Automatica 73 (2016) 200–206

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Equilibrium topology of multi-agent systems with two leaders: A zero-sum game perspective☆

Jingying Ma[a], Yuanshi Zheng[a], Bin Wu[b], Long Wang[c]

[a] Center for Complex Systems, School of Mechano-electronic Engineering, Xidian University, Xi'an 710071, PR China

[b] Department of Evolutionary Theory, Max-Planck-Institute for Evolutionary Biology, August-Thienemann-Str. 2, 24306 Plön, Germany

[c] Center for Systems and Control, College of Engineering, Peking University, Beijing 100871, PR China

ARTICLE INFO

Article history:
Received 5 November 2014
Received in revised form
6 December 2015
Accepted 27 June 2016

Keywords:
Multi-agent systems
Zero-sum game
Containment control
Equilibrium topology

ABSTRACT

It is typical in containment control problems to assume that there is no conflict of interest among leaders. In this paper, we consider the situation where there is conflict between leaders; namely, the leaders compete to attract followers. The strategies of each leader are defined by choosing at most k followers to propagate their information. Then, we formulate a standard two-player zero-sum game by using graph theory and matrix theory. We further prove that each player will choose exactly k followers when the game achieves a Nash equilibrium. It is noteworthy that the interaction graph here is generated from the conflict between leaders and then the Nash equilibrium point of the game corresponds to the equilibrium topology. For the case of choosing one follower, a necessary and sufficient condition for an interaction graph to be the equilibrium topology is derived. Moreover, we can obtain the equilibrium topology directly if followers' interaction graph is a circulant graph or a graph with a center vertex. Simulation examples are provided to validate the effectiveness of the theoretical results.

1. Introduction

In recent years, distributed control of multi-agent systems (MASs) has attracted intensive attention in the scientific community. This is due to its diverse applications in many areas, such as formation control in unmanned aerial vehicles (Gu, 2008), flocking in biology (Jing, Zheng, & Wang, 2014), rendezvous problem of mobile autonomous robots (Xiao, Wang, & Chen, 2012), and so on.

Consensus seeking is a basic problem of MASs which aims to design appropriate distributed protocols or algorithms such that a group of agents can converge to the same state. There have been extensive studies and results under various circumstances, to name but a few, consensus problems in networks of dynamic agents with switching topology (Olfati-Saber & Murray, 2004), second-order consensus (Xie & Wang, 2007), consensus of heterogeneous MASs (Zheng, Zhu, & Wang, 2011), consensus of switched MASs (Zheng & Wang, 2016) and finite-time consensus (Wang & Xiao, 2010; Zheng & Wang, 2012), etc.

Leaders are ubiquitous in nature. Therefore, researchers have paid great attention to problems for MASs with leaders, such as leader-following consensus (Ma, Zheng, & Wang, 2015; Ni & Cheng, 2010), containment control (Ji, Ferrari-Trecate, Egerstedt, & Buffa, 2008; Liu, Xie, & Wang, 2012; Notarstefano, Egerstedt, & Haque, 2011; Zheng & Wang, 2014) and controllability analysis (Guan, Ji, Zhang, & Wang, 2013; Rahmani, Ji, Mesbahi, & Egerstedt, 2009; Wang, Jiang, Xie, & Ji, 2009). Containment control of MASs means that the states of the followers will converge to the convex hull spanned by the leaders. Ji et al. (2008) and Notarstefano et al. (2011) investigated containment control with fixed undirected topology and switching topologies, respectively. Liu et al. (2012) obtained some necessary and sufficient conditions for solving containment control of MASs with directed networks.

In the multi-agent system, each agent is an individual who exchanges information with its neighbors and then makes decision independently. If we further define the utility of agents and assume that individuals adjust their behaviors by promoting utility, game theory can be used to study distributed multi-agent coordination. Bauso, Giarré, and Pesenti (2006) proposed a game theoretic interpretation of consensus problems as mechanism

☆ This work was supported by NSFC (Grant Nos. 61533001, 61375120, 61304160 and 61563043), Max-Planck Society, and the Fundamental Research Funds for the Central Universities (Grant No. JB160419). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Claudio De Persis under the direction of Editor Christos G. Cassandras.

E-mail addresses: majy1980@126.com (J. Ma), zhengyuanshi2005@163.com (Y. Zheng), bin.wu@evolbio.mpg.de (B. Wu), longwang@pku.edu.cn (L. Wang).

http://dx.doi.org/10.1016/j.automatica.2016.07.005
0005-1098/

design problems. Gu (2008) investigated formation control via linear–quadratic Nash differential game. Moreover, cooperative game theory is utilized to ensure team cooperation by considering a combination of individual cost as the team cost in Semsar-Kazerooni and Khorasani (2009). For leader-following MASs, the notion of graphical game was formulated in Vamvoudakis, Lewis, and Hudas (2012). Gharesifard and Cortés (2013) considered the distributed convergence to Nash equilibrium for two networks engaged in a strategic scenario.

Different from the above results, we consider the conflict between leaders for multi-agent systems and formulate a type of game. For multi-agent systems with two leaders, if the followers' interaction subgraph is undirected and connected, then each follower will converge to a convex combination of two leaders' states (Liu et al., 2012). Based on this result, we define the average distance to the followers as the payoff function of each leader. Every leader's strategy is to select at most $k\,(\geq 1)$ followers to connect with. Then, we formulate a noncooperative game in which each leader independently chooses its strategy to minimize payoff function. Noticing that two leaders' decisions will determine the interaction topologies of the system, the Nash equilibrium point corresponds to the equilibrium topology of the system. The main contribution of this paper is threefold. Firstly, we consider the conflict between leaders and formulate it as a zero-sum game. Secondly, we prove that each leader will choose as many as possible followers to minimize its payoff. Finally, for the case selecting one follower, we derive a necessary and sufficient condition for an interaction graph to be the equilibrium topology. Moreover, if followers' interaction graph is a circulant graph or a graph with a center vertex, then the system's equilibrium topology is obtained.

This paper is organized as follows. In Section 2, we introduce some notions and propose our problem. In Section 3, we present our main results and in Section 4, numerical simulations are given to illustrate the effectiveness of the theoretical results. Some conclusions are drawn in Section 5.

