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Topology selection for multi-agent systems with opposite leaders[☆]

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ABSTRACT

For the multi-agent system with a navigational leader and its opponent, the followers cannot converge to the state of the navigational leader. In this paper, we consider the topology selection problem to minimize the opponent's influence which is measured by the tracking error of the system. Firstly, two combinatorial optimization problems are formulated. One is to minimize the tracking error by selecting guided informed-agents (the followers who can obtain the navigational leader's information). The other is to choose minimal number of guided informed-agents under an upper bound constraint of the tracking error. Secondly, for the scenario where the guided informed-agents are preset, we consider the problem of assigning the weights of edges to minimize the tracking error. Three convex optimization problems are proposed to evaluate the upper and lower bounds of the tracking error. Finally, numerical examples are provided to illustrate the effectiveness of the theoretical results.

1. Introduction

In the last decade, cooperative control of multi-agent systems (MASs) has captured tremendous attention from a wide range of academic disciplines, such as biology, physics, and social science etc [1–4]. Consensus seeking is an important issue of cooperative control of MASs which means all agents will converge to the same state. There have been extensive studies and results under various circumstances [5–10]. Multi-agent systems with leaders are also considered which leads to several research hotspots such as the leader-following consensus [11,12], containment control [13,14] and controllability analysis [15]. Shi et al. considered the leader-following consensus problem for MASs with a virtual leader [11]. In [12], the authors considered tracking control under variable topologies and obtained some sufficient conditions for solving the leader-following consensus. As an extension of consensus problem, containment control of multi-agent systems means that the followers will converge to the convex hull spanned by the leaders. Containment control under fixed undirected topology and switching topologies was considered in [13,16], respectively. Some sufficient and/or necessary conditions for solving containment control had also been addressed under varied models, such as containment control of heterogeneous MASs [17] and of MASs with measurement noises [18], etc.

In leader–follower multi-agent systems, followers can be classified into two types: the informed-agents who can receive information from leaders directly and the others who cannot. The leader sends its state information to the informed-agents who will spread this information to other followers by the interaction among the followers. Therefore, the interaction graph of the system is together determined by the subgraph of the followers and the set of informed-agents. Existing studies have shown that followers can converge to the leader under different interaction graphs. Then, a natural question that arises is how to design the interaction graph such that the system can converge quickly. The fast consensus problem is considered by solving semi-definite programming problems [5,19,20]. Based on linear-quadratic regulator theory, [21] proved that the optimal topology of leader-following consensus is a star topology. The problem of topology selection is also studied. In [22], the authors revealed that in order to achieve high accuracy, the more the agents, the smaller the proportion of informed-agents needed. [23] found that the convergence rate for first-order leader–follower MASs can also be determined by the

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maximal distance from the leader to the followers in the interaction graph. In order to minimize this distance, a standard combinatorial optimization problem was proposed. For leader–follower MASs with noises, [24] investigated the problem of minimizing the mean-square deviation via selecting a given number of informed-agents. Meanwhile, the problem of solving containment control in an optimal way was also studied. In [25], the authors gave an explicit expression to estimate the tracking error for multi-agent systems with some moving leaders. Clark et al. [26] formulated a leader selection problem for containment control in order to minimize the convergent error. A supermodular optimization approach was developed to solve this problem.

Competition and conflict are ubiquitous in practice. For the multi-agent system, differences of agents' interests may produce competition and opposition. One example from social science is election. Two candidates run for election and both want to beat their opponent. Based on this fact, [27] formulated the problem of maximizing influence in social networks with competitive ideas as a Stackelberg game. In [28], the authors provided a model to investigate the tension between information aggregation and spread of misinformation in large societies. Another example comes from networks. In a network, some nodes may be compromised by a malicious attacker whose objective is to disrupt the operation of the network. Therefore, a distributed strategy was developed to calculate any arbitrary function of the node values for networks with malicious nodes in [29]. In this paper, we consider the multi-agent system with two leaders who have opposite purposes. One leader named navigational leader propagates the navigational information to drive the followers to follow itself. The other leader named the opponent sends the misinformation to make the followers to keep away from the navigational leader. Considering the existence of the opponent, the followers will never converge to the navigational leader. Therefore, we focus on the problem of topology selection to reduce the opponent's influence. The main contribution of this paper is threefold. Firstly, we define the tracking error to quantify the opponent's influence. Then, we prove that if a follower is added into the set of guided informed-agents (who receive the navigational information), the tracking error will be decreased. Secondly, we formulate two topology selection problems. One is how to choose at most k guided informed-agents to minimize the tracking error. The other is how to select minimal number of guided informed-agents under an upper bound constraint of the tracking error. Since the two problems are *NP*-hard, two algorithms are developed to obtain their suboptimal solutions respectively. Finally, the problem of assigning the weights of edges is considered to reduce the opponent's influence for the case that the guided informed-agents are preset. We propose three convex programming problems to obtain the upper and the lower bounds of the minimal tracking error.

This paper is organized as follows. In Section 2, we introduce the graph theory and the system model. In Section 3, we formulate two guided informed-agents selection problems and develop two algorithms. The problem of designing the optimal weights of guided informed-edges to minimize tracking error is given in Section 4. In Section 5, numerical simulations are carried out to illustrate the effectiveness of our results. Some conclusions are drawn in Section 6.

