Systems & Control Letters 125 (2019) 51–58

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Second-order consensus of hybrid multi-agent systems[☆]

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ARTICLE INFO

Article history:
Received 10 December 2017
Received in revised form 17 December 2018
Accepted 25 January 2019
Available online xxxx

Keywords:
Second-order consensus
Hybrid multi-agent system
Discrete-time
Continuous-time

ABSTRACT

It is well known that heterogeneity is an important feature of multi-agent systems. In this paper, we consider the second-order consensus of hybrid multi-agent system which is composed of continuous-time and discrete-time dynamic agents. By analyzing the interactive mode of different dynamic agents, two kinds of effective consensus protocols are proposed for the hybrid multi-agent system. The analysis tool developed in this paper is based on algebraic graph theory and system transformation method. Some necessary and sufficient conditions are established for solving the second-order consensus of hybrid multi-agent system. Two examples are also provided to demonstrate the effectiveness of the theoretical results.

1. Introduction

Multi-agent systems are systems of interconnected autonomous agents, in which the dynamics of each agent are influenced by the behavior of neighboring agents. Examples of multi-agent systems include social networks, biological networks, sensor networks, mobile autonomous robots, and cyber–physical systems [1,2]. The growing importance of multi-agent systems has led to an interest in coordination control to ensure consensus, flocking, containment, formation, rendezvous, etc. [3–10]. Over the past two decades, a variety of dynamic models of agents have been developed to better understand multi-agent coordination. Moreover, lots of mathematical methods are employed to analysis and control of multi-agent systems. For more details, one can refer to survey paper [11] and references therein.

Consensus is a fundamental problem of multi-agent coordination, which implies that certain quantities of autonomous agents, such as opinions, positions, velocities, or headings, reach an agreement based on local information. By virtue of graph theory, Jadbabaie et al. [12] studied the consensus of multi-agent systems with discrete-time dynamic agents, which provided a theoretical explanation for the behavior of the Vicsek model [13]. Following the work in [12], some realistic and effective protocols have been designed for discrete-time multi-agent systems. In [14], the authors studied the state consensus of discrete-time multi-agent systems with time-delays. Gossip algorithms [15] were employed to analyze the consensus behavior. Consensus of discrete-time multi-agent systems with time-varying topologies and stochastic communication noises was also considered in [16]. On a parallel line of research, consensus of multi-agent systems with continuous-time dynamic agents was investigated in [17]. And some criteria were given for solving the average consensus problem. In [18], the authors presented some more relaxable criteria for solving the consensus of continuous-time multi-agent systems.

Note that the previously mentioned results focus on the consensus of multi-agent systems with first-order dynamic agents. However, with the consideration that the motion of robots is governed by Newton's laws, the second-order consensus was considered by lots of researchers [19–26]. The authors in [19,20] studied the second-order consensus of multiple continuous-time dynamic agents with fixed and switching topologies. In [21], the authors considered the second-order consensus of discrete-time multi-agent systems with absolute velocity information. Based on infinite products of stochastic matrices, the authors in [22] considered the second-order consensus of multiple discrete-time dynamic agents with switching topologies. Second-order consensus with nonuniform time-delays was also studied in [23]. Second-order consensus of sampled-data multi-agent systems was considered in [24]. Ren [25] investigated the second-order consensus with

☆ This work was supported by NSFC, China (Grant Nos. 61773303, 61751301, 61703325 and 61563043) the Fundamental Research Funds for the Central Universities, China (Grant Nos. JB180409 and JBF180402) and the Young Talent Fund of University Association for Science and Technology in Shaanxi of China (20160208).

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https://doi.org/10.1016/j.sysconle.2019.01.009
0167-6911/

bounded control input, reference velocity and without velocity measurement.

Heterogeneity is an important feature of multi-agent systems, especially for cyber–physical systems. The authors in [27] considered the output consensus of heterogeneous linear multi-agent systems. Some criteria were given for solving the coordination control of heterogeneous multi-agent system which is composed of first-order and second-order dynamic agents [28–31]. Meanwhile, it is well known that hybrid means heterogeneous in nature or composition. The theory of hybrid systems has received significant attention in the control community for the past decades. As a special class of hybrid systems, switched systems have been studied by a large number of researchers [32]. For multi-agent systems, most of the results concerned with the coordination control under switching topologies were presented in [16,17,22,33]. Recently, the coordination control of a class of multi-agent systems with switching dynamics was also considered in [34–36]. Some sufficient and/or necessary conditions were given for solving the coordination control under arbitrary switching.

Another topic that is closely related to hybrid multi-agent systems is the coexisting of discrete-time and continuous-time dynamic agents. For example, in the real world, natural and artificial individuals can show collective decision-making. Halloy et al. in [37] used autonomous robots to control self-organized behavioral patterns in group-living cockroaches. However, it is difficult to understand the interactive mode of different dynamic agents and analyze the coordination control of such hybrid multi-agent system. In [38], the authors designed several consensus protocols and obtained the consensus criteria for the hybrid multi-agent system, which is composed of first-order dynamic agents. The objective of this paper is to extend the results in [38] to the case of second-order consensus by graph theory and system transformation method. The main contribution of this paper is threefold. First, two kinds of consensus protocols are designed for the hybrid multi-agent system. Second, the necessary and sufficient conditions are obtained for solving the second-order consensus. Third, the unified framework is established in second-order consensus of the discrete-time and the sampled-data multi-agent system.

The remainder of this paper is organized as follows. In Section 2, we present some notions and results in graph theory and propose the hybrid multi-agent system. In Section 3, we present the main results of this paper. In Section 4, numerical simulations are given to illustrate the effectiveness of theoretical results. Finally, some conclusions are drawn in Section 5.

Notation: Throughout this paper, we let $\mathbb{R}$ be the set of real number, $\mathbb{R}^n$ denotes the n-dimensional real vector space. $\mathcal{I}_m = \{1, 2, \ldots, m\}$, $\mathcal{I}_n/\mathcal{I}_m = \{m+1, m+2, \ldots, n\}$. For a given vector or matrix X, X^T denotes its transpose, $\|X\|$ denotes the Euclidean norm of a vector X. A vector is nonnegative if all its elements are nonnegative. Denote by $\mathbf{1}_n$ (or $\mathbf{0}_n$) the column vector with all entries equal to one (or all zeros). I_n is an n-dimensional identity matrix. $\mathrm{diag}\{a_1, a_2 \cdots, a_n\}$ defines a diagonal matrix with diagonal elements being $a_1, a_2 \cdots, a_n$.

2. Preliminaries

2.1. Algebraic graph theory

The interactions among the agents are described by weighted directed graphs. We introduce some basic concepts regarding graphs and their properties. A more detailed exposition can be found in textbooks on algebraic graph theory [39].