Notation: Throughout this paper, the following notations will be used: $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices. Denote by $\mathbf{1}_n$ (or $\mathbf{0}_n$) the column vector with all entries equal to one (or all zeros). I_n is an n-dimensional identity matrix. For a column vector $\mathbf{b} = [b_1, b_2, \dots, b_n]^T$, $\mathrm{diag}\{\mathbf{b}\}$ is a diagonal matrix with b_i on its diagonal and $\|\mathbf{b}\|_1 = \sum_{i=1}^{n} |b_i|$ is 1-norm of $\mathbf{b}$. For a matrix $A \in \mathbb{R}^{n\times n}$, $\mathrm{adj}A$ and $\det A$ are the adjugate and the determinant of A, respectively. Denote $A_{(i_1,i_2,\dots,i_k;\, j_1,j_2,\dots,j_k)}$ as the matrix obtained by deleting rows $i_1, i_2, \dots, i_k$ and columns $j_1, j_2, \dots, j_k$ from A. $A_{[i_1,i_2,\dots,i_k]}$ is the $k \times k$ principal submatrix of A by keeping rows and columns $i_1, i_2, \dots, i_k$. $\mathcal{I}_n = \{1, \dots, n\}$ is an index set. $|S|$ is the cardinality of a set S. For two sets S_1 and S_2, denote $S_1 \times S_2$ as the Cartesian product and $S_1 \setminus S_2 = S_1 - S_2$. Let $\mathbf{e}_i$ denote the canonical vector with a 1 in the ith entry and 0's elsewhere. The notation $A \Leftrightarrow B$ means that A holds true if and only if B holds true.

2. Preliminaries

2.1. Graph theory

Let $G = \{V, E\}$ be an undirected graph consisting of a vertex set $V = \{1, 2, \dots, n\}$ and an edge set $E = \{(i, j) \in V \times V\}$. A graph $G_W = \{W, E(W)\}$ is called an induced subgraph of G if it is obtained by deleting some vertexes from V, along with any edges that contain a deleted vertex. The adjacency matrix A of G is a symmetric matrix such that for all $i \in V$, $a_{ii} = 0$ and for all $i \neq j$, $(i, j) \in E \Leftrightarrow a_{ij} = a_{ji} = 1$, while $a_{ij} = 0$ otherwise. For $(i, j) \in V \times V$, if G is a connected graph, then there exists an integer $z \geq 1$ such that the ijth entry of A^z is positive. The neighbor set of the vertex i is $\mathcal{N}_i = \{j : (i, j) \in E\}$. The degree matrix $D = \mathrm{diag}\{d_1, d_2, \dots, d_n\} \in \mathbb{R}^{n\times n}$ is a diagonal matrix with $d_i = \sum_{j\in\mathcal{N}_i} a_{ij}$. The Laplacian matrix $L = D - A$. A vertex i is a center vertex if it connects all the other vertexes, i.e., $\mathcal{N}_i = V \setminus \{i\}$. A graph is circulant when the adjacency matrix is a circulant matrix. A connected graph is called a circle if every vertex has exactly two neighbors. A tree is a connected graph where all its subgraphs are not circles. For a connected graph G, a subgraph $G_S = (V, E_S)$ is called a spanning tree of G if $E_S \subseteq E$ and G_S is a tree.

Lemma 1 (Godsil & Royal, 2001). *For a graph G, $\det L = 0$ and $\mathrm{adj}L = \tau(G)\mathbf{1}_n\mathbf{1}_n^T$, where $\tau(G)$ is the number of spanning trees in the graph G.*

Lemma 2. *Suppose that graph G is connected. Then, every principal submatrix of L is positive definite and the inverse matrix of it is a nonnegative matrix.*

Proof. Let $W = \{i_1, i_2, \dots, i_k\}$ be a subset of V. $G_W = \{W, E(W)\}$ is the induced subgraph of G where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Let L_W and A_W be the Laplacian matrix and the adjacency matrix of G_W, respectively. Denoting $\theta_{i_m} = -\sum_{j\notin W} l_{i_m j}$, $m = 1, 2, \dots, k$, we have $A_W = A_{[i_1,i_2,\dots,i_k]}$ and $L_{[i_1,i_2,\dots,i_k]} = L_W + \mathrm{diag}\{\theta_{i_1}, \dots, \theta_{i_k}\}$. Therefore, by Lemma 4 in Ni and Cheng (2010), $L_{[i_1,i_2,\dots,i_k]}$ is positive definite. Moreover, we have

$$L_{[i_1,i_2,\dots,i_k]} = \mathrm{diag}\{l_{i_1i_1}, \dots, l_{i_ki_k}\} - A_W = \eta\left(I_n - \left(\Delta + \frac{A_W}{\eta}\right)\right),$$

where $\eta = \max_{1\leq m\leq k} l_{i_mi_m}$ and $\Delta = \frac{1}{\eta}\mathrm{diag}\{\eta - l_{i_1i_1}, \dots, \eta - l_{i_ki_k}\}$. It is easy to find that $\Delta + \frac{A_W}{\eta}$ is a nonnegative matrix with spectral radius $\rho < 1$. Hence, it follows that $L^{-1}_{[i_1,i_2,\dots,i_k]} = \frac{1}{\eta}\sum_{k=0}^{\infty} \left(\Delta + \frac{A_W}{\eta}\right)^k$. Consequently, we have $L^{-1}_{[i_1,i_2,\dots,i_k]}$ is a nonnegative matrix. ■

2.2. Two-player zero-sum games

Consider a zero-sum game of two players, to be referred to as player P_1 and player P_2, in which each player has finite alternatives. Denote the set of strategies of P_1 and P_2 as $S_1 = \{s_1, s_2, \dots, s_m\}$ and $S_2 = \{\hat{s}_1, \hat{s}_2, \dots, \hat{s}_n\}$, respectively. A strategy pair $(s_i, \hat{s}_j) \in S_1 \times S_2$ means that P_1 chooses the strategy s_i and P_2 chooses the strategy $\hat{s}_j$. For $(s_i, \hat{s}_j)$, the payoff of P_1 is $-a_{ij}$ while that of P_2 is a_{ij}. $A = \{a_{ij}\}_{m\times n}$ is the outcome of the game. This type of two-player zero-sum game is called a matrix game A. In a matrix game A, P_1 wants to minimize the outcome of the game, while P_2 seeks to maximize it, by independent decisions. Under such an incentive, P_1 is forced to pick its security strategy s_{i^*} satisfying $\overline{V}(A) \triangleq \max_j a_{i^*j} = \min_i \max_j a_{ij}$. Similarly, P_2 will choose a security strategy $\hat{s}_{j^*}$ determined by $\underline{V}(A) \triangleq \min_i a_{ij^*} = \max_j \min_i a_{ij}$. Denote S_1^* and S_2^* as the set of the security strategies of P_1 and P_2, respectively.

