IET Control Theory & Applications

Research Article

Quantised consensus of heterogeneous multi-agent systems

ISSN 1751-8644
Received on 23rd October 2014
Revised on 19th May 2015
Accepted on 1st August 2015
doi: 10.1049/iet-cta.2014.1135
www.ietdl.org

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Abstract: In this study, the consensus problem of heterogeneous multi-agent systems with quantised interactions is investigated. Distributed control laws are proposed for heterogeneous multi-agent systems by using uniform and logarithmic quantisers, respectively. Based on graph theory and non-smooth analysis, it is shown that the heterogeneous multi-agent system can reach consensus with logarithmic quantisers. For uniform quantisers, some convergence results are also derived. Simulation examples are presented to illustrate the effectiveness of the theoretical results.

1 Introduction

In recent years, distributed cooperative control of multi-agent systems has attracted considerable attentions of researchers from various disciplines due to its wide applications, including cooperative control of multiple robots, formation control of unmanned aerial vehicles, target tracking of sensor networks and so on. The consensus problem, which is a fundamental problem in cooperative control, aims at designing distributed algorithm to make a group of agents reach an agreement upon some quantities of interest asymptotically or in a finite time. Specifically, Jadbabaie *et al.* [1] provided a theoretical explanation for the consensus behaviour of the Vicsek model [2] based on graph theory. A theoretical framework for the consensus problem of continuous-time multi-agent systems was presented in [3]. Ren and Beard [4] extended the results in [1, 3] to the case of directed graphs with dynamically changing interaction topologies. With the development of issue, lots of works have been done for the consensus problem of multi-agent systems under different contexts by virtue of matrix theory, graph theory, frequency-domain analysis method, Lyapunov method and so on [5–9].

In the existing literatures on the consensus problem, the precise information of neighbours is often needed to implement the control input. However, digital communication channel is widely exploited to realise the information exchange among agents in reality. Due to finite memories capacity and limited communication channels in practical applications, the quantisation effects have to be considered in consensus problems. As a result, there have been many results for the quantised consensus problem of discrete-time multi-agent systems [10–15]. In [12], dynamic coding/decoding digital channels with finite-level uniform symmetric quantisers were employed. By a symmetry error-compensation mechanism, quantised consensus of first-order discrete-time multi-agent systems was achieved under undirected switching networks. Li *et al.* [14] extended the results in [12] to the case with directed switching communication graphs. Based on the notion of input-to-output stability theorem, the authors showed that consensus can be achieved under unidirectional and unbalanced communication networks. For continuous-time multi-agent systems, the quantised consensus problem has attracted more and more attention [16–22]. Dimarogonas and Johansson [16] studied the consensus problem of first-order multi-agent systems under the distributed consensus protocol using the quantised values of the relative states in the case of a tree topology. Based on non-smooth analysis, some convergence results were established in [17] for both first-order and second-order multi-agent systems with quantised information and the results for first-order multi-agent systems were derived under undirected connected graph, which was less conservative than that in [16]. By constructing a novel Lyapunov function, Liu *et al.* [18] improved the results in [17] for second-order multi-agent systems and proved that the consensus can been achieved for any quantiser accuracy under undirected connected graph. Zhu *et al.* [19] considered the quantised consensus problem of multi-agent systems with non-linear dynamics under continuous-time and impulsive control laws, respectively. In [20], the authors dealt with the adaptive coordinated tracking problem for continuous-time first-order multi-agent systems in the presence of quantised information under switching undirected and fixed directed communication graphs, respectively. Recently, the quantised consensus problem of continuous-time multi-agent systems via sampled-data also began to attract much attention. Sampled-data consensus of second-order multi-agent systems was studied in [21], where the consensus protocols were based on the relative quantised states measurements. Wu and Wang [22] considered sampled-data consensus of multi-agent systems with quantised relative states measurements.

However, most work on the consensus problem considered the case where agents have the same dynamics. Owing to various restrictions or the common goals with mixed agents, the dynamics of the agents coupled with each others are really different [23]. Therefore, it is more practical to study the consensus problem of heterogeneous multi-agent systems. The output consensus problem for heterogeneous uncertain linear multi-agent systems was investigated in [24]. By using Lyapunov method, we proposed the heterogeneous multi-agent system composed of first-order and second-order agents and discussed the consensus problem under undirected network in [23]. Zheng and Wang [25] improved the results in [23] to directed network. Based on homogeneous method, the finite-time consensus problem of heterogeneous multi-agent systems was also considered in [26]. Inspired by the work above, we investigate the consensus problem of heterogeneous multi-agent systems involving quantised information. Due to the heterogeneous feature and the discontinuity on the right-hand side of the control input, it is difficult to analyse the quantised consensus of heterogeneous multi-agent systems. We use the tool of from non-smooth analysis to discuss the quantised consensus problem of heterogeneous multi-agent systems under undirected connected and leader-following networks, respectively. We prove that when logarithmic quantisers are used, the heterogeneous multi-agent system can reach consensus asymptotically under the given distributed protocols. We also show that when uniform quantisers are chosen, due to the constraint of uniform quantisation, the differences on their positions just converge to a bounded set asymptotically.

This paper is organised as follows. In Section 2, we provide some definitions and results in graph theory and non-smooth analysis. The consensus problem of heterogeneous multi-agent systems with logarithmic and uniform quantisers are, respectively, discussed in Section 3. In Section 4, the simulation results are also given to show the effectiveness of the obtained results. Section 5 is a brief conclusion.

Notation: Throughout this paper, we let R be the set of real number and Z be the set of integers. $\mathcal{I}_m = \{1, 2, \ldots, m\}$, $\mathcal{I}_n/\mathcal{I}_m = \{m+1, m+2, \ldots, n\}$. The superscript 'T' represents the transpose. Let $B(x, \delta)$ be the open ball of radius δ centred at x, $\mathfrak{B}(R^d)$ be the collection of all subsets of R^d, $\mu(S)$ be the Lebesgue measure of set S, co be the convex hull and $\overline{\text{co}}$ be the convex closure. For a set S, $S \le 0 (S < 0)$ means that $v \le 0 (v < 0)$ for all $v \in S$. For $a \in R$, $\lfloor a \rfloor$ denotes the greatest integer that is less than or equal to a.

2 Preliminaries

In this section, we first present some basic concepts and results in graph theory and non-smooth analysis used in the sequel. For more detailed, see [27–31].

