, we consider to measure the consensus error by the difference of states between two groups. Considering that $\mathcal{G}_c$ and $\mathcal{G}_d$ are directed graphs with a spanning tree, we define the weighted average states for groups $\mathcal{V}_c$ and $\mathcal{V}_d$. Denote $\bar{x}(t) = \mathbf{g}^T x(t)$ be the weighted average state of group $\mathcal{V}_c$, where $\mathbf{g} = [g_1, g_2, \dots, g_n]^T$ is a n-dimensional vector satisfying $\mathcal{L}_c^T \mathbf{g} = 0$. Let $\bar{y}(t_k) = \mathbf{f}^T y(t_k)$ be the weighted average state of group $\mathcal{V}_d$, where $\mathbf{f} = [f_1, f_2, \dots, f_m]^T$ is an m-dimensional vector satisfying $\mathcal{W}^T \mathbf{f} = \mathbf{f}$. Let the consensus error of system (13)-(15) be $\sigma(t) = |\bar{y}(t_k) - \bar{x}(t)|$ for $t \in [kh, (k+1)h)$, $k \in \mathbb{N}$. We have $\lim_{t\to\infty} \sigma(t) = 0$ if and only if system (13)-(15) reaches consensus. Moreover, we have the following.

Theorem 5. *The consensus error of system (13)-(15) is invariant for $t \in [(s-1)H, sH)$. Moreover,*

$$\sigma(sH) = \sigma((s-1)H) - \left(\lambda g_{p_1} + \mu f_{p_2}\right) |y_{p_2}((sH)^-) - x_{p_1}((sH)^-)| \tag{28}$$

if

$$[\bar{y}((sH)^-) - \bar{x}((sH)^-)][y_{p_2}((sH)^-) - x_{p_1}((sH)^-)] \geq 0, \tag{29}$$

holds.

Proof. For $t \in [((s-1)H), sH)$, we know that $x(t) = e^{-\mathcal{L}t} x((s-1)H)$. Because $\mathbf{g}^T e^{-\mathcal{L}t} = \mathbf{g}^T$, we get

$$\begin{cases} \bar{x}(t) = \mathbf{g}^T x((s-1)H) = \bar{x}((s-1)H), t \in [((s-1)H), sH), \\ \bar{x}((sH)^-) = \mathbf{g}^T \lim\limits_{t\to(sH)^-} x(t) = \lim\limits_{t\to(sH)^-} \mathbf{g}^T x(t) = \bar{x}((s-1)H). \end{cases}$$

Since $\mathbf{f}^T \mathcal{W}^k = \mathbf{f}^T$ and (14), we obtain

$$\begin{cases} \bar{y}(t_k) = \bar{y}((s-1)H), t_k \in \{(s-1)H, (s-1)H+h, \dots, sH-h\}, \\ \bar{y}((sH)^-) = \mathbf{f}^T \mathcal{W} y(sH-h) = \bar{y}((s-1)H). \end{cases}$$

Thus, $\sigma(t) = \sigma((s-1)H)$ for all $t \in [((s-1)H), sH)$. By Theorem 1, we have

$$\bar{y}(sH) - \bar{x}(sH) = \bar{y}((sH)^-) - \bar{x}((sH)^-) - (\lambda g_{p_1} + \mu f_{p_2})[y_{p_2}((sH)^-) - x_{p_1}((sH)^-)],$$

where g_{p_1} is the p_1th entry of $\mathbf{g}$ and f_{p_2} is the p_2-entry of $\mathbf{f}$. Therefore, we get

$$\bar{y}(sH) - \bar{x}(sH) = [\bar{y}((s-1)H) - \bar{x}((s-1)H)] - (\lambda g_{p_1} + \mu f_{p_2})[y_{p_2}((sH)^-) - x_{p_1}((sH)^-)].$$

By Lemma 1 and Theorem 1, we know $\lambda g_{p_1} + \mu f_{p_2} > 0$. Thus, (28) holds. □

Remark 9. Theorem 5 reveals that $\sigma(t)$ cannot be decreased without game $\mathbb{G}$. When system (13)-(15) reaches consensus, we have $\lim_{t\to\infty}\sigma(t) = 0$. This means that $\sigma(t)$ is decreased innumerable times when $t \to \infty$. At time sH, if $\sigma(t)$ is declined, it will be decreased by $(\lambda g_{p_1} + \mu f_{p_2})|y_{p_2}((sH)^-) - x_{p_1}((sH)^-)|$. Therefore, the greater $\lambda g_{p_1} + \mu f_{p_2}$ is, the faster system (13)-(15) converge.