Definition 3 (Basar & Olsder, 1995). For a given $(m \times n)$ matrix game $A = \{a_{ij}\}$, if a strategy pair $(s_{i^*}, \hat{s}_{j^*})$ that satisfies $a_{i^*j} \leq a_{i^*j^*} \leq a_{ij^*}$ for all $i \in \mathcal{I}_m$ and $j \in \mathcal{I}_n$, then it is said that the matrix game has a Nash equilibrium point in pure strategies. The corresponding outcome $a_{i^*j^*}$ of the game is called the Nash equilibrium outcome denoted by $V(A)$.

Lemma 4 (Basar & Olsder, 1995). *Let $A = \{a_{ij}\}$ denote an $(m \times n)$ matrix game with $\underline{V}(A) = \overline{V}(A)$. Then, (1) A has a (pure) Nash equilibrium point, (2) the strategy pair $(s_i, \hat{s}_j)$ is a Nash equilibrium point for A if and only if $s_i \in S_1^*$ and $\hat{s}_j \in S_2^*$, (3) $V(A)$ is uniquely given by $\underline{V}(A) = \overline{V}(A)$.*

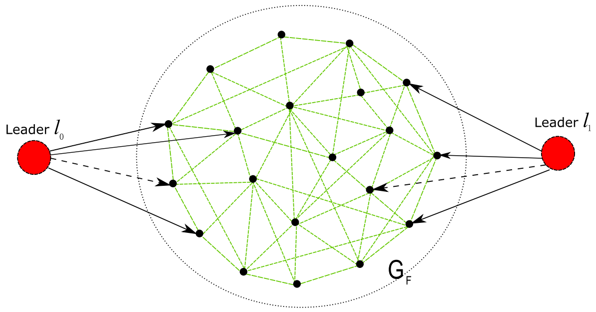


Fig. 1. An interaction graph $\tilde{G}(s_i, s_j)$ determined by the strategy pair (s_i, s_j).

2.3. Problem statement

Consider a multi-agent system consisting of two leaders and n followers. Let the two leaders be l_0 and l_1. Denote the set of followers as $V = \{1, \dots, n\}$. The interaction of the followers is described by an undirected graph $G_F = (V, E)$. The following assumption is given throughout this paper.

Assumption 1 (*Connectivity*). G_F is connected.

The leaders l_0 and l_1 keep static states $y_0,\ y_1 \in \mathbb{R}$ $(y_0 \neq y_1)$, respectively. The state of follower $i \in V$ is $x_i(t) \in \mathbb{R}$. The dynamics of $x_i(t)$ is described as

$$\dot{x}_i = \sum_{j \in 1}^{n} a_{ij}(x_j - x_i) + b_i(y_0 - x_i) + d_i(y_1 - x_i)$$

where

$$b_i = \begin{cases} 1, & i \text{ is connected to } l_0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$d_i = \begin{cases} 1, & i \text{ is connected to } l_1 \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbf{b} = [b_1, \dots, b_n]^T$, $\mathbf{d} = [d_1, \dots, d_n]^T$ and $X_f(t) = [x_1(t), \dots, x_n(t)]^T$. Then, it follows that

$$\dot{X}_f(t) = -(L + \operatorname{diag}\{\mathbf{b} + \mathbf{d}\})X_f(t) + \mathbf{b}y_0 + \mathbf{d}y_1, \tag{1}$$

where L is the Laplacian matrix of G_F. From Liu et al. (2012), we have

Lemma 5. *Suppose that each leader connects to at least one agent in G_F (i.e., $\mathbf{b} \neq \mathbf{0}_n$ and $\mathbf{d} \neq \mathbf{0}_n$) and Assumption 1 holds. Then, $X_f(t)$ will converge to $\lim_{t\to\infty} X_f(t) = \alpha y_0 + \beta y_1$ where*

$$\begin{aligned} \alpha &= [\alpha_1, \alpha_2, \dots, \alpha_n]^T = (L + \operatorname{diag}\{\mathbf{b} + \mathbf{d}\})^{-1}\mathbf{b}, \\ \beta &= [\beta_1, \beta_2, \dots, \beta_n]^T = (L + \operatorname{diag}\{\mathbf{b} + \mathbf{d}\})^{-1}\mathbf{d}, \end{aligned} \tag{2}$$

$\alpha_i + \beta_i = 1$, $\alpha_i > 0$ and $\beta_i > 0$ for all $i \in V$.

Consider the following game regarding multi-agent system (1) (see Fig. 1):

Players: Let l_0 and l_1 be two players.

Strategies: Each player can select at most k $(1 \le k \le n)$ followers from V to connect with, i.e., the set of strategies of each player is $S = \{s_j = (a_1, a_2, \dots, a_n)^T \mid a_i \in \{0, 1\}, \sum_{i=1}^{n} a_i \le k\}$. Obviously, S is a finite set and $|S| = \sum_{m=1}^{k} C_n^m \triangleq N$. Then, let $S = \{s_1, s_2, \dots, s_N\}$.

Payoff: The goal of each player is to steer all followers to move towards itself as closely as possible. As a result, the payoff of each leader can be described as the average distance between the followers and itself. Then, player $l_r (r \in \{0,\ 1\})$ wants to choose a strategy satisfying $\min_{s_i \in S} U_r(s_i, s_j) \triangleq \frac{1}{n}\sum_{m=1}^{n} |\lim_{t\to\infty} x_m(t) - y_r|$.

