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LQR-based optimal topology of leader-following consensus

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SUMMARY

In this paper, we consider the optimal topology for leader-following consensus problem of continuous-time and discrete-time multi-agent systems based on linear quadratic regulator theory. For the first-order multi-agent systems, we propose a quadratic cost function, which is independent of the interaction graph, and find that the optimal topology is a star topology. For the second-order multi-agent systems, a quadratic cost function is also constructed, whereas the optimal topology for second-order leader-following consensus problem is an unevenly weighted star topology. The universality of these findings means that if each follower is connected with the leader, the information exchange between followers is unnecessary and insufficient. Simulation examples are provided to illustrate the effectiveness of the theoretical results. Copyright © 2014 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Recently, more attention has been paid to distributed control of multi-agent systems (MASs), because of its wide applications, such as formation control in unmanned aerial vehicles [1], flocking in biology [2] and rendezvous problem of mobile autonomous robots [3]. Consequently, many research topics about MASs have arisen, to name but a few, consensus problem [4–7], containment control problem [8–11], controllability analysis [12] and optimal control [13–15].

Of these topics, consensus problem of MASs is one of the fundamental problems that have been attracting increasing attentions. Consensus means that a group of agents converge to a common value using information of neighbors. It was originally investigated by Vicsek *et al.* [4] and was explained by graph theory [5]. Some relaxed necessary and sufficient conditions were obtained to solve consensus problem of first-order MASs in [6]. Asynchronous consensus was further considered by Xiao and Wang [16] for MASs with switching topologies. Because second-order dynamics exist widely in nature, MASs with second-order dynamics of agents had aroused wide concern in recent years. Sufficient conditions for second-order consensus were gotten under various circumstances [17, 18]. Consensus problem based on sampled-data control was also investigated in [19, 20]. In [21, 22], the authors further studied consensus of heterogeneous MASs composed of first-order and second-order dynamics of agents.

Omnipresent phenomena in natural systems indicate that there exists at least one leader in MASs, for instance, the navigation aircraft in a fight formation of unmanned aerial vehicles and the queen

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ant in an ant colony. This triggers researchers great interest in leader-following consensus. Some sufficient conditions for solving leader-following consensus were brought up for MASs with an active leader [23] and for high-order MASs [24]. Two different leader roles, the power leader and the knowledge leader, were proposed in [25]. As an extension of leader-following consensus, containment control problem of MASs with multiple leaders was also studied in [8, 9]. Then, some necessary and sufficient conditions were arisen in [10, 11] for different cases.

Along with the consensus, optimality problem is also an active topic for MASs. Relevant studies have corroboratively confirmed that consensus can be achieved under different interaction graphs. A natural question emerges as follows: which is the optimal one under a given performance index? Generally speaking, two classes of performance indexes were considered. One is the convergence rate, and the other is cost function. The convergence rate of the consensus algorithm can be quantified by algebraic connectivity of graph, the second smallest eigenvalue of Laplacian matrix [26]. Moreover, the algebraic connectivity of graph can be increased by designing the weights based on semi-definite convex programming [27]. The fastest convergence graph was proposed in [28] for solving the average consensus problem. Besides, convergence rate can also be measured by convergence time. Hence, achieving consensus within finite-time was considered in [29–31]. Although the convergence rate is a conventional metric of optimality, it usually takes great or even unaffordable control effort to achieve this rate under the limited resources constraint. Cost function, then, is another performance index in need of consideration to get a balance between the state convergence and the control effort. A quadratic cost function was applied in the structured optimal control problem of formation [32]. Differential games were employed to obtain the Nash-bargaining consensus protocol in [33]. Ji *et al.* [13] considered the optimal control of quasi-equilibrium for leader-based formation. Cao and Ren [14] proposed linear quadratic regulator-based (LQR-based) optimal consensus protocol for leaderless MASs and proved that the optimal solution corresponds to a complete (directed) graph. A nonquadratic cost function was constructed for MASs with obstacles in [34]. Hengster-Movric and Lewis [15] used the notion of inverse optimality and partial stability to obtain a sufficient condition for solving a global optimal LQR control problem of MASs.

Motivated by these studies in optimal control of MASs, we consider the optimal control problem for leader-following MASs. Different from some optimal consensus problems in the aforementioned literature [14, 15, 33], we aim to design a global optimal interaction topology for leader-following consensus of MASs without any graphical structure constraints. Basically, leader-following consensus is achieved when the state of each follower $x_i(t)$ converges to the state of the leader $x_{N+1}(t)$. To balance the global consensus error $\sum_{i=1}^{N}\left[q_i(x_i(t)-x_{N+1}(t))^2\right]$ and the control effort $\sum_{i=1}^{N}\left[r_i u_i^2(t)\right]$, we propose a linear quadratic cost function independent of the interaction topology and minimize it by seeking the optimal interaction graph among all consensusable topologies. The main contributions of this paper are twofold. Firstly, we propose a linear quadratic cost function for first-order discrete-time and continuous-time MASs. Based on LQR theory, we prove that the optimal control always corresponds to a star topology. Secondly, for second-order cases, a quadratic cost function is also constructed. However, the optimal topology is an unevenly weighted star topology in which each follower, with different weights in the position and the velocity graph, is connected to no other than the leader. It should be mentioned that these results provide some theoretic explanations for some cooperative games and dictatorship.

The remainder of this paper is organized as follows. In Section 2, we introduce the graph theory and the infinite-time LQR theory. In Section 3, we formulate and solve the optimal control problem of first-order leader-following consensus. In Section 4, the optimal control problem of second-order leader-following consensus is also considered. Simulation examples are provided in Section 5 to illustrate the effectiveness of the theoretical results. Finally, a short conclusion is given in Section 6.

Throughout this paper, the following notations will be used: let $\mathbb{N}$ and $\mathbb{R}$ be the set of nonnegative integral numbers and the set of real numbers, respectively. $\mathbb{R}^{n\times m}$ is the set of $n\times m$ real matrices. Denote by $\mathbf{1}_n$ (or $\mathbf{0}_n$) the column vector with all entries equal to one (or all zeros). I_n denotes an $n-$dimensional identity matrix. $\mathrm{diag}\{A_1,\ \ldots,\ A_n\}$ is a block-diagonal matrix with matrices A_i, $i=1,\ldots,n$, on its diagonal. For a matrix $A\in\mathbb{R}^{n\times n}$, if B satisfies $B^2=A$, then $B=A^{\frac{1}{2}}$ (or $B=\sqrt{A}$). $A\otimes B$ denotes the Kronecker product of matrices A and B. $\mathcal{I}_n=\{1,\ \ldots,\ n\}$ is an index set.

2. PRELIMINARY

2.1. Graph theory

In order to solve distributed coordination problems, graph theory is employed to describe information exchange between agents.

Let $G = \{V, E, A\}$ be a weighted directed graph consisting of a vertex set $V = \{1, 2, \ldots, n\}$, an edge set $E = \{(i, j) \in V \times V\}$ and an adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$. An edge $(j, i) \in E$ implies that the agent i can access the information of the agent j. The adjacency matrix A of G is defined such that for all $i \in \mathcal{I}_n$, $a_{ii} = 0$ and for all $i \neq j$, $(j, i) \in E \Leftrightarrow a_{ij} > 0$, while $a_{ij} = 0$ otherwise. The neighbor set of the agent i is $\mathcal{N}_i = \{j : a_{ij} > 0\}$. The degree matrix $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix with $d_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij}$ and the Laplacian matrix $L = D - A$. A special directed graph is star topology where there is a vertex, labeled i_0, with no neighbors and for all $j \in \mathcal{I}_n \setminus \{i_0\}$, $\mathcal{N}_j = \{i_0\}$.