A weighted directed graph $\mathscr{G}(\mathscr{A}) = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ of order n consists of a vertex set $\mathscr{V} = \{s_1, s_2, \ldots, s_n\}$, an edge set $\mathscr{E} = \{e_{ij} = (s_i, s_j)\} \subset \mathscr{V} \times \mathscr{V}$ and a nonnegative matrix $\mathscr{A} = [a_{ij}]_{n\times n}$. The neighbor set of the agent i is $\mathcal{N}_i = \{j : a_{ij} > 0\}$. A directed path between two distinct vertices s_i and s_j is a finite ordered sequence of distinct edges of $\mathscr{G}$ with the form $(s_i, s_{k_1}), (s_{k_1}, s_{k_2}), \ldots, (s_{k_l}, s_j)$. A directed tree is a directed graph, where there exists a vertex called the root such that there exists a unique directed path from this vertex to every other vertex. A directed spanning tree is a directed tree, which consists of all the nodes and some edges in $\mathscr{G}$. A graph is called undirected if it satisfies $(s_i, s_j) \in \mathscr{E} \Leftrightarrow (s_j, s_i) \in \mathscr{E}$ for $i, j \in \mathcal{I}_n$. An undirected graph is said to be connected if there exists a path between any two distinct vertices of the graph. The degree matrix $\mathscr{D} = [d_{ij}]_{n\times n}$ is a diagonal matrix with $d_{ii} = \sum_{j:s_j\in\mathcal{N}_i} a_{ij}$ and the Laplacian matrix of the graph is defined as $\mathscr{L} = [l_{ij}]_{n\times n} = \mathscr{D} - \mathscr{A}$. It is easy to see that $\mathscr{L}\mathbf{1}_n = 0$.

A nonnegative matrix is said to be a (row) stochastic matrix if all its row sums are 1. A stochastic matrix $P = [p_{ij}]_{n\times n}$ is called indecomposable and aperiodic (SIA) if $\lim_{k\to\infty} P^k = \mathbf{1}y^T$, where y is some column vector. $\mathscr{G}$ is said the graph associated with P when $(s_i, s_j) \in \mathscr{E}$ if and only if $p_{ji} > 0$. The following result proposes the relationship between a stochastic matrix and its associated graph.

Lemma 1 (*[38]*). *Let $H = \mathrm{diag}\{h_1, h_2, \ldots, h_n\}$ and $0 < h_i < \frac{1}{d_{ii}}$, $i \in \mathcal{I}_n$. Then, $I_n - H\mathscr{L}$ is SIA, i.e., $\lim_{k\to\infty} [I_n - H\mathscr{L}]^k = \mathbf{1}_n \nu^T$, if and only if graph $\mathscr{G}$ has a spanning tree. Furthermore, $[I_n - H\mathscr{L}]^T \nu = \nu$, $\mathbf{1}_n^T \nu = 1$ and each element of ν is nonnegative.*

Remark 1. In this paper, we suppose that there exists interaction behavior among the agents. Thus, it is easy to know that $h_i < \frac{1}{d_{ii}}$ is equivalent to $\max_{i\in\mathcal{I}_n}\{h_i d_{ii}\} < 1$ for $i \in \mathcal{I}_n$.

2.2. Hybrid multi-agent system

We consider the hybrid multi-agent system with second-order dynamics which is composed of continuous-time and discrete-time agents. The number of agents is n, labeled 1 through n, where the number of continuous-time agents is m ($m \le n$). Without loss of generality, we assume that agent 1 through agent m are continuous-time agents. Each agent has the dynamics as follows:

$$\begin{cases} \dot{x}_i(t) = v_i(t), & \dot{v}_i(t) = u_i(t), \\ \quad i \in \mathcal{I}_m, & \\ x_i(t_{k+1}) = x_i(t_k) + hv_i(t_k), & v_i(t_{k+1}) = v_i(t_k) + hu_i(t_k), \\ \quad t_k = kh, & k \in \mathbb{N}, \\ \quad i \in \mathcal{I}_n/\mathcal{I}_m, & \end{cases} \tag{1}$$

where $h = t_{k+1} - t_k > 0$ is the sampling period, $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the position-like, velocity-like and control input of agent i, respectively. The initial conditions of agent i are $x_i(0) = x_{i0}$, $v_i(0) = v_{i0}$. Let $x(0) = [x_{10}, x_{20}, \ldots, x_{n0}]^T$, $v(0) = [v_{10}, v_{20}, \ldots, v_{n0}]^T$.

Definition 1 (*Second-Order Consensus*). Hybrid multi-agent system (1) is said to reach second-order consensus if for any initial conditions, we have

$$\lim_{t_k\to\infty} \|x_i(t_k) - x_j(t_k)\| = 0, \quad \lim_{t_k\to\infty} \|v_i(t_k) - v_j(t_k)\| = 0, \quad \text{for } i, j \in \mathcal{I}_n, \tag{2}$$

and

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \quad \lim_{t\to\infty} \|v_i(t) - v_j(t)\| = 0, \quad \text{for } i, j \in \mathcal{I}_m. \tag{3}$$

Next, it is not difficult to verify the following results.

Lemma 2. *Let $f(x)=e^{hx}$, $g(x)=\frac{e^{hx}-1}{x}$ and $h(x)=x-xe^{hx}$. Then, for $x\in(-\infty,0)$, we have $f(x)$, $g(x)$ and $h(x)$ are increasing functions.*

3. Main results

In this section, we will present two kinds of consensus protocols (control inputs) for hybrid multi-agent system (1). The consensus criteria are also established for solving the second-order consensus.

3.1. Case 1

In this subsection, we assume that all agents communicate with their neighbors and update their control inputs in the sampling time t_k with absolute velocity information. Then, the consensus protocol for hybrid multi-agent system (1) is given as

$$\begin{cases} u_i(t)=\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k))-k_1v_i(t_k), \quad t\in(t_k,t_{k+1}], \\ \quad i\in\mathcal{I}_m, \\ u_i(t_k)=\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k))-k_1v_i(t_k), \\ \quad i\in\mathcal{I}_n/\mathcal{I}_m, \end{cases} \tag{4}$$

where $\mathscr{A}=[a_{ij}]_{n\times n}$ is the aforementioned weighted adjacency matrix associated with graph $\mathscr{G}$, $k_1>0$ is the feedback gain.