Remark 1. This game might be used to illuminate some phenomena about social networks which may be modeled by graph theory (Altafini & Lini, 2015; Hegselmann & Krause, 2002). Consider two companies sell similar products in a market. These companies are the leaders and the followers are potential customers. The companies can employ some "spokesmen" from these followers to promote their product. Then, in order to sell more products, each company wants to choose those followers who are more influential than their peers.

3. Main results

3.1. Problem reformulation

In this subsection, we will reformulate the above problem as a zero-sum game.

According to Lemma 5, let

$$u_{ij} = \sum_{m=1}^{n} \frac{\beta_m}{n} = \frac{1}{n}\mathbf{1}_n^T(L + \operatorname{diag}\{s_i + s_j\})^{-1}s_j. \tag{3}$$

It follows that

$$U_0(s_i, s_j) = |y_1 - y_0|u_{ij}, \qquad U_1(s_i, s_j) = |y_1 - y_0|(1 - u_{ij}) \tag{4}$$

and $U_0(s_i, s_j) + U_1(s_i, s_j) = |y_1 - y_0|$ for all $(s_i, s_j) \in S \times S$. Since the two leaders' states are static, $|y_1 - y_0|$ is constant, which produces a conflict between two leaders: the gain of l_1 is exactly equal to the loss of l_0. There is no way a mutually beneficial coalition can be built and consequently the game is noncooperative. Moreover, considering $\min_{s_i \in S} U_0(s_i, s_j) = |y_1 - y_0| \min_{s_i \in S} u_{ij}$, and $\min_{s_j \in S} U_1(s_i, s_j) = |y_1 - y_0| \left(1 - \max_{s_j \in S} u_{ij}\right)$, this game can be reformulated as a two-player zero-sum game where l_0 attempts to minimize u_{ij} while l_1 intends to maximize it.

Definition 6. For a connected graph G_F, there are two players, l_0 and l_1. The player l_0 wants to pick a strategy $s_i \in S$ satisfying $\min_{s_i \in S} \max_{s_j \in S} u_{ij}$, and the player l_1 seeks to choose a strategy $s_j \in S$ such that $\max_{s_j \in S} \min_{s_i \in S} u_{ij}$. This matrix game U is denoted as $\mathcal{G}^k(G_F)$, where $U = \{u_{ij}\}_{N \times N}$, $s_i, s_j \in S$.

Remark 2. It should be mentioned that the game is formulated as a zero-sum game since there are only two players. For multi-agent systems with three leaders, the sum of the leaders' payoffs may be variable and then the game may be cooperative and non-zero-sum.

For system (1), a strategy pair (s_i, s_j) corresponds an interaction graph $\tilde{G}(s_i, s_j)$ which consists of follower interaction graph G_F, two leaders l_0 and l_1 and several directed edges from the leaders to the followers (see Fig. 1).

Definition 7. For $\mathcal{G}^k(G_F)$, if a strategy pair (s_{i^*}, s_{j^*}) is a Nash equilibrium point, i.e., $u_{i^*j} \le u_{i^*j^*} \le u_{ij^*}$, $i, j \in V$. It is said that $\tilde{G}(s_{i^*}, s_{j^*})$ is the equilibrium topology of multi-agent system (1).

Property 8. *For $\mathcal{G}^k(G_F)$, one has*

(1) *if the two players l_0 and l_1 choose the same strategy, then all followers converge to $\frac{y_0+y_1}{2}$;*

(2) *for all* $(s_i, s_j) \in S \times S$, $u_{ij} < (= or\ >) \frac{1}{2}$ *if and only if* $U_0(s_i, s_j) < (= or\ >) U_1(s_i, s_j)$. *Moreover,* $U + U^T = \mathbf{1}_N^T \mathbf{1}_N$, $\overline{V}(U) \geq \frac{1}{2}$ *and* $\underline{V}(U) \leq \frac{1}{2}$;

(3) *if there is a strategy* s_{i^*} *such that* $u_{i^*j} \leq \frac{1}{2}$ *for all* $j \in V$, *then* $\overline{V}(U) = \underline{V}(U) = \frac{1}{2}$.

Proof. The proof is straightforward from the definition of U, Eq. (4) and Lemma 5. ■

Let S_0^* and S_1^* be the set of security strategies of l_0 and l_1, respectively. Then, one can obtain the following results.

Theorem 9. *For* $\mathcal{G}^k(G_F)$, *one has* $S_0^* = S_1^* = S^*$. *Moreover, suppose* $\overline{V}(U) = \frac{1}{2}$, *then* (s_i, s_j) *is a Nash equilibrium point if and only if* $(s_i, s_j) \in S^* \times S^*$.

Proof. If $s_{i^*} \in S_0^*$, $\max_j u_{i^*j} = \underline{V}(U) \leq \max_j u_{ij}$. On the other hand, by Property 8, we have $\max_j u_{ij} = \max_j(1 - u_{ji}) = 1 - \min_j u_{ji}$ for all $i \in \mathcal{I}_N$. Therefore, $\overline{V}(U) = \min_j u_{ji^*} \geq \min_j u_{ji}$. That is to say $s_{i^*} \in S_1^*$ and $\underline{V}(U) = 1 - \overline{V}(U)$. Similarly, if $s_{i^*} \in S_1^*$, we have $s_{i^*} \in S_0^*$. Therefore, $S_0^* = S_1^* = S^*$. If $\overline{V}(U) = \frac{1}{2}$, then $\underline{V}(U) = \frac{1}{2}$. By Lemma 4, (s_i, s_j) is a Nash equilibria point if and only if $(s_i, s_j) \in S^* \times S^*$. ■

Remark 3. By Theorem 9, when (s_{i_1}, s_{j_1}) and (s_{i_2}, s_{j_2}) are Nash equilibrium points, (s_i, s_j) is a Nash equilibrium for any $i, j \in \{i_1, i_2, j_1, j_2\}$. Hence, in the case of multiple Nash equilibrium points, each player does not need to know which security strategy its opponent will use in the game, since all such strategies are in equilibrium and they yield the same value. Therefore, it is not necessary to solve all Nash equilibrium points. As a result, the aim of each player is to seek a strategy s_{i^*} satisfying $u_{i^*j} \leq \frac{1}{2}$ for all $s_j \in S$.