Remark 10. Suppose that $\sigma(t)$ decreases at $t = s_1 H$, and the next decrease of $\sigma(t)$ occurs at $t' = s_2 H$. It follows that $t' - t \geq H$. Recalling $H = rh$, we can conclude that the smaller r is, the faster $\sigma(t)$ converges to 0.

From Remarks 9 and 10, we find that the convergence speed of system (13)-(15) depends on r and $\lambda g_{p_1} + \mu f_{p_2}$. Denote $F(p_1, p_2, \beta_c, \beta_d) = \lambda g_{p_1} + \mu f_{p_2} = \frac{g_{p_1}\beta_c + f_{p_2}\beta_d - (g_{p_1} + f_{p_2})\beta_c\beta_d}{1-\beta_c\beta_d}$. We can accelerate the convergence speed by designing the mechanism of game $\mathbb{G}$:

- decrease r;
- for the scenario, where β_c, β_d are fixed, choose players of game $\mathbb{G}$ by solving the following optimization problem:

$$\begin{aligned} &\max_{p_1, p_2} F(p_1, p_2, \beta_c, \beta_d) \\ &\text{s.t.} \;\; v_{p_1}^c \in \mathcal{R}_c, \;\; \text{or} \;\; v_{p_2}^d \in \mathcal{R}_d. \end{aligned} \tag{30}$$

- For the scenario where p_1, p_2 are fixed, arrange parameters of $\mathbb{G}$ by solving the following optimization problem:

$$\begin{aligned} &\max_{\beta_c,\beta_d} F(p_1,p_2,\beta_c,\beta_d) \\ &\text{s.t.}\ \ \beta_{\min} \le \beta_c \le \beta_{\max},\ \ \beta_{\min} \le \beta_d \le \beta_{\max}. \end{aligned} \tag{31}$$

Compared with (26) and (27), optimization problems (30) and (31) are much easier to solve. Firstly, the solution of problem (30) is $p_1 = \text{argmax}_{i\in\mathcal{I}_n} g_i$ and $p_2 = \text{argmax}_{j\in\mathcal{I}_m} f_j$. Secondly, it is obvious that $F(p_1, p_2, \beta_c, \beta_d)$ is a continuously differentiable function in $\{(\beta_c, \beta_d) | \beta_{\min} \le \beta_c \le \beta_{\max}, \beta_{\min} \le \beta_d \le \beta_{\max}\}$ when p_1, p_2 are fixed. We have

$$\frac{\partial F}{\partial \beta_c} = \frac{(g_{p_1} - f_{p_2}\beta_d)(1-\beta_d)}{(1-\beta_c\beta_d)^2} \quad \text{and} \quad \frac{\partial F}{\partial \beta_d} = \frac{(f_{p_2} - g_{p_1}\beta_c)(1-\beta_c)}{(1-\beta_c\beta_d)^2},$$

which means all critical points satisfy $\beta_c = \frac{1}{\beta_d}$. Therefore, all critical points of $F(p_1, p_2, \beta_c, \beta_d)$ are not in $\{(\beta_c, \beta_d) | \beta_{\min} \le \beta_c \le \beta_{\max}, \beta_{\min} \le \beta_d \le \beta_{\max}\}$. As a result, problem (31) is equivalent with

$$\begin{aligned} &\max_{\beta_c,\beta_d} \frac{g_{p_1}\beta_c + f_{p_2}\beta_d - (g_{p_1} + f_{p_2})\beta_c\beta_d}{1-\beta_c\beta_d} \\ &\text{s.t.}\ \ \beta_c, \beta_d \in \{\beta_{\min}, \beta_{\max}\}. \end{aligned} \tag{32}$$