In MASs, an agent is a leader if it has no neighbors and is a follower if it has at least one neighbor [9]. In this paper, we assume that there are one leader, labeled by $N + 1$, and N followers, labeled by $i \in \mathcal{I}_N$. For the first-order MASs, there is only position information exchange between agents. Hence, we employ a weighted directed graph $G(A)$ with Laplacian matrix L^A to model it. However, for the second-order MASs, both the position and the velocity information are exchanged. Therefore, we still employ $G(A)$ to describe the position information exchange. And besides, we use another weighted directed graph $G(W)$ with Laplacian matrix L^W to model the velocity information exchange. And according to the definition of leader, L^A and L^W can be partitioned as

$$L^A = \begin{pmatrix} L_{ff}^A & -\mathbf{b} \\ \mathbf{0}_N^T & 0 \end{pmatrix} \text{ and } L^W = \begin{pmatrix} L_{ff}^W & -\mathbf{d} \\ \mathbf{0}_N^T & 0 \end{pmatrix},$$

where $\mathbf{b} = [b_1, \ldots, b_N]^T$ and $\mathbf{d} = [d_1, \ldots, d_N]^T$ are N-dimension column vectors. For all $i \in \mathcal{I}_N$, if $b_i > 0$, then there is a directed edge from the leader $N + 1$ to the follower i in the graph $G(A)$ with the weight b_i and as a result, the follower i can access the position information of the leader. Likewise, for all $i \in \mathcal{I}_N$, if $d_i > 0$, then the follower i can have an access to the velocity information of the leader with the weight d_i. For a second-order MAS, if $G(A)$ and $G(W)$ have the same structure (i.e. the (i, j)th entry of L^A equals to zero if and only if the (i, j)th entry of L^W is zero) and different weights, then the system is said to have an unevenly weighted interaction graph.

2.2. Infinite-time linear quadratic regulator theory

In this subsection, some basic concepts and properties in infinite-time LQR theory, which will be used in this paper, are given. For more details, please refer to [35].

Consider a linear system as follows:

$$\dot{X}(t) = GX(t) + HU(t), \tag{1}$$

where $X(t) \in \mathbb{R}^n$, $U(t) \in \mathbb{R}^m$, $G \in \mathbb{R}^{n \times n}$ and $H \in \mathbb{R}^{n \times m}$. Let $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$ be symmetric and be nonnegative and positive definite, respectively. Define the cost function

$$J(U(.), X(0)) = \int_0^\infty \left[X^T(t)QX(t) + U^T(t)RU(t)\right] dt.$$

The physical interpretation of the cost function is the 'total cost' of system (1) during the process from the initial state $X(0)$ to the equilibrium state. The task of the optimal control problem is to find an optimal control $U^*(t) = -K^* X(t)$ minimizing $J(U(.), X(0))$, where K^* is called the optimal feedback gain matrix [35]. In other words, optimal control can drive the state close to the equilibrium state without any excessive cost. Two LQR optimal control lemmas are given, which can be found in [35].

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Lemma 1
Suppose that system (1) is completely controllable or completely stabilizable. Then, the algebraic Riccati equation (ARE)

$$G^T P + PG + Q - PHR^{-1}H^T P = 0$$

has a unique positive-definite solution. Furthermore, the optimal control $U^*(t) = -R^{-1}H^T PX(t)$ minimizes the cost function $J(U(\cdot), X(0))$ and drives the system to achieve asymptotic stability.

Lemma 2
Suppose the discrete-time linear system

$$X(k+1) = G_T X(k) + H_T U(k),\ k \in \mathbb{N}$$

is completely controllable or completely stabilizable, where $X(k) \in \mathbb{R}^n$, $U(k) \in \mathbb{R}^m$, $G_T \in \mathbb{R}^{n\times n}$ and $H_T \in \mathbb{R}^{n\times m}$. Define the cost function

$$J(U(.), X(0)) = \sum_{k=0}^{\infty} \left[X^T(k)QX(k) + U^T(k)RU(k)\right],$$

where $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are symmetric and are nonnegative and positive definite, respectively. Then, the following discrete-time ARE

$$G_T^T \left(P - PH_T\left(R + H_T^T PH_T\right)^{-1} H_T^T P\right) G_T + Q = P$$

has a unique positive-definite solution. Moreover, the optimal control $U^*(k) = -(R+H_T^T PH_T)^{-1} H_T^T PG_T X(k)$ minimizes the cost function $J(U(\cdot), X(0))$ and steers the system to be asymptotically stable.

3. LQR-BASED OPTIMAL TOPOLOGY OF FIRST-ORDER LEADER-FOLLOWING CONSENSUS

3.1. Continuous-time case

Consider first-order leader-following MASs as follows:

$$\begin{aligned} \dot{x}_i(t) &= u_i(t),\ \ i \in \mathcal{I}_N, \\ \dot{x}_{N+1}(t) &= 0, \end{aligned} \tag{2}$$

where $x_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ are the position and the control input of the follower i $(i \in \mathcal{I}_N)$, respectively. And $x_{N+1}(t) \in \mathbb{R}$ is the position of the leader. Let $X(t) = (x_1(t), \ldots, x_N(t))^T$ and $U(t) = (u_1(t), \ldots, u_N(t))^T$. Leader-following consensus means to design $u_i(t)$ $(i \in \mathcal{I}_N)$ based on neighbor information of the follower i to drive the state converging to that of the leader. The precise definition is as follows:

Definition 1 (First-order leader-following consensus)
System (2) is said to achieve leader-following consensus if, for each follower $i \in \mathcal{I}_N$, there exists a local state feedback $u_i(t)$ based on $\mathcal{N}_i^A$, such that the corresponding states of the closed-loop system satisfy

$$\lim_{t\to\infty} |x_i(t) - x_{N+1}(t)| = 0,\ \ i \in \mathcal{I}_N, \tag{3}$$

for any initial state $X(0) \in \mathbb{R}^N$.

A common linear-consensus protocol is considered in [23, 24] as follows:

$$u_i(t) = \sum_{j=1}^{N} a_{ij}[x_j(t) - x_i(t)] - b_i\left[x_i(t) - x_{N+1}(t)\right],\ \ i \in \mathcal{I}_N, \tag{4}$$

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where a_{ij} is the $(i,j)th$ entry of the adjacency matrix A of $G(A)$ and b_i is the ith entry of $\mathbf{b}$. The matrix form of (2) with the protocol (4) is

$$\begin{aligned} \dot{X}(t) &= U(t) \\ U(t) &= -L_{ff}^{A} X(t) + \mathbf{b} x_{N+1}(t). \end{aligned} \tag{5}$$

Then, we propose the cost function

$$J(U(t), X(0)) = \int_0^{\infty} \left\{ \sum_{i=1}^{N} \left[q_i (x_i(t) - x_{N+1}(t))^2 + r_i u_i^2(t) \right] \right\} dt,$$

where $q_i > 0$ and $r_i > 0$ are the weight of consensus error and the weight of control effort of the follower i, respectively. Therefore, the optimal control problem of the leader-following MASs is to find the optimal control $U^*(t)$ minimizing $J(U(t), X(0))$ for any initial state $X(0) \in \mathbb{R}^N$

$$\begin{aligned} &\min_{U(t)} J(U(t), X(0)) \\ &\text{subject to (5) and (3).} \end{aligned} \tag{6}$$

According to $U(t) = -L_{ff}^{A} X(t) + \mathbf{b} x_{N+1}(t)$, seeking $U^*(t)$ is equivalent to finding the optimal topology $G(A^*)$ with Laplacian Matrix L^{A^*}.

Theorem 1
For optimal control problem (6), the optimal topology is a star topology in which the follower i ($i \in \mathcal{I}_N$) is connected only to the leader with the weight $\sqrt{\dfrac{q_i}{r_i}}$.