Theorem 10. *Suppose that a strategy pair is a Nash equilibrium. Then, each strategy of it contains exactly k followers.*

Proof. We reason by contradiction. Suppose that (s_{i^*}, s_{j^*}) is a Nash equilibrium where s_{i^*} contains r follower $f_1, f_2, \ldots, f_r \in V$, $r < k$. It follows that $s_{i^*} = \mathbf{e}_{f_1} + \mathbf{e}_{f_2} + \cdots + \mathbf{e}_{f_r}$ and $u_{i^*j^*} = \min_{s_i \in S} u_{ij^*}$. Let $s_{i'} = s_{i^*} + \mathbf{e}_{f_{r+1}}$ where $f_{r+1} \in V \setminus \{f_1, f_2, \ldots, f_r\}$. According to (3), $u_{i^*j^*} = \frac{1}{n}\mathbf{1}_n^T L_1^{-1} s_{j^*}$ and $u_{i'j^*} = \frac{1}{n}\mathbf{1}_n^T (L_1 + \text{diag}\{\mathbf{e}_{f_{r+1}}\})^{-1} s_{j^*}$ where $L_1 = L + \text{diag}\{s_{i^*} + s_{j^*}\}$. Considering $(L_1 + \text{diag}\{\mathbf{e}_{f_{r+1}}\})^{-1} = L_1^{-1} - \frac{L_1^{-1}\mathbf{e}_{f_{r+1}}\mathbf{e}_{f_{r+1}}^T L_1^{-1}}{1+\mathbf{e}_{f_{r+1}}^T L_1^{-1}\mathbf{e}_{f_{r+1}}}$, one has

$$u_{i^*j^*} - u_{i'j^*} = \frac{1}{n}\mathbf{1}_n^T \frac{L_1^{-1}\mathbf{e}_{f_{r+1}}\mathbf{e}_{f_{r+1}}^T L_1^{-1}}{1 + \mathbf{e}_{f_{r+1}}^T L_1^{-1}\mathbf{e}_{f_{r+1}}} s_{j^*}. \tag{5}$$

Denote η be the maximum number among the diagonal entries of L_1. Similar to the proof of Lemma 2, we can obtain $L_1^{-1} = \frac{1}{\eta}\sum_{z=0}^{\infty}\left(\Delta + \frac{A}{\eta}\right)^z$ where A is the adjacency matrix of G_f and Δ is negative. For every $(n_1, n_2) \in V \times V$, since G_f is connected, there exists $z \geq 1$ such that the (n_1, n_2) entry of A^z is positive. Therefore, L_1^{-1} is a positive matrix, i.e., each entry of L_1^{-1} is positive. Hence, it follows from (5) that $u_{i^*j^*} > u_{i'j^*}$ which conflicts with the fact that $u_{i^*j^*} = \min_{s_i \in S} u_{ij^*}$. Therefore, s_{i^*} contains k followers. By Property 8, we have $u_{i^*j^*} = 1 - u_{j^*i^*}$. Likewise, we can prove s_{j^*} contains k followers. ■

Remark 4. Theorem 10 implies that each leader will choose as many as possible followers to minimize its own payoff. As a result, in order to obtain the equilibrium topology of the game, one only needs to compute all u_{ij} satisfying $\mathbf{1}_n^T s_i = \mathbf{1}_n^T s_j = k$ instead of computing the matrix U.

Remark 5. The results in this subsection can be extended to the case where G_F is a weighted undirected graph.

3.2. Special case: $k = 1$

In this subsection, we investigate game $\mathcal{G}^1(G_F)$. Since both leaders connect with one follower, they have n alternatives, i.e., the strategy set can be denoted as $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$. In order to obtain the equilibrium topology, it is necessary to know whether the inequality $u_{ij} \leq \frac{1}{2}$ holds or not.

Lemma 11. *For game* $\mathcal{G}^1(G_F)$, $u_{ij} < (=or >) \frac{1}{2}$ *if and only if*

$$\|(L + \text{diag}\{\mathbf{e}_i\})^{-1}\mathbf{e}_j\|_1 < (=or >) \|(L + \text{diag}\{\mathbf{e}_j\})^{-1}\mathbf{e}_i\|_1.$$

Proof. By Property 8, we have $u_{ij} < \frac{1}{2} \Leftrightarrow U_0(s_i, s_j) < U_1(s_i, s_j)$. Combining (2) and (4), we get

$$U_0(s_i, s_j) < U_1(s_i, s_j) \Leftrightarrow$$
$$\mathbf{1}_n^T(L + \text{diag}\{\mathbf{e}_i + \mathbf{e}_j\})^{-1}\mathbf{e}_j < \mathbf{1}_n^T(L + \text{diag}\{\mathbf{e}_i + \mathbf{e}_j\})^{-1}\mathbf{e}_i.$$

It follows from $(L + \text{diag}\{\mathbf{e}_i + \mathbf{e}_j\})^{-1} = \frac{\text{adj}(L+\text{diag}\{\mathbf{e}_i+\mathbf{e}_j\})}{\det(L+\text{diag}\{\mathbf{e}_i+\mathbf{e}_j\})}$ that

$$\mathbf{1}_n^T(L + \text{diag}\{\mathbf{e}_i + \mathbf{e}_j\})^{-1}\mathbf{e}_j < \mathbf{1}_n^T(L + \text{diag}\{\mathbf{e}_i + \mathbf{e}_j\})^{-1}\mathbf{e}_i \Leftrightarrow$$
$$\mathbf{1}_n^T\text{adj}(L + \text{diag}\{\mathbf{e}_i + \mathbf{e}_j\})\mathbf{e}_j < \mathbf{1}_n^T\text{adj}(L + \text{diag}\{\mathbf{e}_i + \mathbf{e}_j\})\mathbf{e}_i.$$