Notation: Throughout this paper, the following notations will be used: let $\mathbb{R}$ be the set of real numbers. $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices. Denote $\mathbf{1}_n$ (or $\mathbf{0}_n$) as the column vector with all entries equal to one (or all zeros). For a column vector $\mathbf{b} = [b_1, b_2, \dots, b_n]^T$, $D_{\mathbf{b}}$ is a diagonal matrix with b_i, $i = 1, \dots, n$, on its diagonal and $\|\mathbf{b}\|_p = (\sum_{i=1}^n |b_i|^p)^{\frac{1}{p}}$ is l^p-norm of $\mathbf{b}$. I_n denotes an n-dimensional identity matrix and $0_{n\times m}$ is a matrix with all zero entries. A matrix $A \in \mathbb{R}^{n\times m}$ is nonzero if $A \neq 0_{n\times m}$. For $A, B \in \mathbb{R}^{n\times m}$, denote $A > B$ (resp. $A \geq B$) if $A - B$ is a positive matrix (resp. nonnegative matrix), and let A_{ij} be the ij-th entry of A. For a square matrix A, ρ_A and $\mathrm{tr}(A)$ are the spectral radius and the trace of A, respectively. $\mathrm{adj}(A)$ is the adjugate of A. $\det(A)$ is the determinant of A. For a set S, $|S|$ and 2^S are the cardinality and the power set of S respectively. For two sets S_1 and S_2, denote $S_1 \times S_2$ as the Cartesian product. $S_1 \setminus S_2 = S_1 - S_2$. Let $\mathbf{e}_i$ denote the canonical vector with a 1 in the ith entry and 0's elsewhere.

2. Preliminaries

2.1. Graph theory

In this subsection, we present some basic notions of algebraic graph theory which will be used in this paper. For more details, interested readers are referred to [30] for a thorough study of graph theory.

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ be an undirected graph consisting of a vertex set $\mathcal{V} = \{1, 2, \dots, n\}$ and an edge set $\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V}\}$. The adjacency matrix $\mathcal{A}$ is a matrix such that for all $i \in \mathcal{I}_n$, $a_{ii} = 0$ and for all $i \neq j$, $(i, j) \in \mathcal{E}$ if and only if $a_{ij} = a_{ji} > 0$, while $a_{ji} = 0$ otherwise. A walk of length r in a graph $\mathcal{G}$ is a sequence of vertices $i_0 \sim i_1 \sim \cdots \sim i_r$ where $(i_k, i_{k+1}) \in \mathcal{E}$. If there exists at least one walk from the vertex i to j in $\mathcal{G}$ with length r, then $(\mathcal{A}^r)_{ij} > 0$ [30]. The degree matrix $\mathcal{D} = D_{[d_1,\dots,d_n]^T}$ where $d_i = \sum_{j\in\mathcal{N}_i} a_{ij}$ and the Laplacian matrix $\mathcal{L} = \mathcal{D} - \mathcal{A}$.

Lemma 1. *Let $\mathcal{G}$ be a connected graph. Assume Δ be a nonnegative and nonzero diagonal matrix. Then, $\mathcal{L} + \Delta$ is positive definite and $(\mathcal{L} + \Delta)^{-1}$ is a positive matrix.*

Proof. By Lemma 3 in [12], we obtain that $\mathcal{L} + \Delta$ is positive definite. Let $\theta_i = (\mathcal{L} + \Delta)_{ii}$ and $\hat{\theta} = \max_{1\leq i\leq n} \theta_i$. It follows that

$$\mathcal{L} + \Delta = D_\theta - \mathcal{A} = \hat{\theta}\left[I_n - \left(\tilde{\Delta} + \frac{\mathcal{A}}{\hat{\theta}}\right)\right],$$

where $\theta = [\theta_1, \dots, \theta_n]$ and $\tilde{\Delta} = I_n - \frac{D_\theta}{\hat{\theta}}$. It is easy to show that $\tilde{\Delta} + \frac{\mathcal{A}}{\hat{\theta}}$ is a nonnegative matrix with spectral radius $\rho < 1$. Hence, we have

$$(\mathcal{L} + \Delta)^{-1} = \frac{1}{\hat{\theta}}\sum_{k=0}^{\infty}\left(\tilde{\Delta} + \frac{\mathcal{A}}{\hat{\theta}}\right)^k \geq \frac{1}{\hat{\theta}}\sum_{k=0}^{\infty}\left(\frac{\mathcal{A}}{\hat{\theta}}\right)^k. \tag{1}$$

For every $(i, j) \in \mathcal{V}\times\mathcal{V}$, because $\mathcal{G}$ is connected, there exists at least one walk from i to j with length $k \geq 1$, i.e., $(\mathcal{A}^k)_{ij} > 0$. Therefore, $\sum_{k=0}^{\infty} \mathcal{A}^k > 0$. Together with (1), we have $(\mathcal{L} + \Delta)^{-1} > 0$. ■

Lemma 2. *Let $\mathcal{G}$ be a connected graph. Let Δ_1, Δ_2 be two nonnegative and nonzero diagonal matrices. Suppose that $\Delta_2 - \Delta_1 \geq 0$ and $\Delta_2 \neq \Delta_1$. Then, $(\mathcal{L} + \Delta_1)^{-1} - (\mathcal{L} + \Delta_2)^{-1}$ is a positive matrix.*

Proof. Denote $\Delta_2 - \Delta_1 = D_{[x_1,x_2,\dots,x_n]^T}$ and

$$L_0 = \mathcal{L} + \Delta_1,$$
$$L_1 = L_0 + x_1\mathbf{e}_1\mathbf{e}_1^T,$$
$$\cdots$$
$$L_n = L_{n-1} + x_n\mathbf{e}_n\mathbf{e}_n^T.$$

From the matrix inversion lemma [31], we have

$$L_{k-1}^{-1} - L_k^{-1} = \frac{L_{k-1}^{-1}x_k\mathbf{e}_k\mathbf{e}_k^T L_{k-1}^{-1}}{1 + x_k\mathbf{e}_k^T L_{k-1}^{-1}\mathbf{e}_k}, \quad k = 1, 2, \dots, n.$$