2.1 Graph theory

Let $\mathcal{G}(\mathcal{A}) = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted undirected graph of order n, with a set of vertices $\mathcal{V} = \{s_1, s_2, \ldots, s_n\}$, a set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = (a_{ij}) \in R^{n\times n}$. If $(s_i, s_j) \in \mathcal{E}$, then vertices s_i and s_j can exchange information, namely, they are adjacent. The adjacency matrix $\mathcal{A}$ is a symmetric matrix with adjacency element $a_{ij} > 0$ if s_i, s_j are adjacent, and $a_{ij} = 0$ otherwise. A path from s_i to s_j is a sequence of distinct vertices starting with s_i and ending with s_j such that any two consecutive vertices are adjacent. For $s_i = s_j$, the path is a cycle. An undirected graph $\mathcal{G}$ is called connected if between any two distinct vertices, there is a path between them. A connected graph is a tree if it contains no cycles. An orientation on $\mathcal{G}$ is the assignment of an arbitrary direction to each edge to make it have a head and tail. We make use of $|\mathcal{V}| \times |\mathcal{E}|$ the incidence matrix B for an arbitrary oriented graph. The columns of B are then indexed by the edge set, and the ith row entry takes the value '1' if it is the head of the corresponding edge, '-1' if it is the tail, and zero otherwise. The weighting matrix W is a $|\mathcal{E}| \times |\mathcal{E}|$ diagonal matrix and the ith entry on the diagonal is the adjacency element associated with corresponding edge. For an n-agent system, an agent is called a leader if it only sends the information to other agents and cannot receive any information, that is, when agent n is the leader, $a_{ni} = 0$, $\forall i \in \mathcal{I}_n$ and $\exists j \in \mathcal{I}_{n-1}$, $a_{jn} > 0$.

2.2 Non-smooth analysis

Consider the vector differential equation given by

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where $x \in R^m, f : R^m \to R^m$ is measurable and locally essentially bounded. A Filippov solution of (1) on $[t_0, t_1]$ is defined to be an absolutely continuous function $x : [t_0, t_1] \to R^m$ such that $\dot{x} \in K[f](x)$, where the Filippov set-valued map

$$K[f](x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \overline{\text{co}}\{f(B(x, \delta))\backslash S\}.$$

The Filippov set-valued map obeys the following lemma.

Lemma 1 [29]: i. Assume that $f, g : R^m \to R^n$ are locally bounded, then

$$K[f+g](x) \subseteq K[f](x) + K[g](x).$$

ii. Let $f : R^m \to R^n$ be continuous, then

$$K[f](x) = \{f(x)\}.$$

Let Ω_V be the set of measure zero where the gradient of V with respect to x is not defined. The generalised gradient of V at x is defined by $\partial V(x) \triangleq \text{co}\{\lim_{i\to\infty} \nabla V(x_i) | x_i \to x, x_i \notin \Omega_V\}$. In addition, the set-valued Lie derivative of V with respect to f at x is defined as $\dot{\tilde{V}}(x) \triangleq \bigcap_{\xi\in\partial V(x)} \xi^{\rm T} K[f](x)$. We can use $\dot{\tilde{V}}$ to study the evolution of V along the Filippov solution of (1), which is guaranteed by the following lemma.

Lemma 2 [31]: Let $x(\cdot)$ be a Filippov solution to (1) on an interval containing t, and $V : R^m \to R$ be a Lipschitz and regular function. Then, $V(x(t))$ is absolutely continuous, $(\mathrm{d}/\mathrm{d}t)V(x(t))$ exists almost everywhere and

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \in \dot{\tilde{V}}(x), \quad \text{for} \quad a.e.t \ge 0.$$

Lemma 3 [31] (LaSalle's invariance principle): Let Ω be a compact set such that every Filippov solution to the autonomous system $\dot{x} = f(x), x(0) = x(t_0)$, starting in Ω is unique and remains in Ω for all $t \ge t_0$. Let $V : \Omega \to R$ be a time independent regular function such that $v \le 0$ for all $v \in \dot{\tilde{V}}$ (if $v \in \dot{\tilde{V}}$ is the empty set then this is trivially satisfied). Let $S = \{x \in \Omega | 0 \in \dot{\tilde{V}}\}$. Then every trajectory in Ω converges to the largest invariant set, M, in the closure of S.

3 Main results

Consider a heterogeneous multi-agent system composed of first-order and second-order integrator agents. Without loss of generality, we assume the first m agents are second-order integrator agents while the rest $n - m$ $(n > m)$ agents are first-order integrator agents. The dynamics of agent is described as below

$$\begin{cases} \dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = u_i(t), & i \in \mathcal{I}_m, \\ \dot{x}_i(t) = u_i(t), & i \in \mathcal{I}_n/\mathcal{I}_m, \end{cases} \tag{2}$$

where $x_i, v_i \in R$ are the position and velocity states of agent i, respectively, and $u_i \in R$ is the input of agent i. All results in this paper still hold for $x_i, v_i, u_i \in R^m$ by using the Kronecker product operations.

Definition 1 [23]: The multi-agent system (2) is said to reach consensus asymptotically if for any initial conditions, we have $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0, i, \boldsymbol{j} \in \mathcal{I}_n$ and $\lim_{t\to\infty} |v_i(t) - v_j(t)| = 0, i, \boldsymbol{j} \in \mathcal{I}_m$.

In this paper, we consider two types of quantisers [18]. A uniform quantiser is a map $q_{\rm u} : R \to R$ such that

$$q_{\rm u}(x) = \delta\left(\left\lfloor \frac{x}{\delta} \right\rfloor + \frac{1}{2}\right) \tag{3}$$

where δ is a positive number.

A logarithmic quantiser is a map $q_l : R \to R$ such that

$$q_l(x) = \begin{cases} e^{q_{\rm u}(\ln x)}, & x > 0, \\ 0, & x = 0, \\ -e^{q_{\rm u}(\ln(-x))}, & x < 0. \end{cases} \tag{4}$$

Remark 1: From the definitions above, we know the uniform and logarithmic quantisers have a countable number of levels. A uniform quantiser has equally spaced quantisation levels. Due to the constraint of uniform quantisation, multi-agent systems usually cannot reach exact consensus, but only finite quantisation levels are needed if the quantised variable is bounded. Compared with uniform quantiser, the logarithmic quantiser is capable of adjusting the size of the quantisation step according to the input value. It is possible for multi-agent systems to reach consensus with logarithmic quantisers in certain conditions. However, countable quantisation levels are needed even if the quantised variable is bounded.

We present the protocol for the heterogeneous multi-agent system as follows:

$$u_i(t) = \begin{cases} -\sum_{j=1}^{n} a_{ij} q(x_i - x_j) - p_1 q(v_i), & i \in \mathcal{I}_m, \\ -p_2 \sum_{j=1}^{n} a_{ij} q(x_i - x_j), & i \in \mathcal{I}_n / \mathcal{I}_m, \end{cases} \tag{5}$$

where $q(\bullet)$ is a uniform quantiser or logarithmic qiantiser, $\mathcal{A} = (a_{ij})_{n \times n}$ is the weighted adjacency matrix of the communication graph and $p_1, p_2 > 0$ are control gains.

Owing to the discontinuity of the quantised signals, the solutions in this paper are understood in Filippov sense. A Filippov solution of (2) under protocol (5) is defined as an absolutely continuous solution of the differential inclusion

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) \in K[u_i] \subseteq -K\left[\sum_{j=1}^{n} a_{ij} q(x_i - x_j)\right] - p_1 K[q(v_i)], & i \in \mathcal{I}_m, \end{cases} \tag{6}$$

and

$$\dot{x}_i(t) \in K[u_i] = -p_2 K\left[\sum_{j=1}^{n} a_{ij} q(x_i - x_j)\right], \quad i \in \mathcal{I}_n / \mathcal{I}_m. \tag{7}$$

In the case of uniform quantiser, the Filippov set-valued map for $q_u(x)$ is given as $K[q_u(x)] = q_u(x)$, when $x \neq k\delta, k \in Z$; $K[q_u(x)] = [k\delta - (1/2)\delta, k\delta + (1/2)\delta]$, otherwise. In the case of logarithmic quantiser, for $x \geq 0$, the Filippov set-valued map for $q_l(x)$ is given as $K[q_l(x)] = q_l(x)$, when $x \neq e^{k\delta}, k \in Z$; $K[q_l(x)] = [e^{k\delta - (1/2)\delta}, e^{k\delta + (1/2)\delta}]$, otherwise. Moreover, $K[q_l(-x)] = -K[q_l(x)]$. Note that $aK[q(a)] \geq 0, \forall a \in R$ and the equality holds if and only if $a = 0$.