4 | SIMULATIONS

Suppose that there are 10 CT-agents and 7 DT-agents, ie, $\mathcal{V}_c = \{v_1^c, \dots, v_{10}^c\}$ and $\mathcal{V}_d = \{v_1^d, \dots, v_7^d\}$. Interaction graphs $\mathcal{G}_c$ and $\mathcal{G}_d$ are presented in Figure 1. Let

$$\mathcal{L}_c = \begin{pmatrix} 0.5 & 0 & 0 & 0 & 0 & -0.5 & 0 & 0 & 0 & 0 \\ -2 & 3.3 & 0 & 0 & 0 & -1.3 & 0 & 0 & 0 & 0 \\ 0 & -1.5 & 1.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.5 & -2 & 3.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.2 & 1.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

$$\mathcal{W} = \begin{pmatrix} 0.5 & 0.2 & 0 & 0.3 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.2 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0 & 0.2 & 0.3 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.4 & 0 & 0.2 & 0 & 0.4 \end{pmatrix},$$

$\beta_{\min} = 0.2$, $\beta_{\max} = 0.8$, and $h = 1$.

Example 1. Figure 2 A, B, and C show the state trajectories of all the agents under the following three cases: (a) $r = 4$, $\beta_c = 0.8$, $\beta_d = 0.5$, and v_3^c and v_3^d are players; (b) $r = 2$, $\beta_c = 0.8$, $\beta_d = 0.5$, and v_3^c and v_3^d are players; and (c) $r = 2$, $\beta_c = 0.8$, $\beta_d = 0.8$, and v_1^c and v_1^d are players. Firstly, the state trajectories of the two players have a jump at the playing time instants $H, 2H, \dots$, which are marked by + and * in Figure 2, respectively. This is consistent with the system dynamics in (13)-(15). Secondly, we can observe that consensus among the agents is achieved, which manifests the effectiveness of theoretical results in Theorem 3. Thirdly, we can compare the convergence speed under three cases: (i) let $\mathcal{A}_a$ and $\mathcal{A}_b$ and $\mathcal{A}_c$ be matrix $\mathcal{A}$ in (18) for cases (a)-(c), respectively. Since $\rho(\mathcal{A}_a)^{1/4} = 0.9725 > \rho(\mathcal{A}_b)^{1/2} = 0.9483 > \rho(\mathcal{A}_c)^{1/2} = 0.821$, we can conclude that case (a) is the slowest and case (c) is the fastest, which