Theorem 1. *Consider a directed communication graph $\mathscr{G}$ and suppose that $2\sqrt{\max_{i\in\mathcal{I}_n}\{d_{ii}\}}<k_1<\frac{\sqrt{5}-1}{h}$. Then, hybrid multi-agent system (1) with protocol (4) reaches second-order consensus if and only if graph $\mathscr{G}$ has a directed spanning tree.*

Proof. *(Sufficiency)* Firstly, we will prove that Eq. (2) holds. From (1) and (4), we have

$$\begin{cases} x_i(t)=x_i(t_k)+v_i(t_k)(t-t_k) \\ \quad +\frac{(t-t_k)^2}{2}\left(\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k))-k_1v_i(t_k)\right), \\ v_i(t)=v_i(t_k)+(t-t_k)\left(\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k))-k_1v_i(t_k)\right), \end{cases} \tag{5}$$

for $t\in(t_k,t_{k+1}]$, $i\in\mathcal{I}_m$ and

$$\begin{cases} x_i(t_{k+1})=x_i(t_k)+hv_i(t_k), \\ v_i(t_{k+1})=v_i(t_k)+h\left(\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k))-k_1v_i(t_k)\right), \end{cases} \tag{6}$$

for $i\in\mathcal{I}_n/\mathcal{I}_m$. For (5), when $t=t_{k+1}$, we have

$$\begin{cases} x_i(t_{k+1})=x_i(t_k)+hv_i(t_k) \\ \quad +\frac{h^2}{2}\left(\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k))-k_1v_i(t_k)\right), \\ v_i(t_{k+1})=v_i(t_k)+h\left(\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k))-k_1v_i(t_k)\right). \end{cases} \tag{7}$$

Let $y_{i'}(t_k)=x_i(t_k)+\frac{2}{k_1}v_i(t_k)$. Then, $v_i(t_k)=\frac{k_1}{2}(y_{i'}(t_k)-x_i(t_k))$. For $i\in\mathcal{I}_m$ and $i'\in\mathcal{I}_m$,

$$\begin{aligned} x_i(t_{k+1})&=x_i(t_k)+(h-\frac{h^2}{2}k_1)\frac{k_1}{2}(y_{i'}(t_k)-x_i(t_k)) \\ &\quad+\frac{h^2}{2}\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k)) \\ &=x_i(t_k)+(\frac{hk_1}{2}-\frac{h^2k_1^2}{4})(y_{i'}(t_k)-x_i(t_k)) \\ &\quad+\frac{h^2}{2}\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k)), \end{aligned} \tag{8}$$

$$\begin{aligned} y_{i'}(t_{k+1})&=x_i(t_{k+1})+\frac{2}{k_1}v_i(t_{k+1}) \\ &=x_i(t_k)+hv_i(t_k)+\frac{h^2}{2}\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k))-\frac{h^2k_1}{2}v_i(t_k) \\ &\quad+\frac{2}{k_1}v_i(t_k)+\frac{2h}{k_1}\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k))-2hv_i(t_k) \\ &=y_{i'}(t_k)+(-h-\frac{h^2k_1}{2})v_i(t_k) \\ &\quad+(\frac{h^2}{2}+\frac{2h}{k_1})\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k)) \\ &=y_{i'}(t_k)+(\frac{hk_1}{2}+\frac{h^2k_1^2}{4})(x_i(t_k)-y_{i'}(t_k)) \\ &\quad+(\frac{h^2}{2}+\frac{2h}{k_1})\left[\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-y_{i'}(t_k))\right. \\ &\quad\left.+\sum_{j\in\mathcal{N}_i}a_{ij}(y_{i'}(t_k)-x_i(t_k))\right] \\ &=y_{i'}(t_k)+(\frac{h^2}{2}+\frac{2h}{k_1})\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-y_{i'}(t_k)) \\ &\quad+(\frac{hk_1}{2}+\frac{h^2k_1^2}{4}-\frac{h^2}{2}\sum_{j\in\mathcal{N}_i}a_{ij} \\ &\quad-\frac{2h}{k_1}\sum_{j\in\mathcal{N}_i}a_{ij})(x_i(t_k)-y_{i'}(t_k)). \end{aligned} \tag{9}$$

For $i\in\mathcal{I}_n/\mathcal{I}_m$ and $i'\in\mathcal{I}_n/\mathcal{I}_m$,

$$x_i(t_{k+1})=x_i(t_k)+\frac{hk_1}{2}(y_{i'}(t_k)-x_i(t_k)), \tag{10}$$

$$\begin{aligned} y_{i'}(t_{k+1})&=x_i(t_{k+1})+\frac{2}{k_1}v_i(t_{k+1}) \\ &=x_i(t_k)+\frac{hk_1}{2}(y_{i'}(t_k)-x_i(t_k)) \\ &\quad+\frac{2}{k_1}v_i(t_k)+\frac{2h}{k_1}\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k)) \\ &\quad-hk_1(y_{i'}(t_k)-x_i(t_k)) \\ &=y_{i'}(t_k)+(\frac{hk_1}{2}-\frac{2h}{k_1}\sum_{j\in\mathcal{N}_i}a_{ij})(x_i(t_k)-y_{i'}(t_k)) \\ &\quad+\frac{2h}{k_1}\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-y_{i'}(t_k)). \end{aligned} \tag{11}$$

Let $k_2 = \frac{hk_1}{2} - \frac{h^2k_1^2}{4}$, $k_3 = \frac{hk_1}{2} + \frac{h^2k_1^2}{4} - \frac{h^2}{2}\sum_{j\in\mathcal{N}_i} a_{ij} - \frac{2h}{k_1}\sum_{j\in\mathcal{N}_i} a_{ij}$ and $k_4 = \frac{hk_1}{2} - \frac{2h}{k_1}\sum_{j\in\mathcal{N}_i} a_{ij}$. From (8)–(11), we get a first-order discrete-time multi-agent system with $2n$ agents as follows:

$$\begin{cases} x_i(t_{k+1}) = x_i(t_k) + k_2(y_{i'}(t_k) - x_i(t_k)) + \dfrac{h^2}{2}\displaystyle\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - x_i(t_k)), \\ \quad i \in \mathcal{I}_m, \\ y_{i'}(t_{k+1}) = y_{i'}(t_k) + k_3(x_i(t_k) - y_{i'}(t_k)) \\ \quad + (\dfrac{h^2}{2} + \dfrac{2h}{k_1})\displaystyle\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - y_{i'}(t_k)), \quad i' \in \mathcal{I}_m, \\ x_i(t_{k+1}) = x_i(t_k) + \dfrac{hk_1}{2}(y_{i'}(t_k) - x_i(t_k)), \quad i \in \mathcal{I}_n/\mathcal{I}_m, \\ y_{i'}(t_{k+1}) = y_{i'}(t_k) + k_4(x_i(t_k) - y_{i'}(t_k)) \\ \quad + \dfrac{2h}{k_1}\displaystyle\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - y_{i'}(t_k)), \quad i' \in \mathcal{I}_n/\mathcal{I}_m, \end{cases} \tag{12}$$

where $x_i \in \mathbb{R}$ and $y_{i'} \in \mathbb{R}$ are the states of ith and i'th agents, respectively.