It follows that

$$(\mathcal{L}+\Delta_1)^{-1}-(\mathcal{L}+\Delta_2)^{-1} = \sum_{k=1}^{n}\left[L_{k-1}^{-1}-L_k^{-1}\right] = \sum_{k=1}^{n}\frac{L_{k-1}^{-1}x_k\mathbf{e}_k\mathbf{e}_k^T L_{k-1}^{-1}}{1+x_k\mathbf{e}_k^T L_{k-1}^{-1}\mathbf{e}_k}.$$

Using Lemma 1, $L_{k-1}^{-1} > 0$. Together with $\Delta_2 - \Delta_1 \geq 0$, we get $\frac{L_{k-1}^{-1}x_k\mathbf{e}_k\mathbf{e}_k^T L_{k-1}^{-1}}{1+x_k\mathbf{e}_k^T L_{k-1}^{-1}\mathbf{e}_k} \geq 0$. Since $\Delta_2 \neq \Delta_1$, there exists at least one $\frac{L_{k-1}^{-1}x_k\mathbf{e}_k\mathbf{e}_k^T L_{k-1}^{-1}}{1+x_k\mathbf{e}_k^T L_{k-1}^{-1}\mathbf{e}_k} > 0$. Consequently, we have $\sum_{k=1}^{n}\frac{L_{k-1}^{-1}x_k\mathbf{e}_k\mathbf{e}_k^T L_{k-1}^{-1}}{1+x_k\mathbf{e}_k^T L_{k-1}^{-1}\mathbf{e}_k} > 0$ which implies that $(\mathcal{L}+\Delta_1)^{-1}-(\mathcal{L}+\Delta_2)^{-1} > 0$. ■

2.2. System model

In MASs, an agent who sends its own information and do not receive the information from the others is the leader. An agent is called the follower if it is not the leader. Consider a multi-agent system consisting of two leaders and n followers. In this paper, the following definitions are given.

Definition 1. The leader who sends the navigational information $\rho_0(t)$ is called the navigational leader. The leader who sends the misinformation $\rho_1(t)(\neq \rho_0(t))$ is called the opponent.

Definition 2. The followers who can receive the navigational information are called the guided informed-agents (GI-agents). The followers who can receive the misinformation are called the misguided informed-agents (MI-agents).

Definition 3. For follower i, $\theta_i > 0$ and $\vartheta_i > 0$ are the guided link-weight (GL-weight) and the misguided link-weight (ML-weight), respectively. If follower i is chosen as a GI-agent (resp. MI-agent), then the weight of the corresponding directed edge from the navigational leader (resp. the opponent) to follower i is θ_i (resp. ϑ_i).

The navigational leader wants the followers to track $\rho_0(t)$, while the opponent wants the followers to keep away from $\rho_0(t)$. As a result, the navigational leader propagates $\rho_0(t)$ to GI-agents and the opponent sends $\rho_1(t)$ to MI-agents. In this paper, we assume that the two leaders are static, i.e., $\rho_0(t) = \rho_0$ and $\rho_1(t) = \rho_1$.

Suppose that the interaction of the followers is described by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \ldots, n\}$ is the set of followers. If two followers i and j can communicate with each other, then $(i, j) \in \mathcal{E}$. Let S be the set of GI-agents (GI-set) and T be the set of MI-agents (MI-set). Two Boolean vectors $x = [x_1, \ldots, x_n]^T$ and $y = [y_1, \ldots, y_n]^T$ indicate whether each follower belongs to S and T or not, respectively, i.e.,

$$x_i = \begin{cases}1 & i \in S\\ 0 & \text{otherwise}\end{cases} \quad \text{and} \quad y_i = \begin{cases}1 & i \in T\\ 0 & \text{otherwise,}\end{cases} \qquad i \in \mathcal{V}.$$

Therefore, the dynamics of the follower i is

$$\dot{\xi}^i = \sum_{j=1}^{n} a_{ij}(\xi^j - \xi^i) + x_i\theta_i(\rho_0 - \xi^i) + y_i\vartheta_i(\rho_1 - \xi^i),$$

where a_{ij} is the ijth entry of $\mathcal{A}$. Denote $\theta = [\theta_1, \ldots, \theta_n]^T$, $\vartheta = [\vartheta_1, \ldots, \vartheta_n]^T$ and $\xi(t) = [\xi^1(t), \ldots, \xi^n(t)]^T$. The aggregate dynamics of the system is represented by

$$\dot{\xi}(t) = -(\mathcal{L} + D_\theta D_x + D_\vartheta D_y)\xi(t) + D_\theta x\rho_0 + D_\vartheta y\rho_1, \tag{2}$$

where $\mathcal{L}$ is the Laplacian matrix of $\mathcal{G}$. The following assumptions are given throughout this paper.

A1. (Connectivity) $\mathcal{G}$ is connected.

A2. (Information accessibility) There exist at least one GI-agent and one MI-agent, i.e., $S \neq \emptyset$ and $T \neq \emptyset$.

It follows from [14] that

Lemma 3. *Suppose that* A1 *and* A2 *hold. For multi-agent system (2), one has* $\lim_{t\to\infty}\xi^i(t) = \alpha_i\rho_0 + \beta_i\rho_1$, *where*

$$\begin{aligned}\alpha_i &= \mathbf{e}_i^T(\mathcal{L} + D_\theta D_x + D_\vartheta D_y)^{-1}D_\theta x,\\ \beta_i &= \mathbf{e}_i^T(\mathcal{L} + D_\theta D_x + D_\vartheta D_y)^{-1}D_\vartheta y,\\ \alpha_i &+ \beta_i = 1,\ \alpha_i > 0 \text{ and } \beta_i > 0 \text{ for all } i \in \mathcal{V}.\end{aligned} \tag{3}$$

Remark 1. From Lemma 3, the converging states of the followers are determined by $\mathcal{L}$, S, T, θ and ϑ. Therefore, the navigational leader can reduce the opponent's influence by designing S or θ.