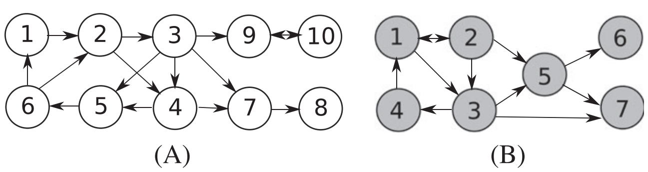

FIGURE 1 Two interaction graphs: A, $\mathcal{G}_c$; B, $\mathcal{G}_d$

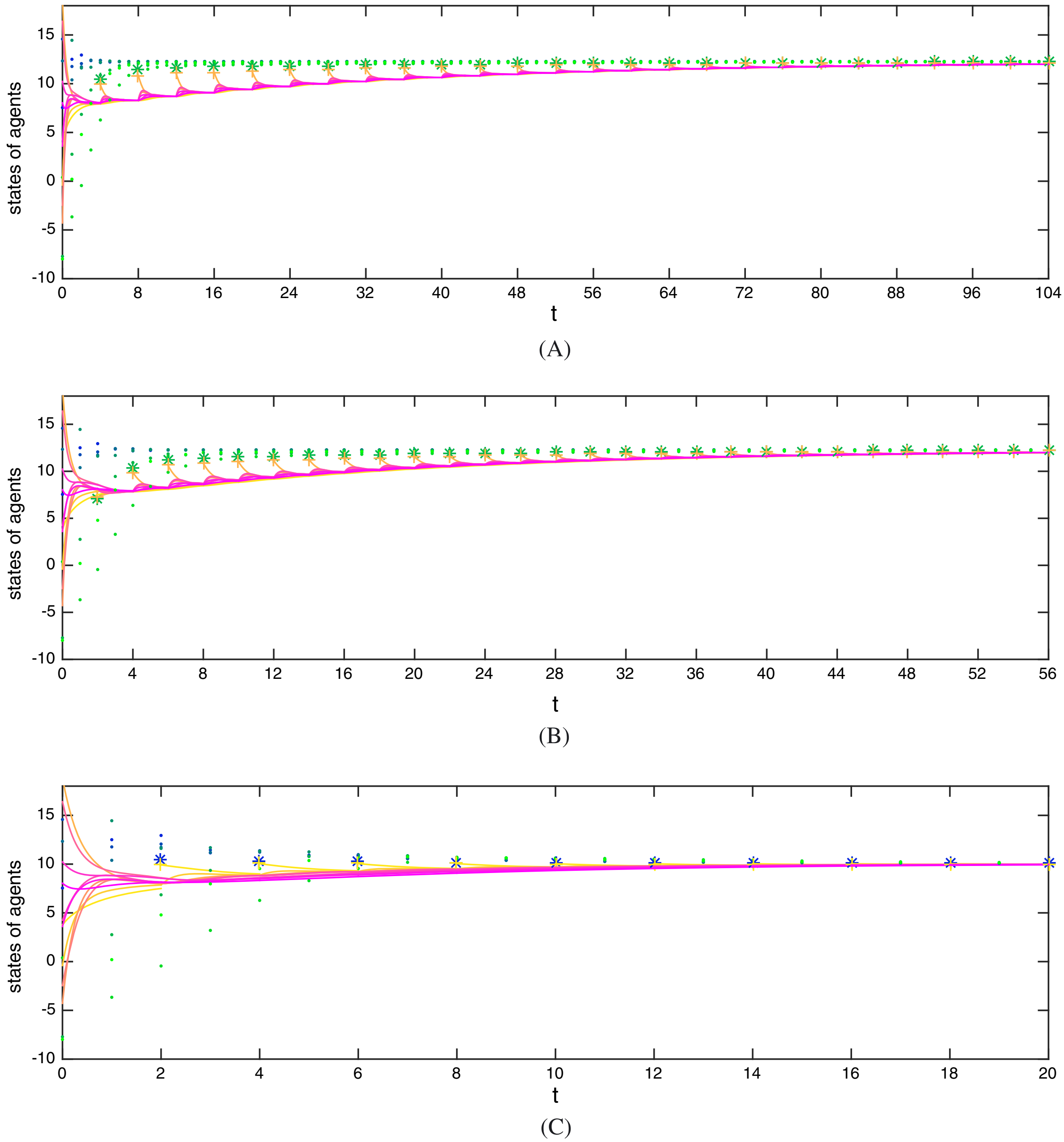

FIGURE 2 State trajectories of all the agents for Example 1. A, State trajectories of all agents ($r = 4, p_1 = v_3^c, p_2 = v_3^d, \beta_c = 0.8, \beta_d = 0.5$); B, State trajectories of all agents ($r = 2, p_1 = v_3^c, p_2 = v_3^d, \beta_c = 0.8, \beta_d = 0.5$); C, State trajectories of all agents($r = 2, p_1 = v_1^c, p_2 = v_1^d, \beta_c = 0.8, \beta_d = 0.5$) [Colour figure can be viewed at wileyonlinelibrary.com]

is manifested in Figure 2; (ii) according to Remark 10, system (13)-(15) would reach consensus more quickly when $r = 2$. The numerical result of Figure 2 also illustrates that cases (b) and (c) ($r = 2$) are faster than case (c) ($r = 4$); (iii) by solving problems (30) and (31), we obtain that the optimal solutions are $p_1 = 1, p_2 = 1$ and $\beta_c = 0.8$, $\beta_d = 0.8$, respectively. It follows that $F_{1,1}(0.8, 0.8) > F_{1,1}(0.5, 0.3) > F_{3,3}(0.8, 0.5)$. Thus, among cases (a)-(c), the system would converge fastest under case (c).

Example 2. Suppose that players of game $\mathbb{G}$ are time-variant and $r = 2$, $\beta_c = 0.8$, $\beta_d = 0.5$. Figure 3A illustrates the state trajectories of all agents and Figure 3B shows the players of game $\mathbb{G}$ at time $H, 2H, \ldots$. By Figure 3, it can be concluded that (i) at least one player is root vertex of $\mathcal{G}_c$ or $\mathcal{G}_d$ at $H, 2H, \ldots$; and (ii) the system achieves consensus. The simulation results are consistent with the theoretical result in Theorem 4.