3.1 Consensus with logarithmic quantisation

First, we consider the consensus problem in the case that the logarithmic quantised information are available and show that the consensus can be achieved.

Theorem 1: Suppose that $\mathcal{G}(\mathcal{A})$ is undirected and connected. Then the heterogeneous multi-agent system (2) with protocol (5) can reach consensus under logarithmic quantisers.

Proof: Let e and v, respectively, denote the column vector formed by all $x_i - x_j \triangleq e_{ij}, (i,j) \in \mathcal{E}$ and all $v_i, i \in \mathcal{I}_m$. Take a Lyapunov function for (6) and (7) as

$$\begin{aligned} V_1(e, v) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \int_0^{e_{ij}} q_l(s)\,\mathrm{d}s + \frac{1}{2}\sum_{i=1}^{m} v_i^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \int_0^{x_i - x_j} q_l(s)\,\mathrm{d}s + \frac{1}{2}\sum_{i=1}^{m} v_i^2. \end{aligned}$$

Because $q_l(t)$ is non-smooth, the time derivative of $V_1(e,v)$ is not defined at some time instants. The function $V_1(e,v)$ is regular because the integral term $\int_0^x q_l(s)\,\mathrm{d}s$ is regular everywhere [18, Lemma 6]. Let $(\partial V_1)_i$ denotes the generalised gradient of V_1 with respect to x_i. Using definition of generalised gradient, one obtains

$$(\partial V_1)_i = K\left[\sum_{j=1}^{n} a_{ij} q_l(x_i - x_j)\right] \triangleq [a_i, b_i] \quad \text{and} \quad \frac{\partial V_1}{\partial v_i} = v_i.$$

Then, $\partial V_1 = [(\partial V_1)_1, (\partial V_1)_2, \ldots, (\partial V_1)_n, v_1, v_2, \ldots, v_m]^{\mathrm{T}}$. Thus (see equation at the bottom of the page)

Note that for arbitrary region $[a_i, b_i]$, there always exists $\xi_i' \in [a_i, b_i]$ such that $-\xi_i'[a_i, b_i] \leq 0$, where the equality holds if and only if $0 \in [a_i, b_i]$. Thus, one has

$$\begin{aligned} \dot{\tilde{V}}_1 \subseteq & \bigcap_{\xi_i \in [a_i, b_i]} \sum_{i=1}^{m} v_i[\xi_i - b_i, \xi_i - a_i] - p_2 \sum_{i=m+1}^{n} \xi_i'[a_i, b_i] \\ & - p_1 \sum_{i=1}^{m} v_i K[q_l(v_i)]. \end{aligned}$$

Hence $\bigcap_{\xi_i \in [a_i, b_i]} [\xi_i - b_i, \xi_i - a_i] = \{0\}$ and $-v_i K[q_l(v_i)] \leq 0$, we get $\dot{\tilde{V}}_1 \leq 0$.

From Lemma 2, we have $\frac{\mathrm{d}}{\mathrm{d}t} V_1(e,v) \leq 0$. It follows that $V_1(e,v) \leq V_1(e(0), v(0))$, which implies that $e(t)$ and $v(t)$ are bounded. Thus, we can apply the non-smooth version of the Lasalle's invariance principle. Define $\Omega \triangleq \{(e,v) \mid V_1(e,v) \leq V_1(e(0), v(0))\}$ and $S \triangleq \{(e,v) \mid 0 \in \dot{\tilde{V}}_1\}$. The solutions converge to the largest weakly invariant set contained

$$\begin{aligned} \dot{\tilde{V}}_1 &= \bigcap_{\xi \in \partial V_1} \xi^{\mathrm{T}} [K[\dot{x}_1], K[\dot{x}_2], \ldots, K[\dot{x}_n], K[\dot{v}_1], K[\dot{v}_2], \ldots, K[\dot{v}_m]]^{\mathrm{T}} \\ &= \bigcap_{\xi_i \in (\partial V_1)_i} \left(\sum_{i=1}^{n} \xi_i K[\dot{x}_i] + \sum_{i=1}^{m} v_i K[\dot{v}_i] \right) \\ &\subseteq \bigcap_{\xi_i \in (\partial V_1)_i} \left(\sum_{i=1}^{m} \xi_i v_i - p_2 \sum_{i=m+1}^{n} \xi_i K\left[\sum_{j=1}^{n} a_{ij} q_l(x_i - x_j)\right] - \sum_{i=1}^{m} v_i \left(K\left[\sum_{j=1}^{n} a_{ij} q_l(x_i - x_j)\right] + p_1 K[q_l(v_i)] \right) \right) \\ &= \bigcap_{\xi_i \in [a_i, b_i]} \left(\sum_{i=1}^{m} v_i[\xi_i - b_i, \xi_i - a_i] - p_2 \sum_{i=m+1}^{n} \xi_i[a_i, b_i] \right) - p_1 \sum_{i=1}^{m} v_i K[q_l(v_i)]. \end{aligned}$$

in $\Omega \cap S$. From $0 \in \dot{\tilde{V}}_1$, we have $0 \in -p_2 \sum_{i=m+1}^{n} \xi_i'[a_i, b_i] - p_1 \sum_{i=1}^{m} v_i K[q_l(v_i)]$. Note that $0 \in v_i K[q_l(v_i)]$ if and only if $v_i = 0$ and $0 \in \xi_i'[a_i, b_i]$ if and only if $0 \in [a_i, b_i]$. Thus, we get that $v_i = 0, i \in \mathcal{I}_m$, and $0 \in [a_i, b_i] = K\left[\sum_{j=1}^{n} a_{ij} q_l(x_i - x_j)\right] \subseteq \sum_{j=1}^{n} a_{ij} K[q_l(x_i - x_j)]$, $i \in \mathcal{I}_n/\mathcal{I}_m$. From $v_i = 0, i \in \mathcal{I}_m$, we have $0 \in \sum_{j=1}^{n} a_{ij} K[q_l(x_i - x_j)], i \in \mathcal{I}_m$ according to (6). Thus, the solutions on the invariant set must satisfy the differential inclusion $0 \in \sum_{j=1}^{n} a_{ij} K[q_l(x_i - x_j)], i \in \mathcal{I}_n$. Then, it can be obtained that $0 \in \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij} K[q_l(x_i - x_j)]$. Note that

$$\begin{aligned}
0 &\in \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i K[q_l(x_i - x_j)] \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i K[q_l(x_i - x_j)] + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ji} x_j K[q_l(x_j - x_i)] \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i K[q_l(x_i - x_j)] - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_j K[q_l(x_i - x_j)] \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} (x_i - x_j) K[q_l(x_i - x_j)].
\end{aligned}$$

Since the communication graph is connected, one has $x_i = x_j$ for all $i, j \in \mathcal{I}_n$. Therefore, according to Lemma 3, we get that $\lim_{t\to\infty} |x_i - x_j| = 0$ for $i, j \in \mathcal{I}_n$ and $\lim_{t\to\infty} v_i(t) = 0$ for $i \in \mathcal{I}_m$. □

We consider the consensus problem for a group of agents without any leader in Theorem 1. However, one leader might exist in multi-agent systems in some practical applications. Therefore, we consider consensus problem in the leader-following network.