3. Minimizing the opponent's influence via GI-agent selection

In this section, assume that $\mathcal{G}, \theta, \vartheta$ and T are fixed. We define the tracking error function to quantify the influence of the opponent. Then, we propose two combinatorial optimization problems to minimize the tracking error function.

3.1. Definition of tracking error function

Given that $\mathcal{G}$, θ and ϑ are fixed, it is easy to know that $\lim_{t\to\infty}\xi_i(t)$ is determined by S and T. Denote $\xi_{S,T}^i = \lim_{t\to\infty}\xi^i(t)$, $i \in \mathcal{V}$. It follows from Lemma 3 that $\xi_{S,T}^i \in (\min\{\rho_0, \rho_1\}, \max\{\rho_0, \rho_1\})$. Therefore, the distance between the follower i and the navigational leader can be employed to quantify the opponent's influence on i.

Definition 4. Let $d_{S,T}^i = |\xi_{S,T}^i - \rho_0|$ be the tracking error of the follower i. Define

$$f_p(S, T) = \left\|d_{S,T}\right\|_p, \quad 1 \leq p \leq \infty$$

as the tracking error function of system (2), where $d_{S,T} = [d_{S,T}^1, \ldots, d_{S,T}^n]^T$.

Remark 2. It is natural to use the p-norm of $d_{S,T}$ to measure the opponent's influence on the whole system. In particular, when $p = 1$, $f_1(S, T)$ is the sum of all agents' tracking error. If $p = \infty$, $f_\infty(S, T)$ is the maximum tracking error among all $d_{S,T}^i$, $i \in \mathcal{V}$.

Theorem 1. *If a follower is added into GI-set, then the tracking error of each follower will be decreased. On the other hand, if a follower is added into MI-set, then the tracking error of each follower will be increased.*

Proof. Denote $S' = S \cup \{v\}$ and $T' = T \cup \{v'\}$, where $v \in \mathcal{V} \setminus S$ and $v' \in \mathcal{V} \setminus T$. It is easy to find that S' is a subset of $\mathcal{V}$. It suffices to prove $d_{S,T} - d_{S',T} > 0$ and $d_{S,T'} - d_{S,T} > 0$.

Let x, x' and y be the indicative vectors of S, S' and T respectively. Using Lemma 3, we have

$$d_{S,T} = c(\mathcal{L} + D_\theta D_x + D_\vartheta D_y)^{-1}D_\vartheta y \tag{4}$$

and

$$d_{S,T} - d_{S',T} = c\left[(\mathcal{L} + D_\theta D_x + D_\vartheta D_y)^{-1} - (\mathcal{L} + D_\theta D_{x'} + D_\vartheta D_y)^{-1}\right]D_\vartheta y,$$

where $c = |\rho_1 - \rho_0|$. Due to $S \subsetneq S'$, we have $x' - x$ is nonnegative and nonzero which implies $(D_\theta D_{x'} + D_\vartheta D_y) - (D_\theta D_x + D_\vartheta D_y)$ is also nonnegative and nonzero. Recalling Lemma 2, we can obtain that

$(\mathcal{L}+D_\theta D_x+D_\vartheta D_y)^{-1}-(\mathcal{L}+D_\theta D_{x'}+D_\vartheta D_y)^{-1}>0$. Considering $c>0$, $D_\vartheta y\geq 0$ and $D_\vartheta y\neq 0$, we have $d_{S,T}-d_{S',T}>0$. Therefore, the tracking error of each follower will be decreased if a follower is added into GI-set.

From (3), we obtain $d_{S,T}=c\left[\mathbf{1}_n-(\mathcal{L}+D_\theta D_x+D_\vartheta D_y)^{-1}D_\theta x\right]$. Similar to the proof of $d_{S,T}-d_{S',T}>0$, one can derive $d_{S,T'}-d_{S,T}>0$ which implies that the tracking error of each follower is increasing with the increasing of GI-agents. ■

Theorem 2. *If some followers are added into GI-set, then the tracking error of system (2) will be decreased. On the other hand, if some followers are added into MI-set, then the tracking error of system (2) will be increased.*

Proof. Denote the new GI-set $S'=S\cup\{v_1,v_2,\dots,v_r\}$ and $T'=T\cup\{n_1,n_2,\dots,n_l\}$, where $\{v_1,v_2,\dots,v_r\}\subseteq\mathcal{V}\setminus S$ and $\{n_1,n_2,\dots,n_l\}\subseteq\mathcal{V}\setminus T$. It suffices to prove $f_p(S,T)>f_p(S',T)$ and $f_p(S,T')>f_p(S,T)$. Letting $S_j=S\cup\{v_1,v_2,\dots,v_j\}$, $j=1,2,\dots,r$, we have $S'=S_r\supsetneq S_{r-1}\supsetneq\cdots\supsetneq S_1\supsetneq S_0=S$. For every $j\in\{1,2,\dots,r\}$, it follows from $S_j=S_{j-1}\cup\{v_j\}$ that $d^i_{S_{j-1},T}-d^i_{S_j,T}>0$. Thus, from the definition of $f_p(S)$, we have $f_p(S_{j-1})>f_p(S_j)$. Consequently, $f_p(S)>f_p(S_1)>\cdots>f_p(S')$. Therefore, we know that if some followers are added into GI-set, the tracking error of system (2) will be decreased. Likewise, we can prove that if some followers are added into MI-set, the tracking error of system (2) will be increased. ■

Corollary 1. *For all $S\in 2^{\mathcal{V}}$, $f_p(\mathcal{V},T)\leq f_p(S,T)$ and the equation holds if and only if $S=\mathcal{V}$.*

Proof. For all $S\in 2^{\mathcal{V}}$, we have $S\subseteq\mathcal{V}$. Therefore, the proof is straightforward from Theorem 2. ■

Theorem 3. *Suppose that GL-weight θ_i is not less than ML-weight ϑ_i for every follower $i\in\mathcal{V}$. Let MI-set be the subset of GI-set. Then, all followers' tracking errors are not greater than $\frac{1}{2}|\rho_1-\rho_0|$.*

Proof. Without loss of generality, let $\rho_0>0$.