Proof
Denote the consensus error of the follower i ($i \in \mathcal{I}_N$) by $\varepsilon_i(t) = x_i(t) - x_{N+1}(t)$ and the consensus error vector by $\varepsilon(t) = (\varepsilon_1(t), \ldots, \varepsilon_N(t))^T$. The error system of (5) is

$$\begin{aligned} \dot{\varepsilon}(t) &= U(t), \\ U(t) &= -L_{ff}^{A} \varepsilon(t). \end{aligned} \tag{7}$$

By substituting $\varepsilon(t)$ into $J(U(t), X(0))$, the optimal control problem (6) can be converted into a standard LQR problem as follows:

$$\begin{aligned} &\min_{U(t)} \int_0^{\infty} \left[\varepsilon^T(t) Q \varepsilon(t) + U^T(t) R U(t) \right] dt \\ &\text{subject to (7) and } \lim_{t \to \infty} |\varepsilon(t)| = 0, \end{aligned}$$

where $Q = \operatorname{diag}\{q_1, \ldots, q_N\}$ and $R = \operatorname{diag}\{r_1, \ldots, r_N\}$. Because the controllability matrix is $[I_N, \ldots]$, system (7) is controllable. Hence, it follows from Lemma 1 that there exists a positive-definite matrix $P \in \mathbb{R}^{N \times N}$ satisfied the following ARE:

$$P R^{-1} P = Q, \tag{8}$$

and the optimal feedback gain matrix $L_{ff}^{A^*} = R^{-1} P$ can stabilize system (7). By premultiplying R^{-1} on both sides of (8), we can easily obtain

$$L_{ff}^{A^*} = (R^{-1} Q)^{\frac{1}{2}} = \operatorname{diag}\left\{ \sqrt{\frac{q_1}{r_1}}, \ldots, \sqrt{\frac{q_N}{r_N}} \right\}.$$

Obviously, we have

$$L^{A^*} = \begin{pmatrix} L_{ff}^{A^*} & -\mathbf{b}^* \\ \mathbf{0}_N^T & 0 \end{pmatrix}$$

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and $\mathcal{N}_i^{A^*} = \{N+1\}$ for all $i \in \mathcal{I}_N$, where $\mathbf{b}^* = \left[\sqrt{\frac{q_1}{r_1}}, \ldots, \sqrt{\frac{q_N}{r_N}}\right]^T$. Thus, the optimal topology is a star topology in which each follower i $(i \in \mathcal{I}_N)$ is connected to no other than the leader with the weight $\sqrt{\frac{q_i}{r_i}}$. Finally, owing to asymptotic stability of close-loop system (7) with the feedback gain matrix $L_{ff}^{A^*}$, system (5) achieves leader-following consensus. □

Remark 1
Star topology $G(A^*)$ has N edges. In [23], the authors proved if the interaction topology $G(A)$ is connected, system (5) can achieve leader-following consensus. It is easy to prove that if $G(A)$ is connected, then the edge number is greater than or equal to N. Therefore, star topology $G(A^*)$ has the minimal edge number among all connected interaction topologies. It should be noted that star topology is not the only connected graph with the minimal edge number. For example, a tree graph has the minimal edge number too. However, $G(A^*)$ is the only one with the minimal edge number as well as the minimal cost.

Remark 2
Suppose there are $M(M > 1)$ leaders, labeled by $N+1, \ldots, N+M$. Then, the Laplacian Matrix of $G(A)$ is

$$L^A = \begin{pmatrix} L_{ff}^A & L_{fr}^A \\ \mathbf{0}_{M\times N} & \mathbf{0}_{M\times M} \end{pmatrix},$$

where $L_{ff}^A \in \mathbb{R}^{N\times N}$ and $L_{fr}^A \in \mathbb{R}^{N\times M}$. Denote $X_f(t) = [x_1(t), \ldots, x_N(t)]^T$ and $X_r(t) = [x_{N+1}(t), \ldots, x_{N+M}(t)]^T$. Then, it follows that

$$\begin{aligned} \dot{X}_f(t) &= -L_{ff}^A X_f(t) - L_{fr}^A X_r(t), \\ \dot{X}_r(t) &= \mathbf{0}_M. \end{aligned} \tag{9}$$

From [10], we have if L_{ff}^A is invertible, then $X_f(t)$ will converge to $X_f^c = -(L_{ff}^A)^{-1} L_{fr}^A X_r(0) \in co(X_r(0))$, where $co(X_r(0))$ is the convex hull spanned by the initial states of the leaders $X_r(0)$. Define a cost function as follows:

$$J(U(t), X(0)) = \int_0^\infty \left[\left(X_f(t) - X_f^c\right)^T Q \left(X_f(t) - X_f^c\right) + U(t)^T R U(t)\right] dt,$$

where $Q = \operatorname{diag}\{q_1, q_2, \ldots, q_N\}$ and $R = \operatorname{diag}\{r_1, r_2, \ldots, r_N\}$ are N-dimension positive-definite diagonal matrices. Similar to Theorem 1, we show that the optimal control problem

$$\begin{gathered} J(U^*(\cdot), X(0)) = \min_{U(\cdot)} J(U(\cdot), X(0)) \\ \text{subject to (9) and } \lim_{t\to\infty} X_f(t) = X_f^c \end{gathered}$$

can be solved by $L_{ff}^{A^*} = \operatorname{diag}\left\{\sqrt{\frac{q_1}{r_1}}, \ldots, \sqrt{\frac{q_N}{r_N}}\right\}$ and

$$L_{fr}^* = \begin{pmatrix} \mathbf{b}_1^* \\ \cdots \\ \mathbf{b}_N^* \end{pmatrix},$$

where $\mathbf{b}_i^* \in \mathbb{R}^{1\times M}$ $(i \in \mathcal{I}_N)$ is the solution of the following convex programming:

$$\begin{gathered} \min_{\mathbf{b}_i} \left\{x_i(0) + \sqrt{\frac{r_i}{q_i}} \mathbf{b}_i X_r(0)\right\}^2 \\ \text{subject to } \mathbf{b}_i \mathbf{1}_M = \sqrt{\frac{q_i}{r_i}}\,. \end{gathered} \tag{10}$$

In other words, the optimal control is associated with an interaction graph in which each follower can only access the information from leaders determined by (10).

3.2. Discrete-time case

By sampling the continuous-time MASs (2) in period $T > 0$, we can formulate first-order discrete-time MASs as

$$\begin{aligned} x_i(kT+T) &= x_i(kT) + u_i(kT)T, \ \ i \in \mathcal{I}_N \\ x_{N+1}(kT+T) &= x_{N+1}(kT), \end{aligned} \tag{11}$$

where $k \in \mathbb{N}$. The discrete-time MAS (11) is said to achieve leader-following consensus if

$$\lim_{k\to\infty} |x_i(kT) - x_{N+1}(kT)| = 0, \ \ i \in \mathcal{I}_N, \tag{12}$$

for any initial state $X(0) \in \mathbb{R}^N$. In order to guarantee (12), let the linear-consensus protocol be

$$u_i(kT) = \sum_{j=1}^{N} a_{ij}[x_j(kT) - x_i(kT)] - b_i\left[x_i(kT) - x_{N+1}(kT)\right], \ i \in \mathcal{I}_N.$$

Then, the matrix form of system (11) is

$$\begin{aligned} X(kT+T) &= X(kT) + U(kT)T, \\ U(kT) &= -L_{ff}^{A}X(kT) + \mathbf{b}x_{N+1}(kT). \end{aligned} \tag{13}$$

Similar to continuous case, we propose the following optimal control problem for system (11):

$$\begin{gathered} \min_{U(kT)} J(U(kT), X(0)) \\ \text{subject to (13) and (12),} \end{gathered} \tag{14}$$

where

$$J(U(kT), X(0)) = \sum_{k=0}^{\infty}\sum_{i=1}^{N}\left[q_i(x_i(kT) - x_{N+1}(kT))^2 + r_i u_i^2(kT)\right],$$

$q_i > 0$ and $r_i > 0$.

Theorem 2
For optimal control problem (14), the optimal topology is a star topology in which the follower i ($i \in \mathcal{I}_N$) is only connected to the leader with the weight $\frac{T}{2}\left[\sqrt{\left(\frac{q_i}{r_i}\right)^2 + \frac{4q_i}{T^2 r_i}} - \frac{q_i}{r_i}\right]$.