Theorem 2: Suppose that the heterogeneous multi-agent system (2) has a leader and $n-1$ followers, and the communication topology among the followers is undirected and connected. Then, the heterogeneous multi-agent system (2) with protocol (5) can reach consensus under logarithmic quantisers if the leader is a first-order integrator agent.

Proof: Without loss of generality, we assume agent n is the leader. To analyse the leader-following consensus problem, we denote the state error between agent i and the leader as $\hat{x}_i(t) = x_i(t) - x_n(t)$, $i \in \mathcal{I}_{n-1}$. Then, the dynamics of $\hat{x}_i$ can be described as (see (8)) and

$$\dot{\hat{x}}_i(t) \in K[u_i] = -p_2 K\left[\sum_{j=1}^{n-1} a_{ij} q(\hat{x}_i - \hat{x}_j) + a_{in} q(\hat{x}_i)\right], \quad i \in \mathcal{I}_n/\mathcal{I}_m. \tag{9}$$

Consider the Lyapunov function

$$\begin{aligned}
V_2(\hat{x}, v) = {}& \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} a_{ij} \int_0^{\hat{x}_i - \hat{x}_j} q_l(s)\,\mathrm{d}s \\
&+ \sum_{i=1}^{n-1} a_{in} \int_0^{\hat{x}_i} q_l(s)\,\mathrm{d}s + \frac{1}{2}\sum_{i=1}^{m} v_i^2,
\end{aligned}$$

where $\hat{x} = [\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{n-1}]^{\mathrm{T}}$ and $v = [v_1, v_2, \ldots, v_m]^{\mathrm{T}}$.

Let $(\partial V_2)_i$ denotes the generalised gradient of V_2 with respect to $\hat{x}_i$ and $(\partial V_2)_i = K[\sum_{j=1}^{n-1} a_{ij} q_l(\hat{x}_i - \hat{x}_j) + a_{in} q_l(\hat{x}_i)] \triangleq [a_i, b_i]$. Then, $\partial V_2 = [(\partial V_2)_1, (\partial V_2)_2, \ldots, (\partial V_2)_{n-1}, v_1, v_2, \ldots, v_m]^{\mathrm{T}}$. Thus (see equation at the bottom of the page)

Since $(\mathrm{d}/\mathrm{d}t)V_2(\hat{x}, v) \le 0$ from Lemma 2, we obtain $V_2(\hat{x}, v) \le V_2(\hat{x}(0), v(0))$ which implies that $\hat{x}(t)$ and $v(t)$ are bounded when the communication topology among followers are connected. Define $\Omega \triangleq \{(\hat{x}, v) \mid V_2(\hat{x}, v) \le V_2(\hat{x}(0), v(0)\}$ and $S \triangleq \{(\hat{x}, v) \mid 0 \in \dot{\tilde{V}}_2\}$. According to Lemma 3, the trajectory of (2) under protocol (4) converges to the largest invariant set contained in $\Omega \cap S$. Note that $0 \in \dot{\tilde{V}}_2$ implies that $v_i = 0, i \in \mathcal{I}_m$ and $0 \in [a_i, b_i] \subseteq \sum_{j=1}^{n-1} a_{ij} K[q_l(\hat{x}_i - \hat{x}_j)] + a_{in} K[q_l(\hat{x}_i)]$, $i \in \mathcal{I}_n/\mathcal{I}_m$, which in turn implies that the solutions on the invariant set must satisfy the differential inclusion $0 \in \sum_{j=1}^{n-1} a_{ij} K[q_l(\hat{x}_i - \hat{x}_j)] + a_{in} K[q_l(\hat{x}_i)]$, $i \in \mathcal{I}_{n-1}$. Note that

$$\begin{aligned}
0 &\in \sum_{i=1}^{n-1} \hat{x}_i \left(\sum_{j=1}^{n-1} a_{ij} K[q_l(\hat{x}_i - \hat{x}_j)] + a_{in} K[q_l(\hat{x}_i)]\right) \\
&= \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} a_{ij} (\hat{x}_i - \hat{x}_j) K[q_l(\hat{x}_i - \hat{x}_j)] + \sum_{i=1}^{n-1} a_{in} \hat{x}_i K[q_l(\hat{x}_i)].
\end{aligned}$$

$$\begin{cases}
\dot{\hat{x}}_i(t) = v_i(t), \\
\dot{v}_i(t) \in K[u_i] \subseteq -K\left[\sum_{j=1}^{n-1} a_{ij} q(\hat{x}_i - \hat{x}_j) + a_{in} q(\hat{x}_i)\right] - p_1 K[q](v_i), \quad i \in \mathcal{I}_m,
\end{cases} \tag{8}$$

$$\begin{aligned}
\dot{\tilde{V}}_2 &= \bigcap_{\xi \in \partial V_2} \xi^{\mathrm{T}} \left[K[\dot{\hat{x}}_1], K[\dot{\hat{x}}_2], \ldots, K[\dot{\hat{x}}_{n-1}], K[\dot{v}_1], K[\dot{v}_2], \ldots, K[\dot{v}_m]\right]^{\mathrm{T}} \\
&= \bigcap_{\xi_i \in (\partial V_2)_i} \left(\sum_{i=1}^{n-1} \xi_i K[\dot{\hat{x}}_i] + \sum_{i=1}^{m} v_i K[\dot{v}_i]\right) \\
&\subseteq \bigcap_{\xi_i \in [a_i, b_i]} \left(\sum_{i=1}^{m} \xi_i v_i - p_2 \sum_{i=m+1}^{n-1} \xi_i K\left[\sum_{j=1}^{n-1} a_{ij} q_l(\hat{x}_i - \hat{x}_j) + a_{in} q_l(\hat{x}_i)\right] - \sum_{i=1}^{m} v_i \left(K\left[\sum_{j=1}^{n-1} a_{ij} q_l(\hat{x}_i - \hat{x}_j) + a_{in} q_l(\hat{x}_i)\right] + p_1 K[q_l(v_i)]\right)\right) \\
&= \bigcap_{\xi_i \in [a_i, b_i]} \left(\sum_{i=1}^{m} v_i \left(\xi_i - [a_i, b_i]\right) - p_2 \sum_{i=m+1}^{n-1} \xi_i [a_i, b_i] - p_1 \sum_{i=1}^{m} v_i K[q_l(v_i)]\right) \\
&\le 0.
\end{aligned}$$