Firstly, we will prove the case of $\rho_1=-\rho_0$. According to Lemma 3, we have $\xi^i_{S,T}=\rho_0\mathbf{e}_i^T(\mathcal{L}+D_\theta D_x+D_\vartheta D_y)^{-1}(D_\theta x-D_\vartheta y)$. It follows from $S\supseteq T$ and $\theta-\vartheta\geq 0$ that $(D_\theta x-D_\vartheta y)\geq 0$. Together with $(\mathcal{L}+D_\theta D_x+D_\vartheta D_y)^{-1}>0$ and $\rho_0\mathbf{e}_i^T\geq 0$, we get $\xi^i_{S,T}\geq 0$. Thus, $d^i_{S,T}=\rho_0-\xi^i_{S,T}\leq\rho_0=\frac{1}{2}|\rho_1-\rho_0|$.

Secondly, if $\rho_1\neq-\rho_0$, we can translate the coordinate system so that its origin coincides with the middle point between ρ_0 and ρ_1. Consequently, system (2) is converted into the case of $\rho_1=-\rho_0$. ■

In the following context of this section, suppose that MI-set is given. Then, denote $f_p(S,T)$ as $f_p(S)$.

3.2. Minimizing $f_p(S)$ via selecting up to k GI-agents

Minimizing $f_p(S)$ by selecting up to k GI-agents is given by the optimization problem

$$\min_S f_p(S)\quad \text{s.t. } |S|\leq k. \tag{5}$$

Theorem 4. *Optimization problem (5) is equivalent to the combinatorial optimization problem*

$$\min_S f_p(S)\quad \text{s.t. } |S|=k. \tag{6}$$

Moreover, the optimal value of (6) is a strictly monotone decreasing function of k.

Proof. Denote the optimal solution to (5) as S^*. Suppose $|S^*|<k$. By Theorem 2, we known $f_p(S^*)<f_p(S^*\cup\{v\})$ where $v\in\mathcal{V}\setminus S^*$. Thus, it has a conflict with the fact that S^* is the optimal solution to (5). Therefore, $|S^*|=k$ which implies the optimal solution to (6) is the optimal solution to (5).

For two positive integers $k_1<k_2\in\{1,\dots,n\}$, denote the corresponding optimal solutions to (6) as S_1^* and S_2^*, respectively. For a set $\Lambda=\{v_1,v_2,\dots,v_{k_2-k_1}\}\subseteq\mathcal{V}\setminus S_1^*$, one has $f_p(S_1^*)<f_p(S_1^*\cup\Lambda)$. It follows from $|S_1^*\cup\Lambda|=k_2$ that $f_p(S_1^*\cup\Lambda)\leq f_p(S_2^*)$. Thus, $f_p(S_1^*)<f_p(S_2^*)$.

Therefore, optimization problem (5) is equivalent to combinatorial optimization problem (6) which is a strictly monotone decreasing function of k. ■

Remark 3. From

$$(\mathcal{L}+D_\theta D_x+D_\vartheta D_y)^{-1}=\frac{\operatorname{adj}(\mathcal{L}+D_\theta D_x+D_\vartheta D_y)}{\det(\mathcal{L}+D_\theta D_x+D_\vartheta D_y)},$$

we have $d^i_{S,T}=\frac{q_i(x)}{\det(x)}$, where

$$q_i(x)=c\mathbf{e}_i^T\operatorname{adj}(\mathcal{L}+D_\theta D_x+D_\vartheta D_y)D_\vartheta y$$

and

$$\det(x)=\det(\mathcal{L}+D_\theta D_x+D_\vartheta D_y).$$

It is shown that $q_i(x)$ and $\det(x)$ are two polynomials with degree $m(\leq n-1)$ and with the highest term $\Pi_{i=1}^n\theta_i x_i$, respectively. Since $f_p^p(S)=\frac{\Sigma_{i=1}^n q_i^p(x)}{\det^p(x)}$ ($p<\infty$), optimization problem (6) is equivalent to

$$\min_x \frac{\Sigma_{i=1}^n q_i^p(x)}{\overset{p}{\det}(x)}$$
$$\text{s.t. } \mathbf{1}_n^T x=k,$$
$$x_i\in\{0,1\},\quad i=1,2,\dots,n,$$

which is *NP*-hard. As a result, (6) is *NP*-hard for $p<\infty$. Similarly, (6) is also *NP*-hard for $p=\infty$.

Since (6) is *NP*-hard, it is hard to find the optimal solution for the case that the system contains large numbers of followers. However, we propose a *Greedy Algorithm for Selecting k GI-agents* to obtain a suboptimal solution to (6). The algorithm initializes $S=\emptyset$. At the jth iteration, the algorithm chooses $v_j\in\mathcal{V}\setminus S$ such that $f_p(S\cup\{v_j\})$ is minimized and sets $S=S\cup\{v_j\}$. The pseudocode description is

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Greedy Algorithm for Selecting k GI-agents: Algorithm for selecting up to k GI-agents to minimize the tracking error $f_p(S)$.