Proof
Denote $\varepsilon(kT) = X(kT) - \mathbf{1}_N \otimes x_{N+1}(kT)$. From system (13), we get

$$\begin{aligned} \varepsilon(kT+T) &= \varepsilon(kT) + TU(kT), \\ U(kT) &= -L_{ff}^{A}\varepsilon(kT), \ k \in \mathbb{N}. \end{aligned} \tag{15}$$

Thus, the optimal control problem (14) is converted into the following standard LQR problem:

$$\begin{gathered} \min_{U(kT)} \sum_{k=0}^{\infty}\left[\varepsilon(kT)^T Q\varepsilon(kT) + U(kT)^T RU(kT)\right] \\ \text{subject to (15) and } \lim_{k\to\infty} |\varepsilon(kT)| = 0, \end{gathered}$$

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where $Q = \operatorname{diag}\{q_1, \ldots, q_N\}$ and $R = \operatorname{diag}\{r_1, \ldots, r_N\}$. By using Lemma 2 and noticing that system (15) is controllable, we can obtain that the discrete-time ARE

$$Q = PT(R + T^2P)^{-1}TP \tag{16}$$

has a unique positive-definite matrix solution. Moreover, the optimal feedback gain matrix $L_{ff}^{A^*} = (R + T^2P)^{-1}TP$ can stabilize system (15). Next, we solve (16). Premultiplying by R^{-1} on both sides of (16) leads to

$$R^{-1}Q = R^{-1}PT\left(I_N + T^2R^{-1}P\right)^{-1}TR^{-1}P. \tag{17}$$

From the fact that $\left(I_N + T^2R^{-1}P\right)^{-1} = I_N - T^2R^{-1}P\left(I_N + T^2R^{-1}P\right)^{-1}$, we get

$$\begin{aligned} R^{-1}PT\left(I_N + T^2R^{-1}P\right)^{-1}TR^{-1}P = {} & \left[TR^{-1}P\right]^2 - T^2R^{-1}P \\ & \times \left[R^{-1}PT\left(I_N + T^2R^{-1}P\right)^{-1}TR^{-1}P\right]. \end{aligned} \tag{18}$$

Together with (17), it follows that

$$R^{-1}Q = T^2\left[R^{-1}P\right]^2 - T^2\left[R^{-1}P\right]R^{-1}Q. \tag{19}$$

Hence, we have $R^{-1}P = \frac{1}{2}\left[R^{-1}Q + \sqrt{(R^{-1}Q)^2 + \frac{4R^{-1}Q}{T^2}}\right]$. Consequently, the feedback gain matrix is

$$L_{ff}^{A^*} = \frac{T}{2}\left[\sqrt{(R^{-1}Q)^2 + \frac{4R^{-1}Q}{T^2}} - R^{-1}Q\right] = \operatorname{diag}\{b_1^*, \ldots, b_N^*\},$$

where $b_i^* = \frac{T}{2}\left[\sqrt{\left(\frac{q_i}{r_i}\right)^2 + \frac{4q_i}{T^2r_i}} - \frac{q_i}{r_i}\right]$ for $i \in \mathcal{I}_N$. Thus, we have

$$L^{A^*} = \begin{pmatrix} L_{ff}^{A^*} & -\mathbf{b}^* \\ \mathbf{0}_N^T & 0 \end{pmatrix},$$

where $\mathbf{b}^* = [b_1^*, \ldots, b_N^*]^T$ and $\mathcal{N}_i^{A^*} = \{N+1\}$ for all $i \in \mathcal{I}_N$. In other words, the optimal topology G^{A^*} is a star topology in which each follower i ($i \in \mathcal{I}_N$) just has one connection to the leader with the weight $\frac{T}{2}\left[\sqrt{\left(\frac{q_i}{r_i}\right)^2 + \frac{4q_i}{T^2r_i}} - \frac{q_i}{r_i}\right]$. Finally, because of the asymptotic stability of (15) with the optimal feedback gain matrix $L_{ff}^{A^*}$, the discrete-time MAS (13) achieves leader-following consensus. □

Remark 3

Obviously, $\frac{T}{2}\left[\sqrt{\frac{q_i^2}{r_i^2} + \frac{4q_i}{T^2r_i}} - \frac{q_i}{r_i}\right] \to \sqrt{\frac{q_i}{r_i}}$ as $T \to 0$.

4. LQR-BASED OPTIMAL TOPOLOGY OF SECOND-ORDER LEADER-FOLLOWING CONSENSUS

4.1. Continuous case

Second-order leader-following MASs can be formulated as

$$\begin{aligned} \dot{x}_i(t) = v_i(t),\ \dot{v}_i(t) = u_i(t),\ i \in \mathcal{I}_N, \\ \dot{x}_{N+1}(t) = v_{N+1}(t),\ \dot{v}_{N+1}(t) = 0, \end{aligned} \tag{20}$$

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where $x_i(t) \in \mathbb{R}$, $v_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ are the position, the velocity and the control input of the follower i, respectively. $x_{N+1}(t) \in \mathbb{R}$ and $v_{N+1}(t) \in \mathbb{R}$ are the position and the velocity of the leader, respectively. Denote $X(t) = (x_1(t), \dots, x_N(t), v_1(t), \dots, v_N(t))^T$ and $U(t) = (u_1(t), \dots, u_N(t))^T$. System (20) is said to achieve leader-following consensus if for any initial state $X(0) \in \mathbb{R}^{2N}$, there exists a local state feedback $U(t)$ where each $u_i(t)$ based on $\mathcal{N}_i^A$ and $\mathcal{N}_i^W$, such that the corresponding states of the closed-loop system satisfy

$$\lim_{t\to\infty} |x_i(t) - x_{N+1}(t)| = 0, \ \lim_{t\to\infty} |v_i(t) - v_{N+1}(t)| = 0, \ i \in \mathcal{I}_N. \tag{21}$$

In this paper, we propose a protocol as follows:

$$\begin{aligned} u_i(t) = & \sum_{j=1, j\neq i}^{N} \left\{a_{ij}\left[x_j(t) - x_i(t)\right] + w_{ij}\left[v_j(t) - v_i(t)\right]\right\} \\ & - b_i\left[x_i(t) - x_{N+1}(t)\right] - d_i\left[v_i(t) - v_{N+1}(t)\right], \end{aligned} \tag{22}$$

where a_{ij} and w_{ij} are the (i, j)th entry of adjacent matrices A and W of $G(A)$ and $G(W)$, respectively. b_i and d_i are the ith entry of $\mathbf{b}$ and $\mathbf{d}$, respectively. The matrix form of system (20) with the protocol (22) is

$$\begin{aligned} \dot{X}(t) &= GX(t) + HU(t), \\ U(t) &= -\left[L_{ff}^A, \ L_{ff}^W\right] X(t) + \mathbf{b}x_{N+1}(t) + \mathbf{d}v_{N+1}(t), \end{aligned} \tag{23}$$

where

$$G = \begin{pmatrix} 0_{N\times N} & I_N \\ 0_{N\times N} & 0_{N\times N} \end{pmatrix} \text{ and } H = \begin{pmatrix} 0_{N\times N} \\ I_N \end{pmatrix}.$$

Then, we propose the optimal control problem of (20) as follows:

$$\begin{aligned} &\min_{U(t)} J(U(t), X(0)) \\ &\text{subject to (23) and (21),} \end{aligned} \tag{24}$$

where

$$J(U(t), X(0)) = \int_0^\infty \left\{\sum_{i=1}^{N}\left[q_i\left((x_i(t) - x_{N+1}(t))^2 + (v_i(t) - v_{N+1}(t))^2\right) + r_i u_i^2(t)\right]\right\} dt,$$

$q_i > 0$ and $r_i > 0$. It follows from (23) that seeking optimal control $U^*(t)$ is equivalent to finding the optimal topology with the position graph $G(A^*)$ and the velocity graph $G(W^*)$ of associating the Laplacian Matrices L^{A^*} and L^{W^*}, respectively.

Theorem 3
For optimal control problem (24), the optimal topology is an unevenly weighted star topology. More specifically, each follower i ($i \in \mathcal{I}_N$) is only connected to the leader with the weights $\sqrt{\frac{q_i}{r_i}}$ in the position graph and $\sqrt{2\sqrt{\frac{q_i}{r_i}} + \frac{q_i}{r_i}}$ in the velocity graph.