Since the communication topology among followers is connected, one has $\hat{x}_i = \hat{x}_j \ \forall i,j \in \mathcal{I}_{n-1}$, and $\hat{x}_i = 0$ if $a_{in} > 0$. Hence, $\hat{x}_i = 0$ for all $i \in \mathcal{I}_{n-1}$. According to Lemma 3, we conclude that $\lim_{t\to\infty} |x_i - x_n| = 0$ for $i \in \mathcal{I}_{n-1}$ and $\lim_{t\to\infty} v_i(t) = 0$ for $i \in \mathcal{I}_m$. □

If agents can get quantised relative velocity measurements, we use $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$ to, respectively, describe the position and velocity information exchange relationship between agents and $\mathcal{A} = (a_{ij})_{n\times n}$, $\mathcal{B} = (b_{ij})_{m\times m}$ are weighted adjacency matrices. The protocol can also be designed as

$$u_i(t) = \begin{cases} -\sum_{j=1}^{n} a_{ij} q(x_i - x_j) - p_1 \sum_{j=1}^{m} b_{ij} q(v_i - v_j), & i \in \mathcal{I}_m, \\ -p_2 \sum_{j=1}^{n} a_{ij} q(x_i - x_j), & i \in \mathcal{I}_n / \mathcal{I}_m \end{cases} \tag{10}$$

where $p_1, p_2 > 0$ are control gains.

Corollary 1: Suppose that $\mathcal{G}(\mathcal{A})$ is a tree and $\mathcal{G}(\mathcal{B})$ is an undirect connected graph. The heterogeneous multi-agent system (2) with protocol (10) can reach consensus under logarithmic quantisers.

Proof: Similar to the proof of Theorem 1, differentiating V_1, gives

$$\begin{aligned}
\dot{\tilde{V}}_1 &= \bigcap_{\xi\in\partial V_1} \xi^{\mathrm{T}} [K[\dot{x}_1], K[\dot{x}_2], \ldots, K[\dot{x}_n], K[\dot{v}_1], K[\dot{v}_2], \ldots, K[\dot{v}_m]]^{\mathrm{T}} \\
&= \bigcap_{\xi_i\in(\partial V_1)_i} \left(\sum_{i=1}^{n} \xi_i K[\dot{x}_i] + \sum_{i=1}^{m} v_i K[\dot{v}_i] \right) \\
&\subseteq \bigcap_{\xi_i\in(\partial V_1)_i} \left(\sum_{i=1}^{m} \xi_i v_i - p_2 \sum_{i=m+1}^{n} \xi_i K\left[\sum_{j=1}^{n} a_{ij} q_l(x_i - x_j) \right] \right. \\
&\quad \left. - \sum_{i=1}^{m} v_i \left(K\left[\sum_{j=1}^{n} a_{ij} q_l(x_i - x_j) \right] + p_1 \sum_{j=1}^{m} b_{ij} K[q_l(v_i - v_j)] \right) \right) \\
&= \bigcap_{\xi_i\in[a_i,b_i]} \left(\sum_{i=1}^{m} v_i [\xi_i - b_i, \xi_i - a_i] - p_2 \sum_{i=m+1}^{n} \xi_i [a_i, b_i] \right) \\
&\quad - \frac{1}{2} p_1 \sum_{i=1}^{m} \sum_{j=1}^{m} b_{ij} (v_i - v_j) K[q_l(v_i - v_j)] \\
&\le 0,
\end{aligned}$$

where $K\left[\sum_{j=1}^{n} a_{ij} q_l(x_i - x_j)\right] \triangleq [a_i, b_i]$.

Note that from $0 \in \dot{\tilde{V}}_1$, we get $v_i = v_j, \forall i,j \in \mathcal{I}_m$ since $\mathcal{G}(\mathcal{B})$ is connected and $0 \in [a_i, b_i] \subseteq \sum_{j=1}^{n} a_{ij} K[q_l(x_i - x_j)]$, $i \in \mathcal{I}_n/\mathcal{I}_m$. Let $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}$ and $y = [v_1, v_2, \ldots, v_m, x_{m+1}, x_{m+2}, \ldots, x_n]^{\mathrm{T}}$. Thus, the solutions on the invariant set must satisfy the differential inclusion $\dot{y} \in -BWK[q_l(\bar{x})] \triangleq HK[q_l(\bar{x})]$, where B and W are, respectively, the incidence matrix and weighting matrix of $\mathcal{G}(\mathcal{A})$ and $\bar{x} = B^{\mathrm{T}} x$. Since $\mathcal{G}(\mathcal{A})$ is a tree, we have $\mathrm{rank}(H) = \mathrm{rank}(B) = n - 1$. Let $\bar{y} = [v_2 - v_1, \ldots, v_m - v_1, x_{m+1}, x_{m+2}, \ldots, x_n]^{\mathrm{T}}$, one has $\dot{\bar{y}} \in QK[q_l(\bar{x})]$, where Q can be obtained by omitting the first row of the matrix which is obtained using the elementary line row transformation of matrix H. Thus, $\mathrm{rank}(Q) = n - 1$, that is, the matrix Q is non-singular. Note that from $v_i = v_j$, $\forall i,j \in \mathcal{I}_m$, and $0 \subseteq \sum_{j=1}^{n} a_{ij} K[q_l(x_i - x_j)]$, $i \in \mathcal{I}_n/\mathcal{I}_m$, we can obtain that $\mathbf{0} \in QK[q_l(\bar{x})]$. Since Q is non-singular, we obtain $\mathbf{0} \in K[q_l(\bar{x})]$, which implies that $\bar{x} = \mathbf{0}$. Thus, one has $x_i = x_j$ for all $i,j \in \mathcal{I}_n$ from the connected of $\mathcal{G}(\mathcal{A})$. Then $v_i = 0, i \in \mathcal{I}_m$ from (10). Thus, it follows Lemma 3 that the trajectory of (2) under protocol (10) converges to $\lim_{t\to\infty} |x_i - x_j| = 0$ for $i,j \in \mathcal{I}_n$ and $\lim_{t\to\infty} v_i(t) = 0$ for $i \in \mathcal{I}_m$. □

3.2 Consensus with uniform quantisation

Compared with logarithmic quantisers, uniform quantisers are easy to realise and computationally cheap. Next, we discuss the consensus problem in the case that uniform quantised information is available. The following lemma is needed for the derivation of the results in this section.

Lemma 4 [18]: (a) For $x \in R$ and $|x| \le \delta$, it holds that $x(K[q_{\mathrm{u}}(x)] + K[q_{\mathrm{u}}(0)]) \subseteq [0, \infty)$. (b) For $x \in R$ and $|x| > \delta$, it holds that $x(K[q_{\mathrm{u}}(x)] + K[q_{\mathrm{u}}(0)]) \subseteq (0, \infty)$.

The following results provide the convergence results for the heterogeneous multi-agent with uniform quantisers.

Theorem 3: Suppose that $\mathcal{G}(\mathcal{A})$ is an undirected connected graph and $p_1 \le \min_{i\in\mathcal{I}_m} \sum_{j=1}^{n} a_{ij}$. For the heterogeneous multi-agent system (2) with protocol (5), the positions of all the agents asymptotically converge to the set $\{x \mid |x_i - x_j| < \delta, (i,j) \in \mathcal{E}\}$ and the velocities of all the second-order agents asymptotically converge to 0 under uniform quantisers.