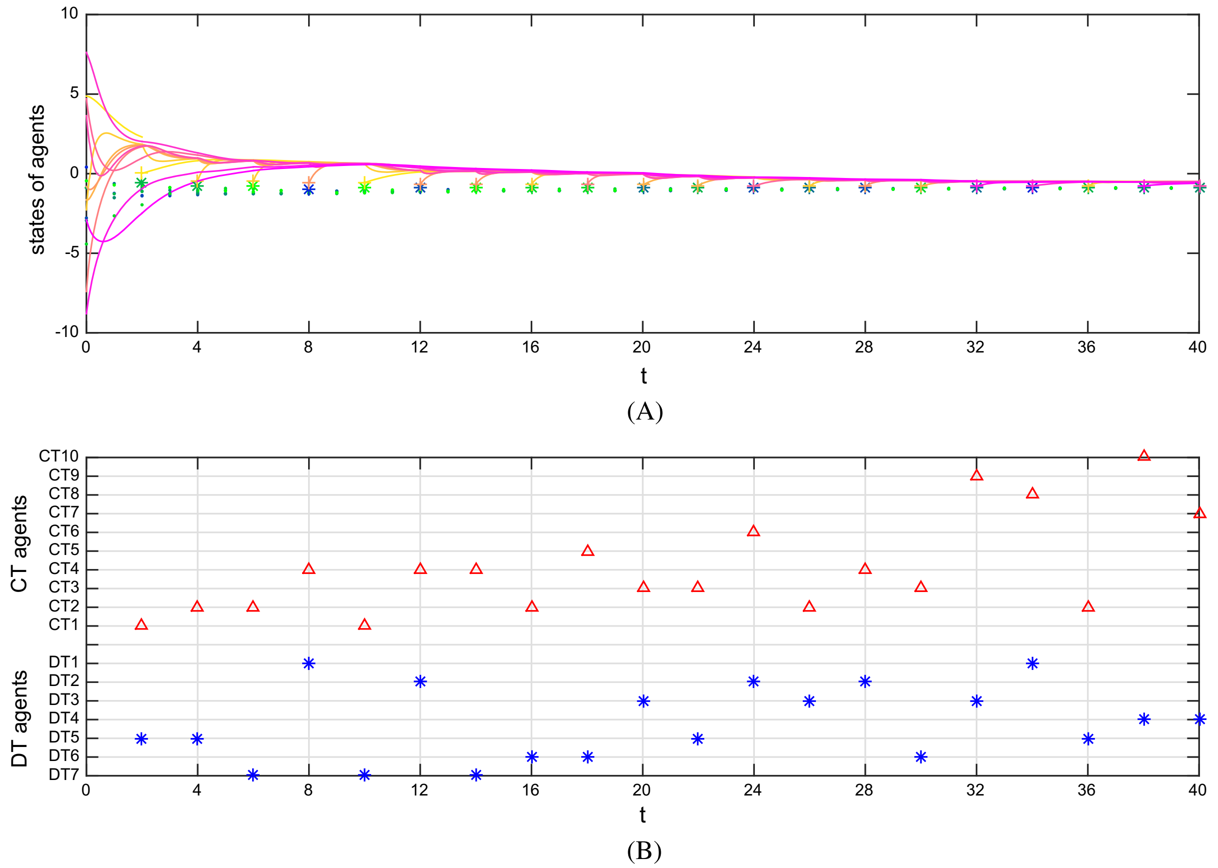

FIGURE 3 State trajectories and players for Example 2. A, State trajectories of all agents; B, Players at time $t = t_k = sH$ [Colour figure can be viewed at wileyonlinelibrary.com]

5 | CONCLUSION

In this paper, consensus problem of hybrid multiagent systems was considered. Agents are categorized into two groups by their dynamics: a group of the CT-agents and a group of the DT-agents. They need to achieve a global task, reaching consensus, by collaboration. However, different dynamics might lead to difference in interests. Therefore, they need to negotiate with each other and make a balance between the global task and the individual interests. This process was modeled by a game. Firstly, we proved that this game has a unique Nash equilibrium solution. Secondly, we obtained that the system can reach consensus if the interaction graph of the system has a spanning tree. Thirdly, we analyzed the convergence speed for the system. We found that the convergence speed depends on the mechanism of game. Therefore, some methods of improving the convergence speed were proposed. In the future, we might consider the containment control and formation control for hybrid systems based on game theory.

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