Proof
Define the consensus error vector of system (20) by

$$\varepsilon(t) = X(t) - \left[\mathbf{1}_N^T \otimes x_{N+1}(t), \mathbf{1}_N^T \otimes v_{N+1}(t)\right]^T.$$

Then, the associated error dynamics system is

$$\begin{aligned} \dot{\varepsilon}(t) &= G\varepsilon(t) + HU(t), \\ U(t) &= -K\varepsilon(t), \end{aligned} \tag{25}$$

where $K = \left[L_{ff}^{A},\ L_{ff}^{W}\right]$. Consequently, the optimal control problem (24) is converted into a standard LQR problem as follows:

$$\begin{aligned}&\min_{U(t)} \int_0^{\infty} \left[\varepsilon^T(t)(I_2 \otimes Q)\varepsilon(t) + U^T(t)RU(T)\right] dt \\ &\text{subject to (25) and } \lim_{t\to\infty} |\varepsilon(t)| = 0\,,\end{aligned} \tag{26}$$

where $Q = \operatorname{diag}\{q_1, \ldots, q_N\}$ and $R = \operatorname{diag}\{r_1, \ldots, r_N\}$. The linear system (25) is controllable because the rank of controllability matrix

$$[H, GH, G^2H, \ldots, G^{2N}H] = \begin{pmatrix} 0_{N\times N} & I_N & \ldots \\ I_N & 0_{N\times N} & \ldots \end{pmatrix}$$

is $2N$. Therefore, according to Lemma 1, the associated ARE of (26)

$$PG + G^TP + I_2 \otimes Q - PHR^{-1}H^TP = 0 \tag{27}$$

has a unique positive-definite solution. Furthermore, the optimal control $U^*(t) = -R^{-1}H^TP\varepsilon(t)$ can drive (25) to achieve asymptotic stability. Next, we solve ARE (27). Denote P by

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix},$$

where $P_{11} \in \mathbb{R}^{N\times N}$ and $P_{22} \in \mathbb{R}^{N\times N}$ are symmetric, and $P_{12} \in \mathbb{R}^{N\times N}$. Then,

$$K^* = \left[L_{ff}^{A*},\ L_{ff}^{W*}\right] = R^{-1}H^TP = \left[R^{-1}P_{12},\ R^{-1}P_{22}\right],$$

and it follows from (27) that

$$\begin{aligned} P_{12}R^{-1}P_{22} = P_{11} &= P_{22}R^{-1}P_{12}^T, \\ Q &= P_{12}R^{-1}P_{12}^T, \\ P_{12}^T + P_{12} + Q &= P_{22}R^{-1}P_{22}. \end{aligned}$$

Supposing $P_{12} = P_{12}^T$ and premultiplying by R^{-1} on both side of the above equations gives

$$\begin{aligned} L_{ff}^{A*} &= \left(R^{-1}Q\right)^{\frac{1}{2}} = \operatorname{diag}\left\{b_1^*, \ldots, b_N^*\right\}, \\ L_{ff}^{W*} &= \left(R^{-1}(2P_{12} + Q)\right)^{\frac{1}{2}} = \operatorname{diag}\left\{d_1^*, \ldots, d_N^*\right\}, \end{aligned}$$

where $b_i^* = \sqrt{\frac{q_i}{r_i}}$ and $d_i^* = \sqrt{2\sqrt{\frac{q_i}{r_i}} + \frac{q_i}{r_i}}$. Thus, we have

$$L^{A^*} = \begin{pmatrix} L_{ff}^{A*} & -\mathbf{b}^* \\ \mathbf{0}_N^T & 0 \end{pmatrix} \text{ and } L^{W^*} = \begin{pmatrix} L_{ff}^{W*} & -\mathbf{d}^* \\ \mathbf{0}_N^T & 0 \end{pmatrix},$$

where $\mathbf{b}^* = [b_1^*, \ldots, b_N^*]^T$ and $\mathbf{d}^* = [d_1^*, \ldots, d_N^*]^T$. This implies that the optimal topology is an unevenly weighted star topology in which each follower i ($i \in \mathcal{I}_N$) is connected to no other than the leader with different weights $\sqrt{\frac{q_i}{r_i}}$ and $\sqrt{2\sqrt{\frac{q_i}{r_i}} + \frac{q_i}{r_i}}$ in the position and the velocity graph, respectively. The asymptotic stability of system (25) with feedback gain matrix K^* means that system (20) achieves leader-following consensus. □

Remark 4

Consider a general linear leader-following MAS as follows:

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i,\ i \in \mathcal{I}_N, \\ \dot{x}_{N+1} &= Ax_{N+1}, \end{aligned} \tag{28}$$

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where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Let the cost function be

$$J(U(t), X(0)) = \int_0^{\infty} \left\{ \sum_{i=1}^{N} \left[q_i \parallel x_i(t) - x_{N+1}(t) \parallel^2 + r_i \parallel u_i(t) \parallel^2 \right] \right\} dt.$$

Assume that the pair (A, B) is completely controllable. Similar to Theorem 3, we obtain that the optimal control problem

$$\begin{aligned} &J(U^*(\cdot), X(0)) = \min_{U(\cdot)} J(U(\cdot), X(0)) \\ &\text{subject to (28) and } \lim_{t\to\infty} X_i(t) = X_{N+1}(t) \end{aligned} \tag{29}$$

can be solved by $u_i^* = -r_i^{-1} B^T P_i (x_i - x_{N+1}),\ i \in \mathcal{I}_N$, where P_i is the solution of the ARE

$$P_i A + A^T P_i + q_i I_n - r_i^{-1} P_i B B^T P_i = 0.$$

Hence, u_i^* is only dependent with x_i and x_{N+1}. That is to say the optimal topology of (29) is a star topology.

4.2. Discrete-time case

By sampling the continuous-time MASs (20) in period $T > 0$, the discrete-time leader-following MASs can be define as

$$\begin{aligned} x_i(kT+T) &= x_i(kT) + v_i(kT)T, \\ v_i(kT+T) &= v_i(kT) + u_i(kT)T,\ i \in \mathcal{I}_N, \\ x_{N+1}((k+1)T) &= x_{N+1}(kT) + v_{N+1}(kT)T, \end{aligned} \tag{30}$$

where $k \in \mathbb{N}$. Denote $X(kT) = [x_1(kT), v_1(kT), \ldots, x_N(kT), v_N(kT)]^T$ and $U(kT) = [u_1(kT), \ldots, u_N(kT)]^T$. System (30) is said to achieve leader-following consensus if

$$\lim_{k\to\infty} |x_i(kT) - x_{N+1}(kT)| = 0,\ \lim_{k\to\infty} |v_i(kT) - v_{N+1}(kT)| = 0,\ i \in \mathcal{I}_N \tag{31}$$

for any initial state $X(0) \in \mathbb{R}^{2N}$. Let the linear-consensus protocol be

$$\begin{aligned} u_i(kT) = &\sum_{j=1, j\neq i}^{N} \left\{ a_{ij} \left[x_j(kT) - x_i(kT) \right] + w_{ij} \left[v_j(kT) - v_i(kT) \right] \right\} \\ &- b_i \left[x_i(kT) - x_{N+1}(kT) \right] - d_i \left[v_i(kT) - v_{N+1}(kT) \right]. \end{aligned}$$

Hence, we have

$$\begin{aligned} &X(kT+T) = (I_N \otimes G_T) X(kT) + (I_N \otimes H_T) U(kT), \\ &U(kT) = -\left\{ L_{ff}^A \otimes [1\ \ 0] + L_{ff}^W \otimes [0\ \ 1] \right\} X(kT) + \mathbf{b} x_{N+1}(kT)) + \mathbf{d} v_{N+1}(kT), \end{aligned} \tag{32}$$

where

$$G_T = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \textit{ and } H_T = \begin{pmatrix} 0 \\ T \end{pmatrix}.$$

Now we propose the optimal control problem of (32) as follows:

$$\begin{aligned} &\min_{U(kT)} J(U(kT), X(0)) \\ &\text{subject to (32) and (31),} \end{aligned} \tag{33}$$

where

$$J(U(kT), X(0)) = \sum_{k=0}^{\infty} \sum_{i=1}^{N} \left[q_i \left((x_i(kT) - x_{N+1}(kT))^2 + (v_i(kT) - v_{N+1}(kT))^2 \right) + r_i u_i^2(kT) \right]$$

and $q_i > 0$ and $r_i > 0$.