Proof: Take a Lyapunov function for (6) and (7) as

$$V_1(e, v) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \int_0^{x_i - x_j} q_{\mathrm{u}}(s)\,\mathrm{d}s + \frac{1}{2} \sum_{i=1}^{m} v_i^2.$$

Similar to the proof in Theorem 1, we have (see equation at the bottom of the page)

where $K\left[\sum_{j=1}^{n} a_{ij} q_u(x_i - x_j)\right] \triangleq [a_i, b_i]$.

Note that $0 \in \dot{\tilde{V}}_1$ implies that $v_i = 0, i \in \mathcal{I}_m$ and $0 \in [a_i, b_i] = K[\sum_{j=1}^{n} a_{ij} q_{\mathrm{u}}(x_i - x_j)] \subseteq \sum_{j=1}^{n} a_{ij} K[q_{\mathrm{u}}(x_i - x_j)] \subseteq \sum_{j=1}^{n} a_{ij} K[q_{\mathrm{u}}(x_i - x_j)] + \sum_{j=1}^{n} a_{ij} K[q_{\mathrm{u}}(0)]$, $i \in \mathcal{I}_n/\mathcal{I}_m$. From $v_i = 0$, one has $0 \in \sum_{j=1}^{n} a_{ij} K[q_{\mathrm{u}}(x_i - x_j)] + p_1 K[q_{\mathrm{u}}(0)] \subseteq \sum_{j=1}^{n} a_{ij} K[q_{\mathrm{u}}(x_i - x_j)] + \sum_{j=1}^{n} a_{ij} K[q_{\mathrm{u}}(0)]$, $i \in \mathcal{I}_m$, according to (6). Then, we obtain the solutions on the invariant set must satisfy the differential inclusion $0 \in \sum_{i=1}^{n} x_i (\sum_{j=1}^{n} a_{ij} K[q_{\mathrm{u}}(x_i - x_j)] + \sum_{j=1}^{n} a_{ij} K[q_{\mathrm{u}}(0)])$,

$$\begin{aligned}
\dot{\tilde{V}}_1 &= \bigcap_{\xi\in\partial V_1} \xi^{\mathrm{T}} [K[\dot{x}_1], K[\dot{x}_2], \ldots, K[\dot{x}_n], K[\dot{v}_1], K[\dot{v}_2], \ldots, K[\dot{v}_m]] \\
&\subseteq \bigcap_{\xi_i\in[a_i,b_i]} \left(\sum_{i=1}^{m} v_i ([\xi_i - b_i, \xi_i - a_i]) - p_2 \sum_{i=m+1}^{n} \xi_i [a_i, b_i] - p_1 \sum_{i=1}^{m} v_i K[q_{\mathrm{u}}(v_i)] \right) \\
&\le 0,
\end{aligned}$$

$i \in \mathcal{I}_n$. Since the communication graph is connected, one has

$$0 \in \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i\left(K[q_u(x_i-x_j)]+K[q_u(0)]\right)$$
$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_i-x_j)\left(K[q_u(x_i-x_j)]+K[q_u(0)]\right).$$

Applying Lemma 4, one can easily get that $|x_i - x_j| < \delta$, $\forall (i,j) \in \mathcal{E}$. Thus, Theorem 3 is proved. □

Theorem 4: Suppose that the heterogeneous multi-agent system (2) has a leader and $n-1$ followers and the communication topology among the followers is undirected and connected. For the heterogeneous multi-agent system (2) with protocol (5), if the leader is a first-order integrator agent and $p_1 \le \min_{i\in\mathcal{I}_m}\sum_{j=1}^{n} a_{ij}$, the positions of all the agents asymptotically converge to the set $\{x \mid |x_i - x_j| < \delta, (i,j) \in \mathcal{E}\}$ and the velocities of second-order agents asymptotically converge to 0 under uniform quantisers.

Proof: Similar to the proof of Theorem 2, consider the Lyapunov function

$$V_2(\hat{x},v)=\frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} a_{ij}\int_0^{\hat{x}_i-\hat{x}_j} q_u(s)\,\mathrm{d}s$$
$$+\sum_{i=1}^{n-1} a_{in}\int_0^{\hat{x}_i} q_u(s)\,\mathrm{d}s+\frac{1}{2}\sum_{i=1}^{m} v_i^2,$$

and we get $\dot{\tilde{V}}_2 \le 0$ and $\dot{\tilde{V}}_2 = 0$ implies that $0 \in \sum_{j=1}^{n-1} a_{ij}K[q_u(\hat{x}_i - \hat{x}_j)] + a_{in}K[q_u(\hat{x}_i)] + \sum_{j=1}^{n} a_{ij}K[q_u(0)]$, $i \in \mathcal{I}_{n-1}$. Thus, the solutions on the invariant set must satisfy the differential inclusion

$$0 \in \sum_{i=1}^{n-1}\hat{x}_i\left(\sum_{j=1}^{n-1} a_{ij}K[q_u(\hat{x}_i-\hat{x}_j)]+a_{in}K[q_u(\hat{x}_i)]+\sum_{j=1}^{n} a_{ij}K[q_u(0)]\right)$$
$$=\frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} a_{ij}(\hat{x}_i-\hat{x}_j)\left(K[q_u(\hat{x}_i-\hat{x}_j)]+K[q_u(0)]\right)$$
$$+\sum_{i=1}^{n-1} a_{in}\hat{x}_i\left(K[q_u(\hat{x}_i)]+K[q_u(0)]\right).$$

From Lemma 4 and the connectivity of the communication topology among followers, we obtain that $|x_i - x_j| < \delta$, $\forall (i,j) \in \mathcal{E}$. Theorem 4 is proved. □

4 Simulations

In this section, two examples are provided to demonstrate the effectiveness of the theoretical results.

Example 1: Consider a heterogeneous multi-agent system with the interaction graph $\mathcal{G}$ shown in Fig. 1 in which the second-order integrator agents are denoted as **1 − 3**, the first-order integrator agents are denoted as **4 − 5**, and each edge weight is assumed to be 1. It is easy to find that $\mathcal{G}$ is connected. The initial positions of the agents are set as $[x_1, x_2, x_3, x_4, x_5] = [12, 6, -8, -14, 8]$, the initial velocities of the second-order integrator agents are set as $[v_1, v_2, v_3] = [2, 3, -2]$. We assume that control gains $p_1 = 2, p_2 = 1$. It is easy to verify that p_1 satisfies the condition in Theorem 3. Figs. 2 and 3 show the simulation results of the heterogeneous multi-agent system (2) with protocol (5) under logarithmic and uniform quantisers with parameter $\delta = 2$, respectively. For the case of without quantiser, the simulation result is shown in Fig. 4. Comparing with the