Due to $2\sqrt{\max_{i\in\mathcal{I}_n}\{d_{ii}\}} < k_1 < \frac{\sqrt{5}-1}{h}$, it is easy to know that $2\sqrt{\max_{i\in\mathcal{I}_n}\{d_{ii}\}} < k_1 < \frac{2}{h}$. Thus, $k_2 = \frac{hk_1}{2} - \frac{h^2k_1^2}{4} > 0$. If $d_{ii} = 0$, we have $0 < k_2 < k_3 = \frac{hk_1}{2} + \frac{h^2k_1^2}{4} < 1$ and $0 < k_4 = \frac{hk_1}{2} < 1$. If $d_{ii} \neq 0$, we have $\frac{h^2k_1^2}{4} - \frac{h^2}{2}\max_{i\in\mathcal{I}_n}\{d_{ii}\} > 0$ and $\frac{k_1h}{2} - \frac{2h}{k_1}\max_{i\in\mathcal{I}_n}\{d_{ii}\} > 0$, which implies that $k_3 > 0$ and $k_4 > 0$. Moreover, $k_2 + \frac{h^2}{2}\sum_{j\in\mathcal{N}_i} a_{ij} < k_3 + (\frac{h^2}{2} + \frac{2h}{k_1})\sum_{j\in\mathcal{N}_i} a_{ij} = \frac{hk_1}{2} + \frac{h^2k_1^2}{4} < 1$ and $\frac{hk_1}{2} = k_4 + \frac{2h}{k_1}\sum_{j\in\mathcal{N}_i} a_{ij} < 1$.

Let $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ be a directed communication graph of first-order multi-agent system (12) with a vertex set $\mathcal{V}' = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \cup \mathcal{V}_4$, which $\mathcal{V}_1 = \{s_1, \ldots, s_m\}$, $\mathcal{V}_2 = \{s_{1'}, \ldots, s_{m'}\}$, $\mathcal{V}_3 = \{s_{m+1}, \ldots, s_n\}$ and $\mathcal{V}_4 = \{s_{(m+1)'}, \ldots, s_{n'}\}$. Suppose that graph $\mathcal{G}$ has a directed spanning tree $T_{\mathcal{G}}$. For each edge $(s_j, s_i) \in T_{\mathcal{G}}$, we have $(s_{j'}, s_j) \in \mathcal{E}'$, $(s_j, s_{i'}) \in \mathcal{E}'$, $(s_{i'}, s_i) \in \mathcal{E}'$. Adding these edges to $T_{\mathcal{G}}$, we get a directed spanning tree for $\mathcal{G}'$. Suppose that graph $\mathcal{G}'$ has a directed spanning tree $T_{\mathcal{G}'}$. For each vertex $s_i \in \mathcal{V}_2 \cup \mathcal{V}_4$, if there exist $s_j, s_k \in \mathcal{V}_1 \cup \mathcal{V}_3$ which make $(s_j, s_i), (s_i, s_k) \in T_{\mathcal{G}'}$, we delete the vertex s_i and add the edge $(s_j, s_k) \in \mathcal{E}$. Thus, we get a directed spanning tree for $\mathcal{G}$. Therefore, graph $\mathcal{G}$ has a directed spanning tree if and only if graph $\mathcal{G}'$ has a directed spanning tree.

By the aforementioned analysis and Lemma 1, since graph $\mathcal{G}$ has a directed spanning tree, it is easy to get that first-order multi-agent system (12) reaches consensus, i.e. $\lim_{t_k\to\infty}\|x_i(t_k) - x_j(t_k)\| = \lim_{t_k\to\infty}\|x_i(t_k) - y_{i'}(t_k)\| = 0$ for $i, i', j \in \mathcal{I}_n$, which implies that $\lim_{t_k\to\infty}\|v_i(t_k)\| = 0$. Thus, we have that Eq. (2) holds if graph $\mathcal{G}$ has a directed spanning tree.

Next, we will prove that Eq. (3) holds. We have

$$\|x_i(t) - x_j(t)\| \leq \|x_i(t) - x_i(t_k)\| + \|x_i(t_k) - x_j(t_k)\| + \|x_j(t_k) - x_j(t)\|$$

and

$$\|v_i(t) - v_j(t)\| \leq \|v_i(t) - v_i(t_k)\| + \|v_i(t_k) - v_j(t_k)\| + \|v_j(t_k) - v_j(t)\|,$$

From (5), it is easy to know that

$$\|x_i(t) - x_i(t_k)\| \leq h\|v_i(t_k)\| + \frac{h^2}{2}\left(\sum_{j\in\mathcal{N}_i} a_{ij}\|x_j(t_k) - x_i(t_k)\| + k_1\|v_i(t_k)\|\right),$$

$$\|v_i(t) - v_i(t_k)\| \leq h\left(\sum_{j\in\mathcal{N}_i} a_{ij}\|x_j(t_k) - x_i(t_k)\| + k_1\|v_i(t_k)\|\right),$$

for $t \in (t_k, t_{k+1}]$, $i \in \mathcal{I}_m$. When $t \to \infty$, we have $t_k \to \infty$. Therefore,

$$\lim_{t\to\infty}\|x_i(t) - x_i(t_k)\| = \lim_{t\to\infty}\|v_i(t) - v_i(t_k)\| = 0,$$

for $i \in \mathcal{I}_m$, which implies that

$$\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = \lim_{t\to\infty}\|v_i(t) - v_j(t)\| = 0, \quad for \ i, j \in \mathcal{I}_m.$$

Thus, hybrid multi-agent system (1) with protocol (4) reaches second-order consensus.

(Necessity) Suppose that graph $\mathcal{G}$ does not have a directed spanning tree. It follows from Lemma 1 that $\lim_{k\to\infty}(I_n - H\mathcal{L})^k \neq \mathbf{1}_n\nu^T$, which means that first-order discrete-time multi-agent system (12) cannot reach consensus. Consequently, hybrid multi-agent system (1) cannot reach consensus. ∎

Remark 2. In this paper, the consensus protocol (4) is designed for hybrid multi-agent system (1) with absolute velocity information, where $-k_1v_i(t_k)$ is the velocity damping term and k_1 is the velocity damping gain. In fact, the results in this paper can also be extended with relative and absolute velocity information as $u_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - x_i(t_k)) + \sum_{j\in\mathcal{N}_i} a_{ij}(v_j(t_k) - v_i(t_k)) - k_1v_i(t_k)$.

Remark 3. In fact, the condition $2\sqrt{\max_{i\in\mathcal{I}_n}\{d_{ii}\}} < k_1 < \frac{\sqrt{5}-1}{h}$ in Theorem 1 can be realized. For a well-connected network (i.e., the $\max_{i\in\mathcal{I}_n}\{d_{ii}\}$ is large), we can choose a small sampling period $h > 0$. Thus, there exists the feedback gain k_1 which satisfies $2\sqrt{\max_{i\in\mathcal{I}_n}\{d_{ii}\}} < k_1 < \frac{\sqrt{5}-1}{h}$.