Theorem 4

For optimal control problem (33), the optimal topology is an unevenly weighted star topology. More specifically, each follower i $(i \in \mathcal{I}_N)$ just has one connection to the leader with different weights b_i^* and d_i^* in the position and the velocity graph, respectively, where

$$\left[b_i^*,\ d_i^*\right] = \left(r_i + H_T^T P_i H_T^T\right)^{-1} H_T^T P_i G_T,$$

and $P_i \in \mathbb{R}^{2\times 2}$ $(i \in \mathcal{I}_N)$ is the solution of the discrete-time ARE

$$G_T^T P_i G_T + q_i I_2 - P_i - G_T^T P_i H_T \left(r_i + H_T^T P_i H_T^T\right)^{-1} H_T^T P_i G_T = 0.$$

Proof

Denote $\varepsilon_i(k) = [x_i(kT) - x_{N+1}(kT),\ v_i(kT) - v_{N+1}(kT)]^T$ and $\varepsilon(k) = \left[\varepsilon_1^T(k), \ldots, \varepsilon_N^T(k)\right]^T$. Then the dynamics of $\varepsilon(k)$ is

$$\begin{aligned} \varepsilon(k+1) &= (I_N \otimes G_T)\varepsilon(k) + (I_N \otimes H_T)U(k), \\ U(k) &= -K\varepsilon(k), \end{aligned} \tag{34}$$

where $K = \left[L_{ff}^A \otimes [1\ 0] + L_{ff}^W \otimes [0\ 1]\right]$. Obviously, the optimal control problem (33) is equivalent to

$$\begin{aligned} &\min_{U(k)} \sum_{k=0}^{\infty} \left[\varepsilon(k)^T (Q \otimes I_2)\varepsilon(k) + U(k)^T R U(k)\right] \\ &\text{subject to (34) and } \lim_{k\to\infty} |\varepsilon(k)| = 0\ , \end{aligned} \tag{35}$$

where $Q = \operatorname{diag}\{q_1, \ldots, q_N\}$ and $R = \operatorname{diag}\{r_1, \ldots, r_N\}$. It is easy to show that the controllability matrix of (34) is of row full rank:

$$\begin{aligned} &\operatorname{rank}\left((I_N \otimes H_T), (I_N \otimes G_T)(I_N \otimes H_T), \ldots, (I_N \otimes G_T)^{2N-1}(I_N \otimes H_T)\right) \\ &= \operatorname{rank}\left(I_N \otimes [0,\ T]^T, I_N \otimes [T^2,\ T]^T, \ldots\right) \\ &= \operatorname{rank}\left(I_N \otimes \begin{pmatrix} T^2 & 0 \\ T & T \end{pmatrix}, \ldots\right) = 2N. \end{aligned}$$

Therefore, we can conclude from Lemma 2 that the discrete-time ARE

$$\begin{aligned} &Q \otimes I_2 + \left(I_N \otimes G_T^T\right) P \left(I_N \otimes G_T\right) - P \\ &- \left(I_N \otimes G_T^T\right) P \left(I_N \otimes H_T\right) \left[R + \left(I_N \otimes H_T^T\right) P \left(I_N \otimes H_T\right)\right]^{-1} \left(I_N \otimes H_T^T\right) P \left(I_N \otimes G_T\right) = 0 \end{aligned} \tag{36}$$

has a unique positive-definite solution P. And the optimal control $U^*(k) = -K^*\varepsilon(k)$ can steer the close-loop system (34) to achieve asymptotic stability, where

$$K^* = \left[R + \left(I_N \otimes H_T^T\right) P \left(I_N \otimes H_T\right)\right]^{-1} \left(I_N \otimes H_T^T\right) P \left(I_N \otimes G_T\right).$$

Our next goal is to solve P and K^* by decomposing the LQR problem (35) into N LQR problems. Observing that R, $Q \otimes I_2$, $I_N \otimes G_T$ and $I_N \otimes H_T$ are all block-diagonal matrices, we suppose $P = \operatorname{diag}\{P_1, \ldots P_N\}$ with $P_i \in \mathbb{R}^{2\times 2}$, $i \in \mathcal{I}_N$, on its diagonal. As a result, the ARE (36) is decomposed into N matrix equations as follows:

$$G_T^T P_i G_T + q_i I_2 - P_i - G_T^T P_i H_T \left(r_i + H_T^T P_i H_T\right)^{-1} H_T^T P_i G_T = 0,\ i \in \mathcal{I}_N. \tag{37}$$

Moreover, it gives rise to $K^* = \operatorname{diag}\left\{K_1^*, \ldots, K_N^*\right\}$, where $K_i^* = \left[b_i^*, d_i^*\right] = \left(r_i + H_T^T P_i H_T^T\right)^{-1} H_T^T P_i G_T$. It should be noted that (37) is AREs of the following LQR problems for all $i \in \mathcal{I}_N$:

$$\begin{aligned} &\min_{u_i(k)} \sum_{k=0}^{\infty} \left[q_i \varepsilon_i(k)^T \varepsilon_i(k) + r_i u_i(k)^2\right] \\ &\text{subject to } \varepsilon_i(k+1) = G_T \varepsilon_i(k) + H_T u_i(k). \end{aligned} \tag{38}$$

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Therefore, the optimal control problem (35) is decomposed into N optimal control problems (38). Thus, each $u_i^*(k)$ $(i \in \mathcal{I}_N)$ is independent of $\varepsilon_j(k)$, for all $j \neq i$. That is to say, there is no information exchange between followers. Using the fact that (G_T, H_T) is controllable, we further obtain the ARE (37) has a unique solution $P_i \in \mathbb{R}^{2\times 2}$ for each $i \in \mathcal{I}_N$, and the optimal feedback matrix of (38) is K_i^*. In addition, the optimal control of (38)

$$u_i^* = -b_i^* \left(x_i(kT) - x_{N+1}(kT)\right) - d_i^* \left(v_i(kT) - v_{N+1}(kT)\right)$$

gives rise to the asymptotical stability of the associated close-loop of $\varepsilon_i(k)$, and therefore, leads to the leader-following consensus of the ith follower. Thus, we conclude that the optimal topology is an unevenly weighted star topology in which each follower i $(i \in \mathcal{I}_N)$ is only connected to the leader with different weights b_i^* and d_i^* in the position and the velocity graph, respectively. □

The results of this paper mean that if each follower can access the information from the leader, then any information from other followers is unnecessary and insufficient. This explains some cooperation games. Consider a cooperative team, if the goal of the team is a "common knowledge" for all members (in other words, each member has full knowledge about the goal of the team and knows how to fulfill it and moreover, knows that all others also know the goal and the method), then information exchange and cooperation between members become unnecessary. However, cooperation always decreases the "total payoff" because either the goal information is not fully available to all members or some members cannot achieve the goal independently. Besides the cooperation games, the result of the paper is also sociologically applicable. In dictatorship, the dictator would prefer a network where all the followers can access no other than the dictator himself. This type of network prevents the followers from communicating with each other, thus, safeguards the ruling power of the dictator.

5. SIMULATIONS

In this section, we give two numerical simulations to illustrate the theoretical results in Sections 3 and 4, respectively.

Example 1
Consider a first-order continuous-time MAS that consists of four followers, labeled by 1, 2, 3 and 4, and a leader, labeled by 5. Suppose $Q = R = I_4$ and $x_5(0) = 0$. Four interaction topologies G^*, G_2, G_3 and G_4 are depicted in Figures 1, 2, 3 and 4, respectively. Table I gives the costs $J(U(t), X(0))$ of G^*, G_2 and G_3 with same initial states. It shows that the cost of G^* is always minimal, which is consistent with the result of Theorem 1. Consider the star topology G_4 with the weights 1, 1, y and z $(y,\ z \in \mathbb{R}^+)$. Note that for a given initial state $X(0) \in \mathbb{R}^4$, we have the cost function $J(U(t), X(0)) = J_{X(0)}(y, z)$ for G_4. $J_{X(0)}(y, z)$ is minimal when $G_4 = G^*$ $(y = z = 1)$

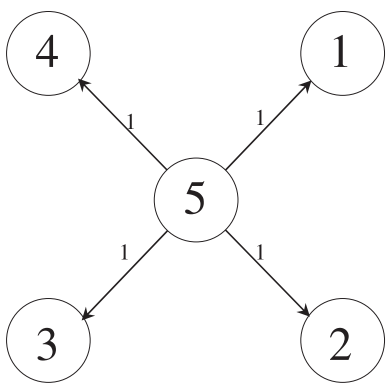

Figure 1. The topology G^*.