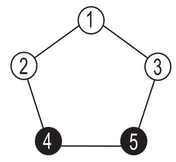

Fig. 1 *Undirected graph*

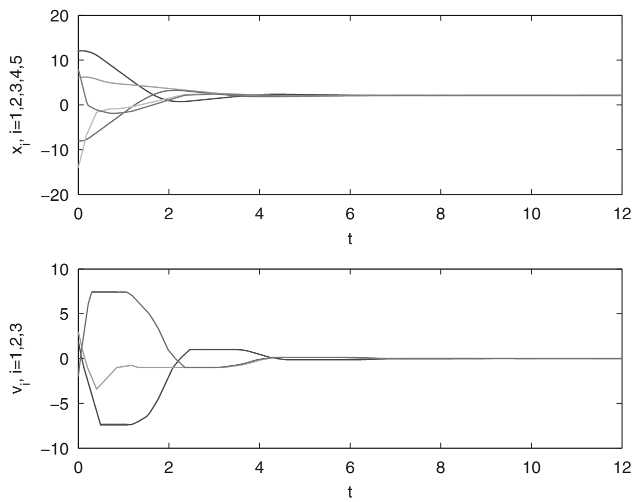

Fig. 2 *State trajectories of the agents under logarithmic quantisers*

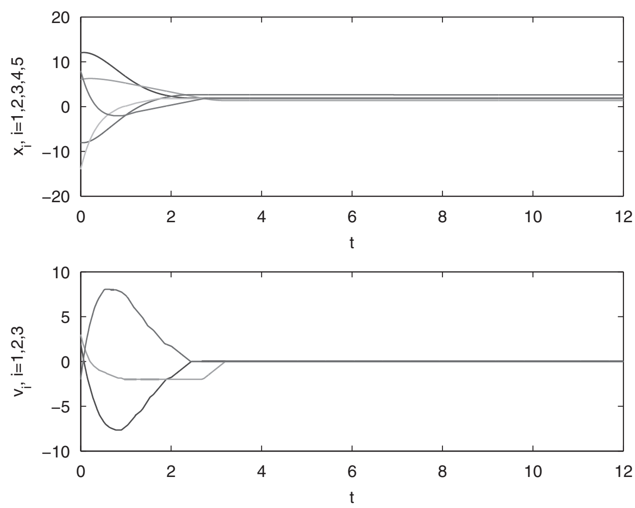

Fig. 3 *State trajectories of the agents under unform quantisers*

resuts without quantisation, we can see that when uniform quantisers are used, the positions of all the agents just convege to a bound set and when logarithmic quantisers are used, the consensus is achieved, but the quantisation process leads to slower convegence speed. Furthermore, we look the effect of δ on the convergence performance of the system. Fig. 5 shows the position trajectories of the agents under logarithmic quantisers with different values of $\boldsymbol{\delta}$. We can see that the bigger $\boldsymbol{\delta}$ is, the more time it takes for the quantised system to converge.

Example 2: When the communication topology is depicted in Fig. 6, where the first-order agent 5 is the leader, Figs. 7 and 8 show the simulation results of the heterogeneous multi-agent system (2) with protocol (5) under with the logarithmic and uniform quantisers, respectively. From Fig. 7, we can see that the leader-following consensus is achieved when logarithmic quantisers are used, which

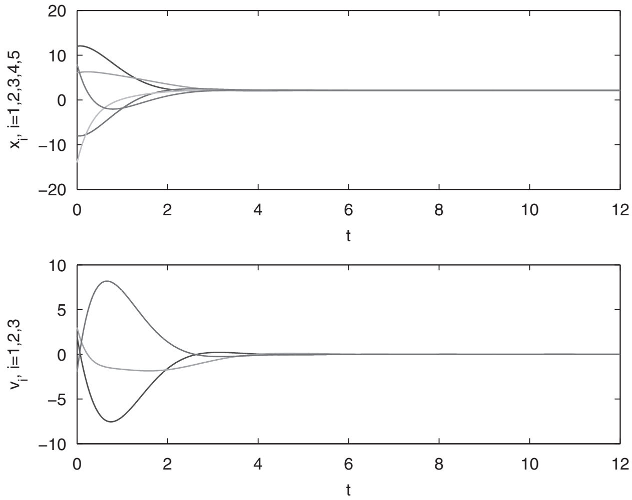

Fig. 4 *State trajectories of the agents without quantisers*

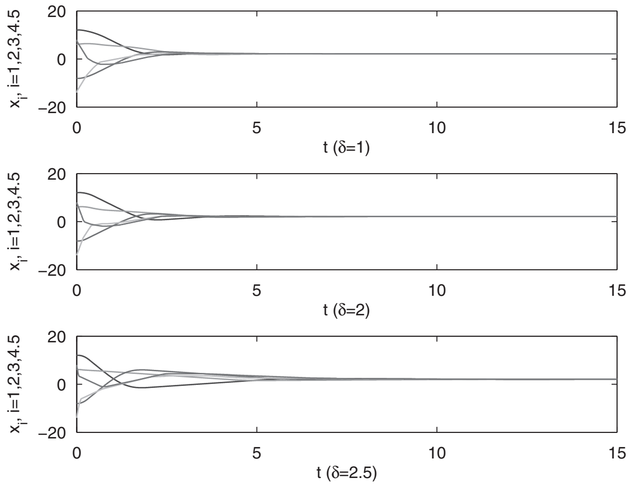

Fig. 5 *Position trajectories of the agents under logarithmic quantisers*

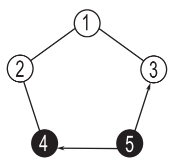

Fig. 6 *Leader-following network*

verify the correctness of Theorem 2. The simulation results shown in Fig. 8 confirm the results of Theorem 4. It can be seen that the position of all the agents convege to a bound set under unform quantisers.

5 Conclusion

By using graph theory and non-smooth analysis, the consensus of heterogeneous multi-agent systems has been investigated in the presence of quantised information under undirected connected and leader-following communication networks. It has been shown that when logarithmic quantisers are used, the heterogeneous multi-agent system can reach consensus asymptotically for any quantiser accuracy. For the case of uniform quantisers, it has been proved that the position differences of the agents converge to a bounded set and the velocities converge to zero. The future work will focus

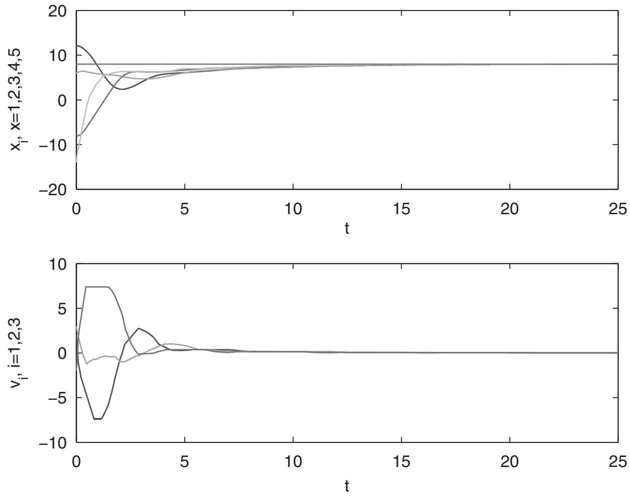

Fig. 7 *State trajectories of the agents under logarithmic quantisers*

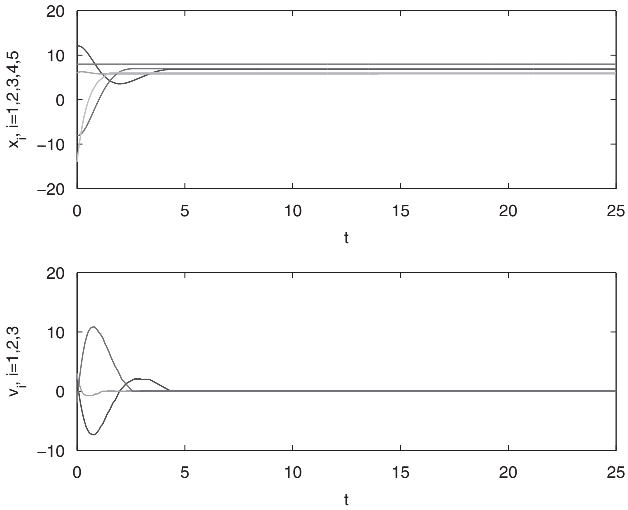

Fig. 8 *State trajectories of the agents under uniform quantisers*

on the quantised consensus of heterogeneous multi-agent systems under directed and switching networks.