Input: Number of GI-agents k, the followers' interaction graph $\mathcal{G}$, MI-set T, GL-weight vector θ, ML-weight vector ϑ.
Output: GI-set S
Initialization: $S\leftarrow\emptyset, j\leftarrow 1$
while $(j<k)$
  $v_j\leftarrow\arg\min\{f_p(S\cup\{v\}):v\in\mathcal{V}\setminus S\}, S\leftarrow S\cup\{v_j\}, j\leftarrow j+1$
end while
return S

3.3. Minimizing the size of GI-set under an upper bound constraint on $f_p(S)$

In this subsection, we consider the problem of selecting minimum number of GI-agents under an upper bound constraint on $f_p(S)$. The problem can be described as

$$\min_S |S| \quad \text{s.t.}\ f_p(S) \le \alpha. \tag{7}$$

Remark 4. From Corollary 1, we have if $\alpha \le f_p(\mathcal{V})$, then optimization problem (7) is no solution.

Remark 5. Similar to the analysis in Remark 3, combinatorial optimization problem (7) is *NP*-hard.

Because (7) is *NP*-hard, an algorithm is proposed to solve a suboptimal solution of (7), namely *Algorithm to Select Minimal-size GI-set* (ASMG). S is initialized by $S = \emptyset$. At each iteration, the node $v^* \in \mathcal{V} \setminus S$ is selected such that $f_p(S \cup \{v^*\})$ is minimal and S is set by $S = S \cup \{v^*\}$. The pseudocode description of the algorithm is

Algorithm to Select Minimal-size GI-set: Algorithm for selecting the minimum-size GI-set S such that $f_p(S) \le \alpha$.

Input: Upper bound α, the followers' interaction graph $\mathcal{G}$, MI-set T, GL-weight vector θ, ML-weight vector ϑ.
Output: GI-set S
Initialization: $S \leftarrow \emptyset$
while $(f_p(S) > \alpha)$
  $v^* \leftarrow \arg\min\{f_p(S \cup \{v\}) : v \in \mathcal{V} \setminus S\}, S \leftarrow S \cup \{v^*\},$
end while
return S

4. Minimizing the opponent's influence via GL-weight assigning

In this section, suppose that $\mathcal{G}$, ϑ, S and T is known by navigational leader. Hence, the tracking error of system (2) can be reduced by assigning GL-weight vector.

4.1. Bounds of tracking error

In this section, the following assumption is given.

A3. For follower $i \in \mathcal{V}$, there exists a threshold $\bar{\theta}_i > 0$ for its GL-weight, i.e., $\theta_i \in [0, \bar{\theta}_i]$.

Denote $S = \{i_1, i_2, \dots, i_k\}$ and $\theta_S = D_x\theta$ where $i_1 < i_2 < \cdots < i_k$. Noticing that θ_S is a vector with θ_{i_r} in the i_r-th entry, $r = 1, 2, \dots, k$, and 0's elsewhere. Denote $\xi^i_{\tilde{\theta}_S} = \lim_{t\to\infty} \xi^i(t)$ where $\tilde{\theta}_S = [\theta_{i_1}, \theta_{i_2}, \dots, \theta_{i_k}]$. Recalling Lemma 3, one has $\xi^i_{\tilde{\theta}_S}$ is determined by $\tilde{\theta}_S$.

Definition 5. Define $d^i_{\tilde{\theta}_S} = |\xi^i_{\tilde{\theta}_S} - \rho_0|$ as the tracking error of follower i. Let $f_p(\tilde{\theta}_S) = \left\| d_{\tilde{\theta}_S} \right\|_p$ be the tracking error function of system (2), where $1 \le p \le \infty$ and $d_{\tilde{\theta}_S} = [d^1_{\tilde{\theta}_S}, \dots, d^n_{\tilde{\theta}_S}]^T$.

Similar to the proof of Theorem 1, we have

Theorem 5. *If GL-weight of one GI-agent is increased, then the tracking error of every follower and of system (2) will be decreased.*

Denote $\tilde{y} = D_\vartheta y$. It follows from Lemma 3 that

$$d_{\tilde{\theta}_S} = c(\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}\tilde{y}. \tag{8}$$

Owing to the fact that $d^i_{\tilde{\theta}_S}$ is a fraction of two polynomials of $\tilde{\theta}_S$, it is difficult to know whether $f_p(\tilde{\theta}_S)$ is a convex function or not. The following results give some convex relaxations.

Theorem 6. (1) *For $f_1(\tilde{\theta}_S)$, one has*

$$\frac{c}{\|\tilde{y}\|_\infty}\tilde{y}^T(\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}\tilde{y} \le f_1(\tilde{\theta}_S) \le c\|\tilde{y}\|_\infty \mathbf{1}_n^T(\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}\mathbf{1}_n,$$

where $\tilde{y}^T(\mathcal{L}+D_{\theta_S}+D_{\tilde{y}})^{-1}\tilde{y}$ and $\mathbf{1}_n^T(\mathcal{L}+D_{\theta_S}+D_{\tilde{y}})^{-1}\mathbf{1}_n$ are convex functions of $\tilde{\theta}_S$.

(2) *For $f_2(\tilde{\theta}_S)$, one has*

$$f_2(\tilde{\theta}_S) \le c\|\tilde{y}\|_2 \mathrm{tr}((\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}),$$

where $\mathrm{tr}((\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1})$ is a convex function of $\tilde{\theta}_S$.