Remark 4. From [21,24], it is easy to find that the discrete-time multi-agent system has the form as (6) and the sampled-data multi-agent system has the form as (7). Owing to the hybrid feature of consensus protocol (4), the analysis of hybrid multi-agent system (1) is more difficult than the discrete-time multi-agent system. The result in Theorem 1 establishes a unified framework for the second-order consensus of discrete-time and sampled-data multi-agent systems, i.e., hybrid multi-agent system (1) becomes a sampled-data multi-agent system if $m = n$, and hybrid multi-agent system (1) becomes a discrete-time multi-agent system if $m = 0$.

3.2. Case 2

In this subsection, we still assume that the interaction among agents happens in sampling time t_k. However, different from Case 1, we assume that each continuous-time dynamic agent can obtain its own state in real time. Thus, the consensus protocol for hybrid multi-agent system (1) is given as

$$\begin{cases} u_i(t) = \displaystyle\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - x_i(t)) - k_1v_i(t), \quad t \in (t_k, t_{k+1}], \quad i \in \mathcal{I}_m, \\ u_i(t_k) = \displaystyle\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - x_i(t_k)) - k_1v_i(t_k), \quad i \in \mathcal{I}_n/\mathcal{I}_m, \end{cases} \tag{13}$$

where $\mathcal{A} = [a_{ij}]_{n\times n}$ is the weighted adjacency matrix associated with graph $\mathcal{G}$, $k_1 > 0$ is the feedback gain.

Theorem 2. *Consider a directed communication graph $\mathcal{G}$ and suppose that $2\sqrt{\max_{i\in\mathcal{I}_n}\{d_{ii}\}} < k_1 < \frac{\sqrt{5}-1}{h}$. Then, hybrid multi-agent system (1) with protocol (13) reaches second-order consensus if and only if graph $\mathcal{G}$ has a directed spanning tree.*

Proof. *(Sufficiency)* From (1) and (13), for $i \in \mathcal{I}_m$ and $t \in (t_k, t_{k+1}]$, if $d_{ii} \neq 0$, we have

$$\ddot{x}_i(t) = \dot{v}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - x_i(t)) - k_1\dot{x}_i(t),$$

i.e.,

$$\ddot{x}_i(t) + k_1\dot{x}_i(t) + \left(\sum_{j\in\mathcal{N}_i} a_{ij}\right) x_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}x_j(t_k).$$

Solving the above second order ordinary differential equation, we get

$$\begin{cases} x_i(t) = c_1 e^{r_1(t-t_k)} + c_2 e^{r_2(t-t_k)} + \dfrac{\sum_{j\in\mathcal{N}_i} a_{ij}x_j(t_k)}{\sum_{j\in\mathcal{N}_i} a_{ij}}, \\ v_i(t) = \dot{x}_i(t) = c_1 r_1 e^{r_1(t-t_k)} + c_2 r_2 e^{r_2(t-t_k)}, \end{cases} \tag{14}$$

where $r_1 = \frac{-k_1+\sqrt{k_1^2-4\sum_{j\in\mathcal{N}_i} a_{ij}}}{2}$, $r_2 = \frac{-k_1-\sqrt{k_1^2-4\sum_{j\in\mathcal{N}_i} a_{ij}}}{2}$, $c_1 = \frac{v_i(t_k)+\frac{r_2\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k)-x_i(t_k))}{\sum_{j\in\mathcal{N}_i} a_{ij}}}{r_1-r_2}$ and $c_2 = \frac{v_i(t_k)+\frac{r_1\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k)-x_i(t_k))}{\sum_{j\in\mathcal{N}_i} a_{ij}}}{r_2-r_1}$, for $t \in (t_k, t_{k+1}]$. It is easy to find that $r_1 + r_2 = -k_1$, $r_1 r_2 = \sum_{j\in\mathcal{N}_i} a_{ij}$.

For $i \in \mathcal{I}_m$, when $t = t_{k+1}$, we have

$$\begin{aligned} x_i(t_{k+1}) &= c_1 e^{r_1 h} + c_2 e^{r_2 h} + \frac{\sum_{j\in\mathcal{N}_i} a_{ij}x_j(t_k)}{\sum_{j\in\mathcal{N}_i} a_{ij}} \\ &= \frac{v_i(t_k) + \frac{r_2\cdot\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k)-x_i(t_k))}{\sum_{j\in\mathcal{N}_i} a_{ij}}}{r_1 - r_2}\cdot e^{r_1 h} \\ &\quad + \frac{v_i(t_k) + \frac{r_1\cdot\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k)-x_i(t_k))}{\sum_{j\in\mathcal{N}_i} a_{ij}}}{r_2 - r_1}\cdot e^{r_2 h} + \frac{\sum_{j\in\mathcal{N}_i} a_{ij}x_j(t_k)}{\sum_{j\in\mathcal{N}_i} a_{ij}} \\ &= \frac{e^{r_1 h} - e^{r_2 h}}{r_1 - r_2} v_i(t_k) + \frac{\frac{r_2 e^{r_1 h} - r_1 e^{r_2 h}}{\sum_{j\in\mathcal{N}_i} a_{ij}}}{r_1 - r_2}\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - x_i(t_k)) \\ &\quad + \frac{\sum_{j\in\mathcal{N}_i} a_{ij}x_j(t_k)}{\sum_{j\in\mathcal{N}_i} a_{ij}} \\ &= x_i(t_k) + \frac{r_2 e^{r_1 h} - r_1 e^{r_2 h} + (r_1 - r_2)}{\sum_{j\in\mathcal{N}_i} a_{ij}(r_1 - r_2)}\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - x_i(t_k)) \\ &\quad + \frac{e^{r_1 h} - e^{r_2 h}}{r_1 - r_2} v_i(t_k), \end{aligned} \tag{15}$$

and

$$\begin{aligned} v_i(t_{k+1}) &= c_1 r_1 e^{r_1 h} + c_2 r_2 e^{r_2 h} \\ &= \frac{r_1 v_i(t_k) + \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - x_i(t_k))}{r_1 - r_2}\cdot e^{r_1 h} \\ &\quad + \frac{r_2 v_i(t_k) + \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - x_i(t_k))}{r_2 - r_1}\cdot e^{r_2 h} \\ &= \frac{r_1 e^{r_1 h} - r_2 e^{r_2 h}}{r_1 - r_2} v_i(t_k) + \frac{e^{r_1 h} - e^{r_2 h}}{r_1 - r_2}\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - x_i(t_k)) \\ &= v_i(t_k) + \frac{e^{r_1 h} - e^{r_2 h}}{r_1 - r_2}\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - x_i(t_k)) \\ &\quad + \frac{r_1 e^{r_1 h} - r_2 e^{r_2 h} - (r_1 - r_2)}{r_1 - r_2} v_i(t_k). \end{aligned} \tag{16}$$