6 Acknowledgments

This work was supported by NSFC (grant nos. 61375120 and 61304160) and the Fundamental Research Funds for the Central Universities (grant nos. JB140406 and NSIY211416).

7 References

1 Jadbabaie, A., Lin, J., Morse, A.S.: ‘Coordination of groups of mobile autonomous agents using nearest neighbor rules’, *IEEE Trans. Autom. Control*, 2003, **48**, (6), pp. 988–1001
2 Vicsek, T., Czirok, A., Jacob, E.B. *et al.*: ‘Novel type of phase transition in a system of self-driven particles’, *Phys. Rev. Lett.*, 1995, **75**, (6), pp. 1226–1229
3 Olfati-Saber, R., Murray, R.: ‘Consensus problems in networks of agents with switching topology and time-delays’, *IEEE Trans. Autom. Control*, 2004, **49**, (9), pp. 1520–1533
4 Ren, W., Beard, R.W.: ‘Consensus seeking in multiagent systems under dynamically changing interaction topologies’, *IEEE Trans. Autom. Control*, 2005, **50**, (5), pp. 655–661
5 Xiao, F., Wang, L.: ‘Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays’, *IEEE Trans. Autom. Control*, 2008, **53**, (8), pp. 1804–1816
6 Gao, Y., Wang, L.: ‘Sampled-data based consensus of continuous-time multi-agent systems with time-varying topology’, *IEEE Trans. Autom. Control*, 2011, **56**, (5), pp. 1226–1231

7 Guan, Z., Wu, Y., Feng, G.: 'Consensus analysis based on impulsive systems in multiagent networks', *IEEE Trans. Circuits Syst. I*, 2012, **59**, (1), pp. 170–178
8 Tang, Y., Gao, H., Zou, W. *et al.*: 'Distributed synchronization in networks of agent systems with nonlinearities and random switchings', *IEEE Trans. Cybern.*, 2013, **43**, (1), pp. 358–370
9 Wang, Q., Gao, H., Alsaadi, F. *et al.*: 'An overview of consensus problems in constrained multi-agent coordination', *Syst. Sci. Control Eng.*, 2014, **2**, (1), pp. 275–284
10 Kashyap, A., Basar, T., Srikant, R.: 'Quantized consensus', *Automatica*, 2007, **43**, (7), pp. 1192–1203
11 Carli, R., Fagnani, F., Frasca, P. *et al.*: 'Gossip consensus algorithms via quantized communication', *Automatica*, 2010, **46**, (1), pp. 70–80
12 Li, T., Xie, L.: 'Distributed consensus over digital networks with limited bandwidth and time-varying topologies', *Automatica*, 2011, **47**, (9), pp. 2006–2015
13 Li, T., Xie, L.: 'Distributed coordination of multi-agent systems with quantized-observer based encoding-decoding', *IEEE Trans. Autom. Control*, 2012, **57**, (12), pp. 3023–3037
14 Li, D., Liu, Q., Wang, X. *et al.*: 'Quantized consensus over directed networks with switching topologies', *Syst. Control Lett.*, 2014, **65**, pp. 13–22
15 Guan, Z.H., Meng, C., Liao, R.Q. *et al.*: 'Consensus of second-order multi- agent dynamic systems with quantized data', *Phys. Lett. A*, 2012, **376**, pp. 387–393
16 Dimarogonas, D.V., Johansson, K.H.: 'Stability analysis for multi-agent systems using the incidence matrix: quantized communication and formation control', *Automatica*, 2010, **46**, (4), pp. 695–700
17 Guo, M., Dimarogonas, D.V.: 'Consensus with quantized relative state measurements', *Automatica*, 2013, **49**, (8), pp. 2531–2537
18 Liu, H., Cao, M., Persis, C.D.: 'Quantization effects on synchronized motion of teams of mobile agents with second-order dynamics', *Syst. Control Lett.*, 2012, **61**, (12), pp. 1157–1167
19 Zhu, Y., Zheng, Y., Wang, L.: 'Quantized consensus of multi-agent systems with nonlinear dynamics', *Int. J. Syst. Sci.*, 2015, **46**, (11), pp. 2061–2071
20 Fu, J., Wang, J.: 'Adaptive coordinated tracking of multi-agent systems with quantized information', *Syst. Control Lett.*, 2014, **74**, pp. 115–125
21 Chen, W., Li, X., Jiao, L.C.: 'Quantized consensus of second-order continuous-time multi-agent systems with a directed topology via sampled data', *Automatica*, 2013, **49**, pp. 2236–2242
22 Wu, Y., Wang, L.: 'Sample-data consensus for multi-agent systems with quantized communication', *Int. J. Control*, 2015, **88**, (2), pp. 413–428
23 Zheng, Y., Zhu, Y., Wang, L.: 'Consensus of heterogeneous multi-agent systems', *IET Control Theory Appl.*, 2011, **16**, (5), pp. 1881–1888
24 Kim, H., Shim, H., Seo, J.H.: 'Output consensus of heterogeneous uncertain linear multi-agent system', *IEEE Trans. Autom. Control*, 2011, **50**, (11), pp. 1689–1711
25 Zheng, Y., Wang, L.: 'Distributed consensus of heterogeneous multi-agent systems with fixed and switching topologies', *Int. J. Control*, 2012, **85**, (12), pp. 1967–1976
26 Zheng, Y., Wang, L.: 'Finite-time consensus of heterogeneous multi-agent systems with and without velocity measurements', *Syst. Control Lett.*, 2012, **61**, (8), pp. 871–878
27 Godsil, C., Royal, G.: 'Algebraic graph theory' (Springer-Verlag, New York, 2001)
28 Clarke, F.: 'Optimization and nonsmooth analysis' (Wiley, New York, 1983)
29 Paden, B., Sastry, S.: 'A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators', *IEEE Trans. Circuits Syst.*, 1987, **34**, (1), pp. 73–82
30 Filippov, A.F.: 'Differential equations with discontinuous righthand sides' (Kluwer Academic, Amsterdam, 1988)
31 Shevitz, D., Paden, B.: 'Lyapunov stability theory of nonsmooth systems', *IEEE Trans. Autom. Control*, 1994, **39**, (9), pp. 1910–1914