Proof. We firstly prove (1). From $0 \le \tilde{y} \le \|\tilde{y}\|_\infty \mathbf{1}_n$ and $(\mathcal{L}+D_{\theta_S}+D_{\tilde{y}})^{-1} > 0$, we have

$$0 \le (\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}\tilde{y} \le \|\tilde{y}\|_\infty(\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}\mathbf{1}_n$$

and

$$\mathbf{1}_n(\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1} \ge \frac{1}{\|\tilde{y}\|_\infty}\tilde{y}^T(\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}.$$

It follows from $f_1(\tilde{\theta}_S) = c\mathbf{1}_n^T(\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}\tilde{y}$ that

$$\frac{c}{\|\tilde{y}\|_\infty}\tilde{y}^T(\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}\tilde{y} \le f_1(\tilde{\theta}_S) \le c\|\tilde{y}\|_\infty \mathbf{1}_n^T(\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}\mathbf{1}_n.$$

For an n-dimensional column vector $\mathbf{b} \ne 0$, $\mathbf{b}_n^T(\mathcal{L}+D_{\theta_S}+D_{\tilde{y}})^{-1}\mathbf{b}_n$ is the composition of a convex function $\mathbf{b}_n^T P^{-1}\mathbf{b}_n$ of a positive matrix P with an affine function $P = \mathcal{L} + D_{\theta_S} + D_{\tilde{y}}$. From [32], we know that $\mathbf{b}_n^T(\mathcal{L}+D_{\theta_S}+D_{\tilde{y}})^{-1}\mathbf{b}_n$ is a convex function of $\tilde{\theta}_S$. Therefore, we have $\tilde{y}^T(\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}\tilde{y}$ and $\mathbf{1}_n^T(\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}\mathbf{1}_n$ are convex functions of $\tilde{\theta}_S$.

Then we give the proof of (2). From the definition of $f_2(\theta_S)$, we get

$$f_2(\theta_S) = c\left[\tilde{y}^T(\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-2}\tilde{y}\right]^{\frac{1}{2}}.$$

Since $(\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}$ is positive define,

$$f_2(\theta_S) \le c\rho_{(\mathcal{L}+D_{\theta_S}+D_{\tilde{y}})^{-1}}\|\tilde{y}\|_2 \le c\|\tilde{y}\|_2 tr((\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}).$$

From [32], we know that $\mathrm{tr}(P^{-1})$ is a convex function of a positive matrix P. Moreover, we have $\mathcal{L} + D_{\theta_S} + D_{\tilde{y}}$ is an affine function. Hence, $\mathrm{tr}((\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1})$ is a convex function of θ_S. ■

4.2. Minimizing tracking error via assigning GL-weights

In this subsection, suppose that there is a budget on the sum of GL-weights, i.e., $\sum_{j=1}^k \theta_{i_r} \le \pi$. The problem of minimizing tracking error via assigning GL-weights is formulated by the optimization problem

$$\min_{\tilde{\theta}_S} f_p(\tilde{\theta}_S) \quad \text{s.t.}\ \sum_{j=1}^k \theta_{i_r} \le \pi, \quad \theta_{i_r} \in [0, \bar{\theta}_{i_r}],\ r = 1, 2, \dots, k. \tag{9}$$

It is not hard to see that the constraint set of (9) is convex and compact. Considering that $f_p(\tilde{\theta}_S)$ is continuous in the constraint set of (9), the optimal solution of (9) exists. By Theorem 6, we can straightly obtain the following results.

Theorem 7. *For $p = 1$, the optimal value of (9) has an upper bound $c\mu\|\tilde{y}\|_\infty$ and a lower bound $\frac{c\tau}{\|\tilde{y}\|_\infty}$, where μ is the optimal value of the convex optimization problem*

$$\begin{aligned} &\min_{\tilde{\theta}_S} \mathbf{1}_n^T(\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}\mathbf{1}_n \\ &\text{s.t. } \mathbf{1}_k^T\tilde{\theta}_S = \pi, \quad \theta_{i_r} \in [0, \bar{\theta}_{i_r}], \quad r = 1, 2, \dots, k, \end{aligned} \tag{10}$$

and τ is the optimal value of the convex optimization problem

$$\begin{aligned} &\min_{\tilde{\theta}_S} \tilde{y}^T(\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}\tilde{y} \\ &\text{s.t. } \mathbf{1}_k^T\tilde{\theta}_S = \pi, \quad \theta_{i_r} \in [0, \bar{\theta}_{i_r}], \quad r = 1, 2, \dots, k. \end{aligned} \tag{11}$$

Theorem 8. *For $p = 2$, the optimal value of (9) has an upper bound $c\gamma\|\tilde{y}\|_2$, where γ is the optimal value of the convex optimization problem*

$$\begin{aligned} &\min_{\tilde{\theta}_S} \operatorname{tr}((\mathcal{L} + D_{\theta_S} + D_{\tilde{y}})^{-1}) \\ &\text{s.t. } \mathbf{1}_k^T\tilde{\theta}_S = \pi, \quad \theta_{i_r} \in [0, \bar{\theta}_{i_r}], \quad r = 1, 2, \dots, k. \end{aligned} \tag{12}$$

Using the Schur complement, convex optimization problems (10) and (11) can be transformed into SDP problems

$$\begin{aligned} &\min_{z,\, \tilde{\theta}_S} z \\ &\text{s.t. } \begin{pmatrix} z & \mathbf{b}_n^T \\ \mathbf{b}_n & \mathcal{L} + D_{\theta_S} + D_{\tilde{y}} \end{pmatrix} \succeq 0 \\ &\mathbf{1}_k^T\tilde{\theta}_S = \pi, \quad \theta_{i_r} \in [0, \bar{\theta}_{i_r}], \quad r = 1, 2, \dots, k, \end{aligned} \tag{13}$$

where $\mathbf{b}_n = \mathbf{1}_n$ and $\mathbf{b}_n = \tilde{y}$, respectively. Likewise, convex optimization problem (12) can be transformed into an SDP problem

$$\begin{aligned} &\min_{Z,\, \tilde{\theta}_S} \operatorname{tr}(Z) \\ &\text{s.t. } \begin{pmatrix} Z & I_n \\ I_n & \mathcal{L} + D_{\theta_S} + D_{\tilde{y}} \end{pmatrix} \succeq 0 \\ &\mathbf{1}_k^T\tilde{\theta}_S = \pi, \quad \theta_{i_r} \in [0, \bar{\theta}_{i_r}], \quad r = 1, 2, \dots, k. \end{aligned} \tag{14}$$

SDP problems (13) and (14) can be solved efficiently using standard SDP solvers, such as SDPT3 [33] etc.