Let $y_{i'}(t_k) = x_i(t_k) + \frac{2}{k_1}v_i(t_k)$. Then, $v_i(t_k) = \frac{k_1}{2}(y_{i'}(t_k) - x_i(t_k))$. For $i \in \mathcal{I}_m$ and $i' \in \mathcal{I}_m$,

$$\begin{aligned} x_i(t_{k+1}) &= x_i(t_k) + \frac{\frac{e^{r_1 h}-1}{r_1} - \frac{e^{r_2 h}-1}{r_2}}{r_1 - r_2}\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - x_i(t_k)) \\ &\quad + \frac{k_1}{2}\frac{e^{r_1 h} - e^{r_2 h}}{r_1 - r_2}(y_{i'}(t_k) - x_i(t_k)) \\ &= x_i(t_k) + \frac{k_1 k_5}{2}(y_{i'}(t_k) - x_i(t_k)) + k_6\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - x_i(t_k)), \end{aligned} \tag{17}$$

$$\begin{aligned} y_{i'}(t_{k+1}) &= x_i(t_k) + k_6\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - x_i(t_k)) + \frac{k_1 k_5}{2}(y_{i'}(t_k) - x_i(t_k)) \\ &\quad + \frac{2}{k_1}\left(v_i(t_k) + k_5\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - x_i(t_k)) + k_7 v_i(t_k)\right) \\ &= y_{i'}(t_k) + (\frac{k_1 k_5}{2} + k_7)(y_{i'}(t_k) - x_i(t_k)) \\ &\quad + (\frac{2k_5}{k_1} + k_6)\left(\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - y_{i'}(t_k)) \right.\\ &\quad \left. + \sum_{j\in\mathcal{N}_i} a_{ij}(y_{i'}(t_k) - x_i(t_k))\right) \\ &= y_{i'}(t_k) + \left(-\frac{k_1 k_5}{2} - k_7 - \sum_{j\in\mathcal{N}_i} a_{ij}(\frac{2k_5}{k_1} + k_6)\right) \\ &\quad \times (x_i(t_k) - y_{i'}(t_k)) + (\frac{2k_5}{k_1} + k_6)\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k) - y_{i'}(t_k)), \end{aligned} \tag{18}$$

where $k_5 = \frac{e^{r_1 h}-e^{r_2 h}}{r_1-r_2}$, $k_6 = \frac{\frac{e^{r_1 h}-1}{r_1} - \frac{e^{r_2 h}-1}{r_2}}{r_1-r_2}$ and $k_7 = \frac{r_1 e^{r_1 h}-r_2 e^{r_2 h}-(r_1-r_2)}{r_1-r_2}$.

Due to $r_2 < r_1 < 0$, from Lemma 2, we have $k_5 > 0$, $k_6 > 0$ and $k_7 < 0$. In addition, we know $k_1 > 2\sqrt{\max_{i\in\mathcal{I}_n}\{d_{ii}\}}$, which implies that $(\frac{k_1}{2} - \frac{2}{k_1}\sum_{j\in\mathcal{N}_i} a_{ij}) > 0$. See the equation in Box I. Thus, $\frac{k_1 k_5}{2} + k_6\sum_{j\in\mathcal{N}_i} a_{ij} = (-\frac{k_1 k_5}{2} - k_7 - \sum_{j\in\mathcal{N}_i} a_{ij}(\frac{2k_5}{k_1} + k_6)) + (\frac{2k_5}{k_1} + k_6)\sum_{j\in\mathcal{N}_i} a_{ij}$. Owing to $r_1 + r_2 = -k_1$,

$$\begin{aligned} \frac{k_1 k_5}{2} + k_6\sum_{j\in\mathcal{N}_i} a_{ij} &= -\frac{k_1 k_5}{2} - k_7 \\ &= -\frac{\frac{k_1}{2}(e^{r_1 h} - e^{r_2 h}) + \big((r_1 e^{r_1 h} - r_1) - (r_2 e^{r_2 h} - r_2)\big)}{r_1 - r_2} \\ &= -\frac{e^{r_1 h} + e^{r_2 h}}{2} + 1 < 1. \end{aligned}$$

For $i \in \mathcal{I}_m$ and $t \in (t_k, t_{k+1}]$, if $d_{ii} = 0$, (14) can be replaced by

$$\begin{cases} x_i(t) = c_3 + c_4 e^{-k_1(t-t_k)}, \\ v_i(t) = -k_1 c_4 e^{-k_1(t-t_k)}, \end{cases} \tag{19}$$

where $c_3 = x_i(t_k) + \frac{v_i(t_k)}{k_1}$ and $c_4 = -\frac{v_i(t_k)}{k_1}$. Thus, it is easy to get that

$$\begin{cases} x_i(t_{k+1}) = x_i(t_k) + \dfrac{1 - e^{-k_1 h}}{2}(y_{i'}(t_k) - x_i(t_k)), \\ y_{i'}(t_{k+1}) = y_{i'}(t_k) + \dfrac{1 - e^{-k_1 h}}{2}(x_i(t_k) - y_{i'}(t_k)), \end{cases} \tag{20}$$

where $i \in \mathcal{I}_m$, $i' \in \mathcal{I}_m$ and $0 < \frac{1-e^{-k_1 h}}{2} < 1$.

Moreover

$$-\frac{k_1k_5}{2}-k_7-\sum_{j\in\mathcal{N}_i}a_{ij}(\frac{2k_5}{k_1}+k_6)$$
$$=\frac{\frac{k_1}{2}(e^{r_1h}-e^{r_2h})+[(r_1e^{r_1h}-r_1)-(r_2e^{r_2h}-r_2)]+\sum_{j\in\mathcal{N}_i}a_{ij}(\frac{e^{r_1h}-1}{r_1}-\frac{e^{r_2h}-1}{r_2})+\frac{2}{k_1}\sum_{j\in\mathcal{N}_i}a_{ij}(e^{r_1h}-e^{r_2h})}{r_2-r_1}$$
$$=\left(\frac{k_1}{2}+r_1+r_2+\frac{2}{k_1}\sum_{j\in\mathcal{N}_i}a_{ij}\right)\frac{(e^{r_1h}-e^{r_2h})}{r_2-r_1}=\left(\frac{k_1}{2}-\frac{2}{k_1}\sum_{j\in\mathcal{N}_i}a_{ij}\right)\frac{(e^{r_1h}-e^{r_2h})}{r_1-r_2}$$
$$=k_5\left(\frac{k_1}{2}-\frac{2}{k_1}\sum_{j\in\mathcal{N}_i}a_{ij}\right)>0.$$

Box I.