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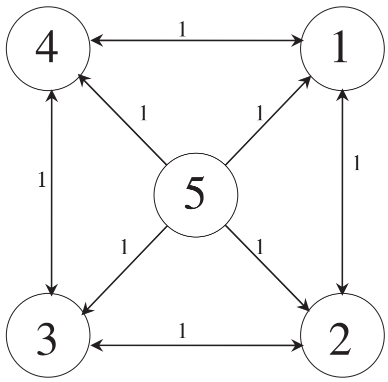


Figure 2. The topology G_2.

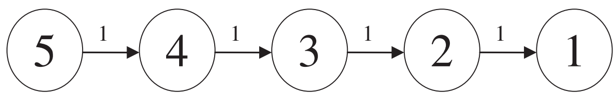


Figure 3. The topology G_3.

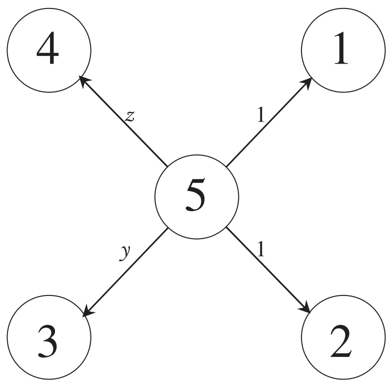


Figure 4. The topology G_4.

Table I. The costs of different interaction topologies.

$X(0)$	$J(U(t), X(0))$ G^*	G_2	G_3
$[6.78 \quad -7.57 \quad -7.43 \quad 3.92]^T$	173.84	288.76	253.52
$[1.10 \quad 0.046 \quad -8.73 \quad -2.47]^T$	83.53	128.69	165.05
$[2.63 \quad 6.54 \quad 6.89 \quad 7.48]^T$	153.11	167.55	189.30

with a random chosen initial state $X(0) = [2.63 \quad 6.54 \quad 6.89 \quad 7.48]^T$ in Figure 5, which is consistent with the result of Theorem 1.

Example 2

Consider a second-order continuous-time MAS that consists of four followers and a leader. Suppose that $Q = R = I_4$, $x_5(0) = 0$ and $v_5(t) = 1$ $(t \geqslant 0)$. We propose a star topology $\hat{G}^*$ in which the position and the velocity graph are depicted in Figure 6. The second interaction topology is $\hat{G}_2$

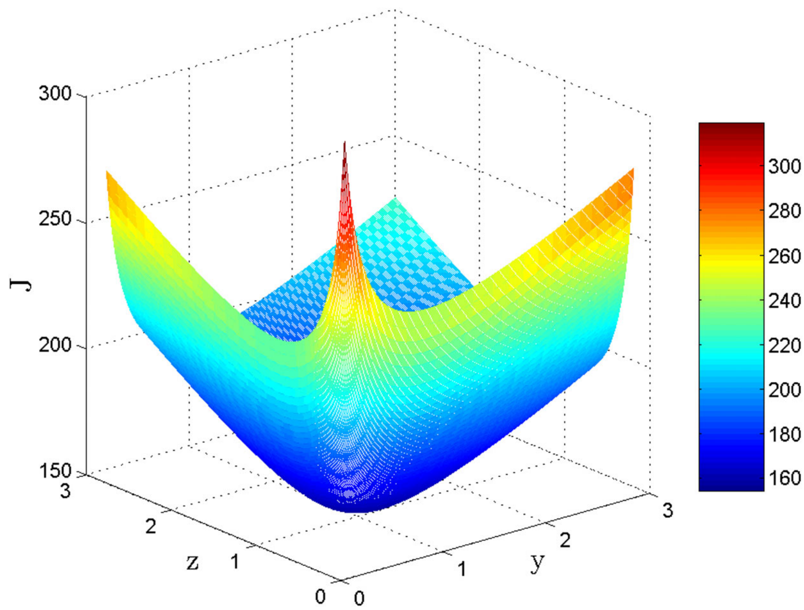
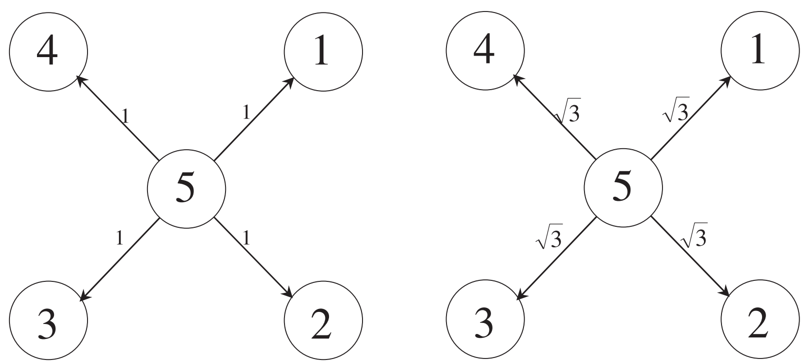
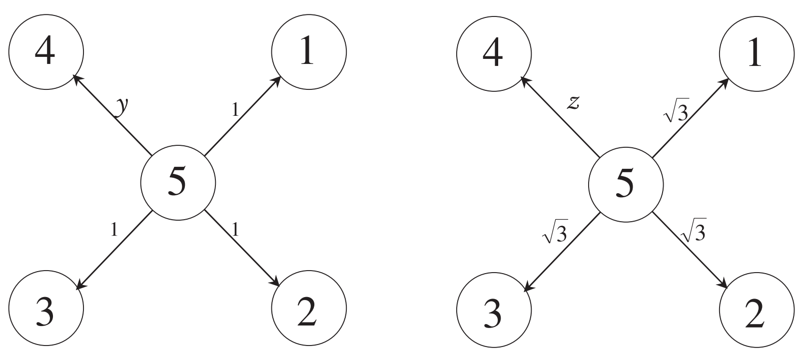


Figure 5. $J_{X(0)}(y,z)$ with the topology G_4 and $X(0) = [2.63 \quad 6.54 \quad 6.89 \quad 7.48]^T$.



(a) Position graph of $\hat{G}^*$ (b) Velocity graph of $\hat{G}^*$

Figure 6. The topology $\hat{G}^*$.



(a) Position graph of $\hat{G}_5$ (b) Velocity graph of $\hat{G}_5$

Figure 7. The topology $\hat{G}_5$.

Table II. The costs of different interaction topologies.

$X(0)$	$J(U(t), X(0))$ $\hat{G}^*$	$\hat{G}_2$	$\hat{G}_3$
$[7.57 \quad 74.31 \quad 7.84 \quad -3.74 \quad 0.83 \quad 9.26 \quad 4.18 \quad 28.69]^T$	12326	18293	22091
$[-3.81 \quad 1.53 \quad 15.90 \quad 9.34 \quad 1.50 \quad -4.34 \quad 10.69 \quad 8.09]^T$	1338.1	1888.4	5941.8
$[0.40 \quad -2.38 \quad 1.80 \quad 0.79 \quad 3.80 \quad 8.30 \quad 7.68 \quad 10.70]^T$	369.78	441.58	2893.9

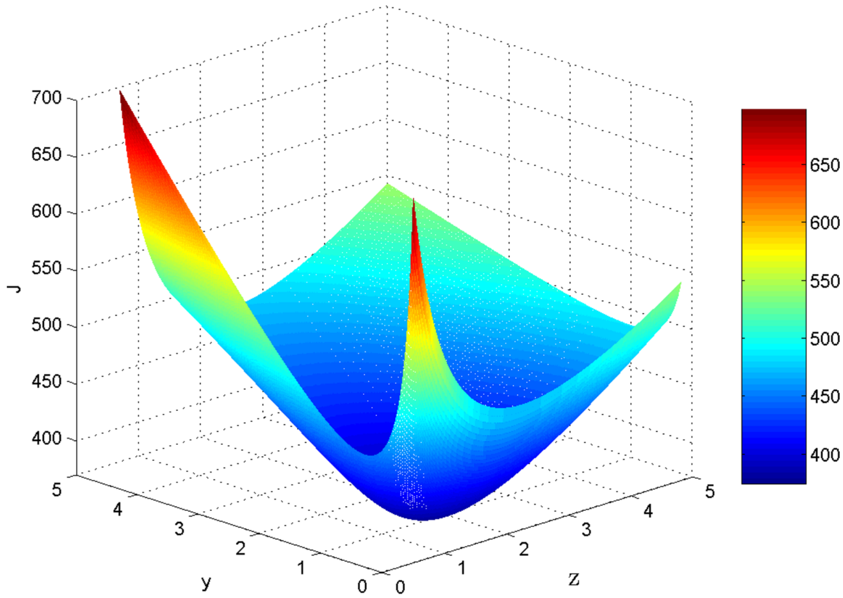


Figure 8. $J_{X(0)}(y, z)$ with $\hat{G}_5$ and $X(0) = [0.40 \quad -2.38 \quad 1.80 \quad 0.79 \quad 3.80 \quad 8.30 \quad 7.68 \quad 10.70]^T$.