5. Simulations

In this section, numerical examples are performed to illustrate the effectiveness of our results in Sections 3 and 4. Consider a network with 100 randomly distributed nodes in a unit square which is depicted in Fig. 1. A pair of nodes communicates with each other if their distance is not greater than 0.2 unit. The edge-weights a_{ij} for each $(i, j) \in \mathcal{E}$ are uniformly selected from 1 to 10. The navigational information $\rho_0 = 0$ and the misinformation $\rho_1 = -1$.

Example 1. Assume that GL-weight vector θ, ML-weight vector ϑ and MI-set T are given. S is the decision variable. The MI-agents are depicted by red spots in Fig. 1. Max degree algorithms are always proposed to solve similar problems. In order to minimize the tracking error $f_\infty(S)$, max degree algorithm and greedy algorithm are used to solve S. Fig. 2 demonstrates that when the number of the GI-agents is varying from 1 to 50, the corresponding tracking errors under the solutions of (6) solved by two algorithms are decreasing. This result manifests the effectiveness of theoretical results in Theorems 2 and 4. We also observe from Fig. 2 that using greedy algorithm results in lower tracking error than employing max degree algorithm. This means that greedy algorithm is more efficient than max degree algorithm for this network.

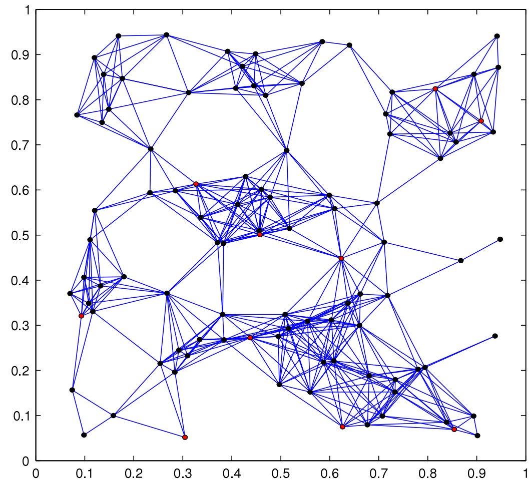

Fig. 1. The 100-nodes random network with 10 MI-agents (red spots). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

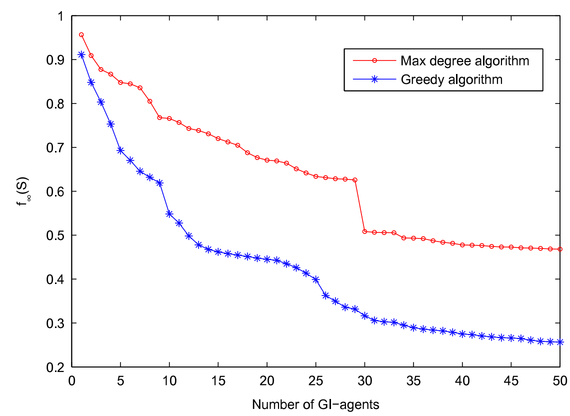

Fig. 2. Comparison of tracking errors by using two algorithms: Max degree algorithm and Greedy algorithm.

Example 2. Assume that MI-set T, GI-set S and DL-weight vector ϑ are given. GL-weight vector θ is the decision variable. We solve convex programming problem (11) to obtain the lower bound of the optimal solution to (9). And by solving (10), we compute the corresponding tracking error $f_1(\tilde{\theta}_S)$ as an upper bound of the optimal solution to (9). The results are shown in Fig. 3. We can observe that the upper bound and lower bound of $f_1(\tilde{\theta}_S)$ are decreasing when π varies from 1 to 50, which illustrates the effectiveness of theoretical results in Theorem 5.

6. Conclusions

Differences of interests produce conflicts. For the multi-agent system, agents with different interests may be opposite. In this paper, we consider the topology selection problem of the multi-agent system with two opposite leaders. We defined the tracking error for the system. It has been proved that the tracking error is decreasing with the increasing of GI-agents or the increasing of the GL-weights. For the case that the GI-agents were not preset, we formulated two combinatorial programming problems. One was to select up to k GI-agents to minimize tracking error. The other was to select minimum-size GI-set under an upper bound constraint of tracking error. Because both of two problems are NP-hard, it is difficult to solve them when the system contains large numbers of followers. Consequently, we proposed two algorithms to obtain their suboptimal solutions respectively. For the case that

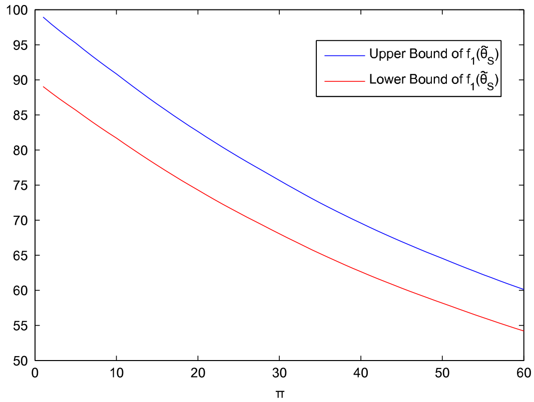

Fig. 3. The upper and the lower bounds of $f_1(\tilde{\theta}_S)$.

GI-agents were fixed, we investigated the problem of designing GL-weights to minimize the tracking error. Three convex optimization problems were formulated to evaluate the upper and the lower bounds of the minimal tracking error. Finally, two numerical examples were given to illustrate the effectiveness of the established results. Future work may consider this problem for MASs under the changing MI-sets or with measurement noises, etc.

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