Noticing that

$$\text{adj}(L + \text{diag}\{\mathbf{e}_i + \mathbf{e}_j\})\mathbf{e}_j = \text{adj}(L + \text{diag}\{\mathbf{e}_i\})\mathbf{e}_j, \tag{6}$$

one has

$$\mathbf{1}_n^T\text{adj}(L + \text{diag}\{\mathbf{e}_i + \mathbf{e}_j\})\mathbf{e}_j < \mathbf{1}_n^T\text{adj}(L + \text{diag}\{\mathbf{e}_i + \mathbf{e}_j\})\mathbf{e}_i$$
$$\Leftrightarrow \mathbf{1}_n^T\text{adj}(L + \text{diag}\{\mathbf{e}_i\})\mathbf{e}_j < \mathbf{1}_n^T\text{adj}(L + \text{diag}\{\mathbf{e}_j\})\mathbf{e}_i.$$

By using Laplace expansion along column i, it follows from Lemma 1 that

$$\det(L + \text{diag}\{\mathbf{e}_i\}) = \det L + \mathbf{e}_i^T\text{adj}L\,\mathbf{e}_i = \tau(G_F). \tag{7}$$

Likewise, we also have $\det(L + \text{diag}\{\mathbf{e}_j\}) = \tau(G_F)$. Thus,

$$\mathbf{1}_n^T\text{adj}(L + \text{diag}\{\mathbf{e}_i\})\mathbf{e}_j < \mathbf{1}_n^T\text{adj}(L + \text{diag}\{\mathbf{e}_j\})\mathbf{e}_i \Leftrightarrow$$
$$\mathbf{1}_n^T(L + \text{diag}\{\mathbf{e}_i\})^{-1}\mathbf{e}_j < \mathbf{1}_n^T(L + \text{diag}\{\mathbf{e}_j\})^{-1}\mathbf{e}_i.$$

Denote $\bar{L}_i = \begin{pmatrix} 1 & -\mathbf{e}_i^T \\ -\mathbf{e}_i & L + \text{diag}\{\mathbf{e}_i\}\end{pmatrix}$. Due to G_F being connected, $\bar{L}_i$ is also the Laplacian matrix of a connected graph with an additional vertex 0 and an edge $(0, i)$ to G_F. Therefore, by Lemma 2, $(L + \text{diag}\{\mathbf{e}_i\})^{-1}$ is a nonnegative matrix. Likewise, $(L + \text{diag}\{\mathbf{e}_j\})^{-1}$ is also nonnegative. Then, we get $\mathbf{1}_n^T(L + \text{diag}\{\mathbf{e}_i\})^{-1}\mathbf{e}_j = \|(L + \text{diag}\{\mathbf{e}_i\})^{-1}\mathbf{e}_j\|_1$ and $\mathbf{1}_n^T(L + \text{diag}\{\mathbf{e}_j\})^{-1}\mathbf{e}_i = \|(L + \text{diag}\{\mathbf{e}_j\})^{-1}\mathbf{e}_i\|_1$. Thus,

$$u_{ij} < \frac{1}{2} \Leftrightarrow \|(L + \text{diag}\{\mathbf{e}_i\})^{-1}\mathbf{e}_j\|_1 < \|(L + \text{diag}\{\mathbf{e}_j\})^{-1}\mathbf{e}_i\|_1.$$

Similarly, we can prove the case of "=" and ">". ■

Let

$$S_e = \{\mathbf{e}_{i^*} | \|(L + \text{diag}\{\mathbf{e}_{i^*}\})^{-1}\mathbf{e}_k\|_1 \leq \|(L + \text{diag}\{\mathbf{e}_k\})^{-1}\mathbf{e}_{i^*}\|_1,$$
$$\mathbf{e}_{i^*} \in S,\ \mathbf{e}_k \in S\}.$$

By Lemma 11, we can derive a necessary and sufficient condition for an interaction graph $\tilde{G}(\mathbf{e}_i, \mathbf{e}_j)$ to be the equilibrium topology.

Theorem 12. *For game* $\mathcal{G}^1(G_F)$, $\tilde{G}(\mathbf{e}_{i^*}, \mathbf{e}_{j^*})$ *is the equilibrium topology if and only if* $\mathbf{e}_{i^*} \in S_e$ *and* $\mathbf{e}_{j^*} \in S_e$.

Proof. Because of Definition 7, it suffices to prove that $(\mathbf{e}_{i^*}, \mathbf{e}_{j^*})$ is a Nash equilibrium point if and only if $\mathbf{e}_{i^*} \in S_e$ and $\mathbf{e}_{j^*} \in S_e$. It follows from Lemma 11 that

$$\mathbf{e}_{i^*} \in S_e \Leftrightarrow u_{i^*k} \le \frac{1}{2}, \quad k \in V. \tag{8}$$

Sufficiency is proved as follows. Due to (8), we have $\max_k u_{i^*k} = \max_k u_{j^*k} = \frac{1}{2}$. Then, it follows from Property 8 that $\overline{V}(U) = \frac{1}{2}$ and moreover, $\mathbf{e}_{i^*}$ and $\mathbf{e}_{j^*}$ are security strategies. By Theorem 9, we have $(\mathbf{e}_{i^*}, \mathbf{e}_{j^*})$ is a Nash equilibrium point.

Necessity is proved as follows. Firstly, we will prove $u_{i^*j^*} = \frac{1}{2}$ by contradiction. Since $(\mathbf{e}_{i^*}, \mathbf{e}_{j^*})$ is a Nash equilibrium point, one has $u_{i^*j} \le u_{i^*j^*} \le u_{ij^*}$ for all $i, j \in V$. Suppose $u_{i^*j^*} > \frac{1}{2}$. Recalling $u_{j^*j^*} = \frac{1}{2}$, it follows that $u_{i^*j^*} > u_{j^*j^*}$ which conflicts with the fact that $u_{i^*j^*} \le u_{ij^*}$. Supposing $u_{i^*j^*} < \frac{1}{2}$, we have $u_{i^*i^*} > u_{ij^*}$ which has a conflict with $u_{i^*j} \le u_{i^*j^*}$. Therefore, we have proved $u_{i^*j^*} = \frac{1}{2}$. Then, it follows that $u_{i^*j} \le \frac{1}{2} \le u_{ij^*}$ for all $i, j \in V$. According to (8), we have $\mathbf{e}_{i^*} \in S_e$. Since $u_{ij^*} = 1 - u_{j^*i}$, we obtain $u_{j^*i} \le \frac{1}{2}$. Hence, $\mathbf{e}_{j^*} \in S_e$. ■

Property 13. *For game $\mathcal{G}^1(G_F)$, one has $\lim_{t\to\infty}(x_i(t) - y_0) = \lim_{t\to\infty}(y_1 - x_j(t))$ under every strategy pair $(\mathbf{e}_i, \mathbf{e}_j) \in S \times S$.*

Proof. For the case of $i = j$, it follows from Property 8 that $\lim_{t\to\infty}(x_i(t) - y_0) = \lim_{t\to\infty}(y_1 - x_j(t)) = \frac{y_1 - y_0}{2}$.