For $i\in\mathcal{I}_n/\mathcal{I}_m$ and $i'\in\mathcal{I}_n/\mathcal{I}_m$, similar to the analysis of Theorem 1,

$$x_i(t_{k+1})=x_i(t_k)+\frac{hk_1}{2}(y_{i'}(t_k)-x_i(t_k)) \quad (21)$$

and

$$y_{i'}(t_{k+1})=y_{i'}(t_k)+k_4(x_i(t_k)-y_{i'}(t_k))+\frac{2h}{k_1}\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-y_{i'}(t_k)). \quad (22)$$

From (17), (18) (or (20)) and (21), (22), we get a first-order discrete-time multi-agent system as follows:

$$\begin{cases} x_i(t_{k+1})=x_i(t_k)+\frac{k_1k_5}{2}(y_{i'}(t_k)-x_i(t_k))+k_6\sum_{j\in\mathcal{N}_i}a_{ij} \\ \quad\times(x_j(t_k)-x_i(t_k)), \qquad i\in\mathcal{I}_m, \\ y_{i'}(t_{k+1})=y_{i'}(t_k)+k_5\left(\frac{k_1}{2}-\frac{2}{k_1}\sum_{j\in\mathcal{N}_i}a_{ij}\right)(x_i(t_k)-y_{i'}(t_k)) \\ \quad+(\frac{2k_5}{k_1}+k_6)\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-y_{i'}(t_k)), \qquad i'\in\mathcal{I}_m, \\ x_i(t_{k+1})=x_i(t_k)+\frac{hk_1}{2}(y_{i'}(t_k)-x_i(t_k)), \qquad i\in\mathcal{I}_n/\mathcal{I}_m, \\ y_{i'}(t_{k+1})=y_{i'}(t_k)+k_4(x_i(t_k)-y_{i'}(t_k)) \\ \quad+\frac{2h}{k_1}\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-y_{i'}(t_k)), \qquad i'\in\mathcal{I}_n/\mathcal{I}_m. \end{cases} \quad (23)$$

By the aforementioned analysis and Lemma 1, since graph $\mathscr{G}$ has a directed spanning tree, it is easy to get that first-order multi-agent system (23) reaches consensus, which implies that $\lim_{t_k\to\infty}\|x_i(t_k)-x_j(t_k)\|=0$, $\lim_{t_k\to\infty}\|v_i(t_k)\|=0$ for $i,j\in\mathcal{I}_n$.

For $i,j\in\mathcal{I}_m$, we have

$$\|x_i(t)-x_j(t)\|\leq\|x_i(t)-x_i(t_k)\|+\|x_i(t_k)-x_j(t_k)\|+\|x_j(t_k)-x_j(t)\|$$

and

$$\|v_i(t)-v_j(t)\|\leq\|v_i(t)-v_i(t_k)\|+\|v_i(t_k)-v_j(t_k)\|+\|v_j(t_k)-v_j(t)\|.$$

If $d_{ii}\neq 0$, from (14), we know that

$$\|x_i(t)-x_i(t_k)\|$$
$$=\|c_1e^{r_1(t-t_k)}+c_2e^{r_2(t-t_k)}+\frac{\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k))}{\sum_{j\in\mathcal{N}_i}a_{ij}}\|$$
$$=\left\|\frac{r_2e^{r_1(t-t_k)}-r_1e^{r_2(t-t_k)}+(r_1-r_2)}{\sum_{j\in\mathcal{N}_i}a_{ij}(r_1-r_2)}\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k))\right.$$
$$\left.+\frac{e^{r_1(t-t_k)}-e^{r_2(t-t_k)}}{r_1-r_2}v_i(t_k)\right\|$$
$$\leq\frac{\frac{e^{r_1(t-t_k)}-1}{r_1}-\frac{e^{r_2(t-t_k)}-1}{r_2}}{r_1-r_2}\sum_{j\in\mathcal{N}_i}a_{ij}\left\|(x_j(t_k)-x_i(t_k))\right\|$$
$$+\frac{e^{r_1(t-t_k)}-e^{r_2(t-t_k)}}{r_1-r_2}\|v_i(t_k)\|$$
$$\leq\frac{1}{\sum_{j\in\mathcal{N}_i}a_{ij}}\left(\sum_{j\in\mathcal{N}_i}a_{ij}\left\|(x_j(t_k)-x_i(t_k))\right\|\right)+\frac{2}{k_1}\|v_i(t_k)\|,$$

$$\|v_i(t)-v_i(t_k)\|=\left\|\frac{e^{r_1(t-t_k)}-e^{r_2(t-t_k)}}{r_1-r_2}\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_k)-x_i(t_k))\right.$$
$$\left.+\frac{r_1e^{r_1(t-t_k)}-r_2e^{r_2(t-t_k)}-(r_1-r_2)}{r_1-r_2}v_i(t_k)\right\|$$
$$\leq\frac{e^{r_1(t-t_k)}-e^{r_2(t-t_k)}}{r_1-r_2}\sum_{j\in\mathcal{N}_i}a_{ij}\|(x_j(t_k)-x_i(t_k))\|$$
$$+\frac{(r_1-r_1e^{r_1(t-t_k)})-(r_2-r_2e^{r_2(t-t_k)})}{r_1-r_2}\|v_i(t_k)\|$$
$$\leq\frac{2}{k_1}\sum_{j\in\mathcal{N}_i}a_{ij}\|(x_j(t_k)-x_i(t_k))\|+2\|v_i(t_k)\|.$$

If $d_{ii}=0$, it follows from (19) that

$$\|x_i(t)-x_i(t_k)\|=\left\|\frac{v_i(t_k)}{k_1}-\frac{v_i(t_k)}{k_1}e^{-k_1(t-t_k)}\right\|$$
$$\leq\frac{1}{k_1}(1-e^{-k_1(t-t_k)})\|v_i(t_k)\|\leq\frac{1}{k_1}\|v_i(t_k)\|,$$

$$\|v_i(t)-v_i(t_k)\|=\|v_i(t_k)e^{-k_1(t-t_k)}-v_i(t_k)\|$$
$$\leq(1-e^{-k_1(t-t_k)})\|v_i(t_k)\|\leq\|v_i(t_k)\|.$$

When $t\to\infty$, we have $t_k\to\infty$. Therefore,

$$\lim_{t\to\infty}\|x_i(t)-x_i(t_k)\|=\lim_{t\to\infty}\|v_i(t)-v_i(t_k)\|=0,$$

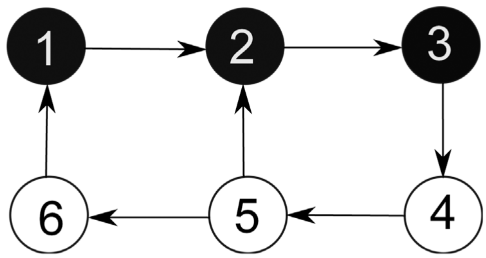

Fig. 1. A directed graph $\mathcal{G}$.

for $i \in \mathcal{I}_m$, which implies that

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = \lim_{t\to\infty} \|v_i(t) - v_j(t)\| = 0, \quad for \ \ i, j \in \mathcal{I}_m.$$

Thus, hybrid multi-agent system (1) with protocol (13) reaches second-order consensus.