in which the position and the velocity topology are G_2. The third one is $\hat{G}_3$ in which the position and the velocity graph are G_3. Figure 7 describes an unevenly weighted star topology $\hat{G}_5$ with the weights 1, 1, 1 and y in the position graph, and the weights $\sqrt{3}$, $\sqrt{3}$, $\sqrt{3}$ and z in the velocity graph, where $y, \ z \in \mathbb{R}^+$. Table II shows the costs of $\hat{G}^*$, $\hat{G}_2$, and $\hat{G}_3$ with same initial states. It is easy to find that the cost of $\hat{G}^*$ is always minimal. In addition, we get the cost function $J(U(t), X(0)) = J_{X(0)}(y, z)$ for $\hat{G}_5$. Notice that when $\hat{G}_5 = \hat{G}^*$ $(y = 1, \ z = \sqrt{3})$, $J_{X(0)}(y, z)$ is minimal with a random chosen initial state $X(0) = [0.4 \quad -2.3816 \quad 1.8 \quad 0.79 \quad 3.8 \quad 8.3 \quad 7.68 \quad 10.7]^T$ in Figure 8. The results of Table II and Figure 8 illustrate the effectiveness of theoretical result in Theorem 3.

6. CONCLUSION

In this paper, we developed LQR-based optimal control for leader-following MASs and found that the optimal control is always associated with star topology. Firstly, we proposed a quadratic cost function, which is independent of the interaction graph. Secondly, by employing LQR method, we can get the optimal topology by solving the associated ARE. For both first-order and second-order leader-following MASs, the optimal control always corresponds to a star topology in which each follower just has an access to information of the leader. The results offer some theoretical explanations to some cooperation games and dictatorship. Future work may consider optimal control problems for some MASs with constrains, such as MASs with fixed and switching topologies and MASs without velocity measurement.

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REFERENCES

1. Xiao F, Wang L, Chen J, Gao Y. Finite-time formation control for multi-agent systems. *Automatica* 2009; **45**(11):2605–2611.
2. Olfati-Saber R. Flocking for multi-agent dynamics systems: algorithms and theory. *IEEE Transactions on Automatic Control* 2006; **51**(3):401–420.
3. Lin J, Morse AS, Anderson BDO. The multi-agent rendezvous problem. *Proceedings of 42nd IEEE Conference on Decision and Control* 2003; **2**(9):1508–1513.
4. Vicsek T, Czirok A, Jacob EB, Cohen I, Schochet O. Novel type of phase transition in a system of self-driven particles. *Physical Review Letters* 1995; **75**(6):1226–1229.
5. Jabdabaie A, Lin J, Morse AS. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control* 2003; **48**(6):988–1001.
6. Ren W, Beard RW. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control* 2005; **50**(5):655–661.
7. Wang L, Xiao F. A new approach to consensus problem in discrete-time multiagent systems with time-delays. *Science in China Series F: Information Sciences* 2007; **50**(4):625–635.
8. Ji M, Ferrari-Trecate G, Egerstedt M, Buffa A. Containment control in mobile networks. *IEEE Transactions on Automatic Control* 2008; **53**(8):1972–1975.
9. Cao Y, Ren W. Containment control with multiple stationary or dynamic leaders under a directed interaction graph. *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, Shanghai, China, 2009; 3014–3019.
10. Liu H, Xie G, Wang L. Necessary and sufficient conditions for containment control of networked multi-agent systems. *Automatica* 2012; **48**(7):1415–1422.
11. Zheng Y, Wang L. Containment control of heterogeneous multi-agent systems. *International Journal of Control* 2014; **87**(1):1–8.
12. Ji Z, Wang Z, Lin H, Wang Z. Interconnection topologies for multi-agent coordination under leader-follower framework. *Automatica* 2009; **45**(12):2857–2863.
13. Ji M, Muhammad A, Egerstedt M. Leader-based multi-agent coordination: controllability and optimal control. *American Control Conference* 2006; **2006**:1358–1363.
14. Cao Y, Ren W. Optimal linear-consensus algorithms: an LQR perspective. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics* 2010; **40**(3):819–830.
15. Hengster-Movric K, Lewis F. Cooperative optimal control for multi-agent systems on directed graph topologies. *IEEE Transactions on Automatic Control* 2014; **59**(3):769–774.
16. Xiao F, Wang L. Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays. *IEEE Transactions on Automatic Control* 2008; **53**(8):1804–1816.
17. Xie G, Wang L. Consensus control for a class of networks of dynamic agents. *International Journal of Robust and Nonlinear Control* 2007; **17**(10-11):941–959.
18. Ren W. On consensus algorithms for double-integrator dynamics. *IEEE Transactions on Automatic Control* 2008; **53**(6):1503–1509.
19. Gao Y, Wang L. Sampled-data based consensus of continuous-time multi-agent systems with time-varying topology. *IEEE Transactions on Automatic Control* 2011; **56**(5):1226–1231.
20. Gao Y, Wang L, Xie G, Wu B. Consensus of multi-agent systems based on sampled-data control. *International Journal of Control* 2009; **82**(12):2193–2205.
21. Zheng Y, Zhu Y, Wang L. Consensus of heterogeneous multi-agent systems. *IET Control Theory and Applications* 2011; **5**(16):1881–1888.
22. Zheng Y, Wang L. Distributed consensus of heterogeneous multi-agent systems with fixed and switching topologies. *International Journal of Control* 2012; **85**(12):1967–1976.
23. Hong Y, Hu J, Gao L. Tracking control for multi-agent consensus with an active leader and variable topology. *Automatica* 2006; **42**(7):1177–1182.
24. Ni W, Cheng D. Leader-following consensus of multi-agent systems under fixed and switching topologies. *Systems and Control Letters* 2010; **59**(3):209–217.
25. Wang W, Slotine JJE. A theoretical study of different leader roles in networks. *IEEE Transactions on Automatic Control* 2006; **51**(7):1156–1161.
26. Olfati-Saber R, Murray RM. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control* 2004; **49**(9):1520–1533.
27. Xiao L, Boyd S. Fast linear iterations for distributed averaging. *Systems and Control Letters* 2004; **53**(1):65–78.
28. Delvenne JC, Carli R, Zampieri S. Optimal strategies in the average consensus problem. *Systems and Control Letters* 2009; **58**(10):759–765.

29. Wang L, Xiao F. Finite-time consensus problems for networks of dynamic agents. *IEEE Transactions on Automatic Control* 2010; **55**(4):950–955.
30. Zheng Y, Wang L. Finite-time consensus of heterogeneous multi-agent systems with and without velocity measurements. *Systems and Control Letters* 2012; **61**(8):871–878.
31. Wang X, Hong Y. Finite-time consensus for multi-agent networks with second-order agent dynamics. *Proceedings of the 17th World Congress The International Federation of Automatic Control*, Seoul, South Korea, 2008; 15185–15190.
32. Lin F, Fardad M, Jovanovic MR. Optimal control of vehicular formations with nearest neighbor interactions. *IEEE Transactions on Automatic Control* 2012; **57**(9):2203–2218.
33. Semsar-Kazerooni E, Khorasani K. Multi-agent team cooperation: a game theory approach. *Automatica* 2009; **45**(10):2205–2213.
34. Wang J, Xin M. Multi-agent consensus algorithm with obstacle avoidance via optimal control approach. *International Journal of Control* 2010; **83**(12):2606–2621.
35. Andreson BDO, Moore JB. *Optimal Control: Linear Quadratic Methods*. Prentice Hall: Englewood Cliffs, NJ, 1990.