For the case of $i \ne j$, one can renumber vertexes of G_F by exchanging the number of 1 and i and that of 2 and j, i.e., $1 \leftrightarrow i$, $2 \leftrightarrow j$. Then, it follows that the new Laplacian matrix $L' = PLP^T$ where $P = [\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_3, \ldots \mathbf{e}_{i-1}, \mathbf{e}_1, \mathbf{e}_{i+1}, \ldots, \mathbf{e}_{j-1}, \mathbf{e}_2, \mathbf{e}_{j+1}, \ldots, \mathbf{e}_n]$ is orthogonal. It is easy to prove that $\mathbf{1}_n^T \mathrm{adj}(L + \mathrm{diag}\{\mathbf{e}_i\})\mathbf{e}_j = \mathbf{1}_n^T \mathrm{adj}(L' + \mathrm{diag}\{\mathbf{e}_1\})\mathbf{e}_2$ and $\mathbf{1}_n^T \mathrm{adj}(L + \mathrm{diag}\{\mathbf{e}_j\})\mathbf{e}_i = \mathbf{1}_n^T \mathrm{adj}(L' + \mathrm{diag}\{\mathbf{e}_2\})\mathbf{e}_1$. Consequently, it suffices to prove the case of $i = 1$ and $j = 2$.

Let $\tilde{L}_1 = L + \mathrm{diag}\{\mathbf{e}_1\}$ and $\tilde{L}_2 = L + \mathrm{diag}\{\mathbf{e}_2\}$, it follows that $\tilde{L}_1 = \tilde{L}_2 + \mathrm{diag}\{1, -1, 0, \ldots, 0\}$. Denote $\tilde{L}_1^{-1} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$ and $\tilde{L}_2^{-1} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n)$ where $\mathbf{p}_i$, $\mathbf{r}_i$ are n-dimension vectors. Thus, it is easy to show that $\tilde{L}_1\tilde{L}_2^{-1} = I_n + (\mathbf{r}_1, -\mathbf{r}_2, \mathbf{0}_n, \ldots, \mathbf{0}_n)^T$, $\tilde{L}_2\tilde{L}_1^{-1} = (\tilde{L}_1\tilde{L}_2^{-1})^{-1} = I_n + (-\mathbf{p}_1, \mathbf{p}_2, \mathbf{0}_n, \ldots, \mathbf{0}_n)^T$, and $\mathbf{p}_1 = \mathbf{r}_2 = \mathbf{1}_n$. Consequently, $\tilde{L}_1\tilde{L}_2^{-1}$ and $\tilde{L}_2\tilde{L}_1^{-1}$ have eigenvalues $\lambda_1, \lambda_2, 1, \ldots, 1$ and $\lambda_1^{-1}, \lambda_2^{-1}, 1, \ldots, 1$, respectively. Noticing from (7) that $\lambda_1\lambda_2 = |\tilde{L}_1\tilde{L}_2^{-1}| = |\tilde{L}_1||\tilde{L}_2|^{-1} = 1$, we have $\lambda_2 = \lambda_1^{-1}$. Therefore, the eigenvalues of $\tilde{L}_2\tilde{L}_1^{-1}$ are same as those of $\tilde{L}_1\tilde{L}_2^{-1}$. Let $\mathbf{p}_2 = (w_1, w_2, \ldots, w_n)^T$ and $\mathbf{r}_1 = (z_1, z_2, .., z_n)$, we obtain $|\lambda I_n - \tilde{L}_2\tilde{L}_1^{-1}| = (\lambda-1)^{n-2}(\lambda^2 - (w_2+1)\lambda + w_1)$ and $|\lambda I_n - \tilde{L}_1\tilde{L}_2^{-1}| = (\lambda-1)^{n-2}(\lambda^2 - (z_1+1)\lambda + z_2)$. Thus, one has $w_1 = z_2 = 1$ and $w_2 = z_1 = \lambda_1 + \lambda_1^{-1}$. By Lemma 5, together with (6) and (7), we get

$$\begin{aligned}\lim_{t\to\infty} X_f(t) &= (L + \mathrm{diag}\{\mathbf{e}_1 + \mathbf{e}_2\})^{-1}(\mathbf{e}_1, \mathbf{e}_2)(y_0, y_1)^T \\ &= \mu\left[\tilde{L}_2^{-1}\mathbf{e}_1, \tilde{L}_1^{-1}\mathbf{e}_2\right](y_0, y_1)^T = \mu[\mathbf{r}_1, \mathbf{p}_2](y_0, y_1)^T,\end{aligned}$$

where $\mu = \frac{\tau(G_F)}{|L + \mathrm{diag}\{\mathbf{e}_1 + \mathbf{e}_2\}|}$. It follows that $\lim_{t\to\infty} x_1(t) = \mu y_0 + \mu(\lambda_1 + \lambda_1^{-1})y_1$ and $\lim_{t\to\infty} x_2(t) = \mu(\lambda_1 + \lambda_1^{-1})y_0 + \mu y_1$. Because $\mu + \mu(\lambda_1 + \lambda_1^{-1}) = 1$, we have $\lim_{t\to\infty}(x_i(t) - y_0) = \lim_{t\to\infty}(y_1 - x_j(t))$. ■

3.3. Graphical results

In this subsection, we will deduce some graphical results for $\mathcal{G}^1(G_F)$.