(Necessity) Similar to the proof of necessity in Theorem 1, we know that if the directed communication graph $\mathcal{G}$ does not have a directed spanning tree, then hybrid multi-agent system (1) cannot achieve consensus. ■

Remark 5. In consensus protocol (13), we assume that the interaction among agents happens in sampling time t_k for $t \in (t_k, t_{k+1}]$. However, each continuous-time dynamic agent can obtain its own state in real time. Thus, the continuous-time dynamic agent i uses its own continuous information $x_i(t)$ ($v_i(t)$) and transfers the information $x_i(t)$ by network. Nevertheless, the neighbor agent j can only receive information in sampling time.

Remark 6. Note that hybrid multi-agent system (1) presents a unified viewpoint for both the discrete-time multi-agent system and the continuous-time multi-agent system. In other words, if $m = 0$, hybrid multi-agent system (1) becomes a discrete-time multi-agent system. And if $m = n$, hybrid multi-agent system (1) becomes a continuous-time multi-agent system.

4. Simulations

In this section, we provided two examples to demonstrate the effectiveness of our theoretical results.

Suppose that there are 6 agents. The continuous-time dynamic agents and the discrete-time dynamic agents are denoted by 1–3 and 4–6, respectively. The dynamics of the agents are described in (1). The communication graph $\mathcal{G}$ is shown in Fig. 1 with 0–1 weights. It can be noted that $\mathcal{G}$ has a directed spanning tree and $\max_{i\in\mathcal{I}_6}\{d_{ii}\} = 2$. Let $x(0) = [-13, 14, 3, -9, 20, 6]^T$ and $v(0) = [-7, 10, -2, 6, 15, -12]^T$.

Example 1. Let the sampling period $h = 0.3$ and the feedback gain $k_1 = 3.8$. It is easy to calculate that $2\sqrt{\max_{i\in\mathcal{I}_n}\{d_{ii}\}} < k_1 < \frac{\sqrt{5}-1}{h}$ holds. By using consensus protocol (4), the state trajectories of all the agents are shown in Fig. 2, which is consistent with Theorem 1.

Example 2. Let the sampling period $h = 0.35$ and the feedback gain $k_1 = 3.5$. It is easy to calculate that $2\sqrt{\max_{i\in\mathcal{I}_n}\{d_{ii}\}} < k_1 < \frac{\sqrt{5}-1}{h}$ holds. By using consensus protocol (13), the state trajectories of all the agents are shown in Fig. 3, which is consistent with Theorem 2.

5. Conclusions

In this paper, we studied the second-order consensus of hybrid multi-agent system which is composed of continuous-time and discrete-time dynamic agents. Two effective consensus protocols were presented. First, we assumed that all agents update their strategies in the sampling time with absolute velocity information. Then, we assumed that each continuous-time agent can observe its own state in real time. When $2\sqrt{\max_{i\in\mathcal{I}_n}\{d_{ii}\}} < k_1 < \frac{\sqrt{5}-1}{h}$, we proved that the hybrid multi-agent system reaches the second-order consensus if and only if the communication graph has a directed spanning tree. In the future, we may consider the second-order consensus of hybrid multi-agent systems with only relative velocity information and time-delays, etc. ■

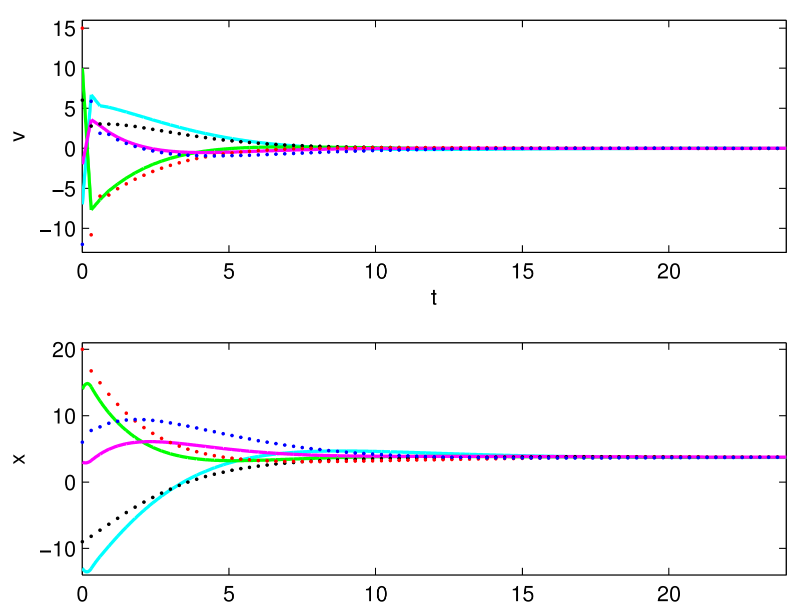

Fig. 2. The state trajectories of all agents with consensus protocol (4) and communication graph $\mathcal{G}$.

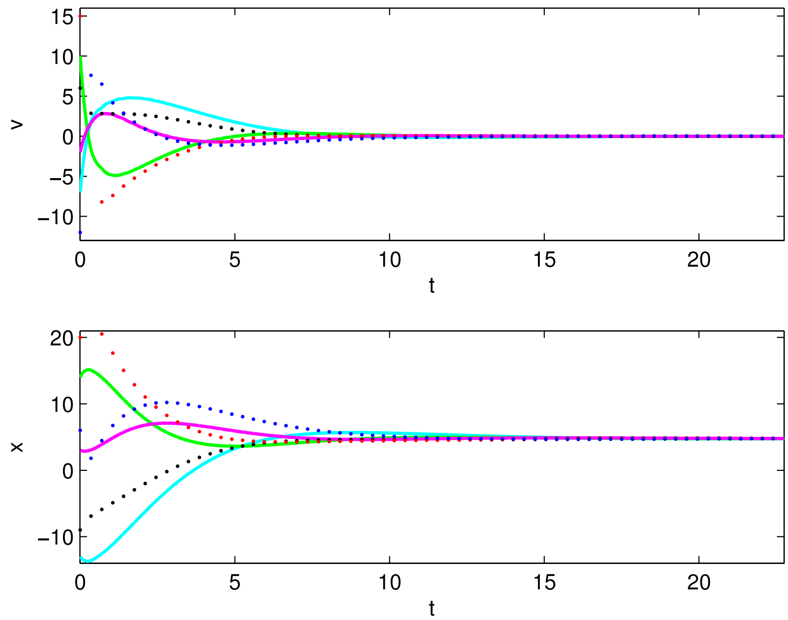

Fig. 3. The state trajectories of all agents with consensus protocol (13) and communication graph $\mathscr{G}$.

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