Lemma 14. *Suppose that Assumption 1 holds. Then, one has*

$$\mathbf{1}_n^T \mathrm{adj}(L + \mathrm{diag}\{\mathbf{e}_i\})\mathbf{e}_j = n\tau(G_F) + \sum_{k\ne i, k\in V} (-1)^{k+j} M_{jk} \tag{9}$$

for $i, j \in V$ and $i \ne j$, where $M_{jk} = \det L_{(i,j;\ i,k)}$.

Proof. Firstly, we will prove the case of $i < j$. Denote $\tilde{L} = L + \mathrm{diag}\{\mathbf{e}_i\}$ and $\Delta^i = \mathrm{diag}\{\mathbf{e}_i\}$, we have

$$\det \tilde{L}_{(j;k)} = \det\left(L_{(j;k)} + \Delta^i_{(j;k)}\right) = \det L_{(j;k)} + M_{jk}.$$

Therefore, it follows from $(-1)^{k+j}\det L_{(j;k)} = \tau(G_F)$ that

$$\mathbf{1}_n^T \mathrm{adj}(L + \mathrm{diag}\{\mathbf{e}_i\})\mathbf{e}_j = n\tau(G_F) + \sum_{k\ne i, k\in V} (-1)^{k+j} M_{jk}.$$

Similarly, we can prove the case of $i > j$. ■

Denoting by $\mathcal{N}_i$ the neighbor set of follower i in G_F, we have

Theorem 15. *For two followers $i, j \in V$, if $\mathcal{N}_i \setminus \{j\} \supseteq \mathcal{N}_j \setminus \{i\}$, then $u_{ij} \le \frac{1}{2}$ and the equality holds if and only if $\mathcal{N}_i \setminus \{j\} = \mathcal{N}_j \setminus \{i\}$.*

Proof. Similar to the proof of Property 13, it suffices to prove the result in the case of $i = 1$ and $j = 2$.

It follows from Lemmas 11 and 14 that $u_{12} \le \frac{1}{2}$ if and only if $\det M^{(21)} - \det M^{(12)} \ge 0$. Since (9), we have $\det M^{(21)} - \det M^{(12)} = \begin{vmatrix} 0 & \mathbf{1}_{n-2}^T \\ \mathbf{r} & Q \end{vmatrix}$ where $Q = L_{[3,4,\ldots,n]}$ and $\mathbf{r} = [l_{31} - l_{32}, l_{41} - l_{42}, \ldots, l_{n1} - l_{n2}]^T$. If $\mathcal{N}_1 \setminus \{2\} = \mathcal{N}_2 \setminus \{1\}$, then $l_{j1} - l_{j2} = 0$, $j \in \{3, 4, \ldots, n\}$. It follows that $\det M^{(21)} - \det M^{(12)} = 0$. Hence $u_{12} = \frac{1}{2}$. If $\mathcal{N}_1 \setminus \{2\} \supset \mathcal{N}_2 \setminus \{1\}$, denote $\mathcal{N}_2 \setminus \{1\} = \{i_1, i_2, \ldots, i_k\}$ and $\mathcal{N}_1 \setminus \{2\} = \{i_1, i_2, \ldots, i_k, i_{k+1}, \ldots, i_{k+h}\}$, then $l_{j1} - l_{j2} = 0$ for $j \notin \{1, 2, i_{k+1}, \ldots, i_{k+h}\}$ and $l_{j1} - l_{j2} = -1$ for $j \in \{i_{k+1}, \ldots, i_{k+h}\}$. Therefore, $\det M^{(21)} - \det M^{(12)} = \sum_{m=k+1}^{k+h} \sum_{j=3}^{n} (-1)^{j+i_m} Q_{(i_m-2;j-2)}$. It is easy to find that $(-1)^{j+i_m} Q_{(i_m-2;j-2)}$ is the $(j-2, i_m-2)$th entry of adj$Q$. By Lemma 2, we know that Q is positive definite and Q^{-1} is a nonnegative matrix. Then, it follows that adj$Q$ is an invertible nonnegative matrix. Hence, for all $j \in \{3, 4, \ldots, n\}$ and $m \in \{k+1, k+2, \ldots, k+h\}$, we have $(-1)^{j+i_m} Q_{(i_m-2;j-2)} \ge 0$ and the inequality holds at least with one $j \in \{3, 4, \ldots, n\}$ which implies that $\det M^{(21)} - \det M^{(12)} > 0$. Consequently, we can make a conclusion that $u_{12} < \frac{1}{2}$. ■

Theorem 16. *For game $\mathcal{G}^1(G_F)$,*

1. *if G_F is a circulant graph, then $\tilde{G}(\mathbf{e}_i, \mathbf{e}_j)$ is the equilibrium topology for every $(\mathbf{e}_i, \mathbf{e}_j) \in S \times S$;*
2. *if G_F has a center vertex i^c, then $\tilde{G}(\mathbf{e}_{i^c}, \mathbf{e}_{i^c})$ is the equilibrium topology.*

Proof. Firstly, we will prove (1). If G_F is a circulant graph, adj$A$ is a circulant matrix. It follows that L is also a circulant matrix. Without loss of generality, we assume that $i < j$. Denote $\tilde{L}_i = L + \mathrm{diag}\{\mathbf{e}_i\}$ and $\tilde{L}_j = L + \mathrm{diag}\{\mathbf{e}_j\}$. Then for a permutation matrix $P = \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n+1-j} \\ \mathbf{0} & I_{j-i} & \mathbf{0} \end{pmatrix}$, we have $\tilde{L}_i = P\tilde{L}_jP^T$. Noticing that P is orthogonal, we obtain that $\mathbf{1}_n^T(\tilde{L}_i)^{-1}\mathbf{e}_j = \mathbf{1}_n^T P(\tilde{L}_j)^{-1}P^T\mathbf{e}_j$. Thus, it follows from $P^T\mathbf{e}_j = \mathbf{e}_i$ and $\mathbf{1}_n^T P = \mathbf{1}_n^T$ that $\mathbf{1}_n^T(\tilde{L}_i)^{-1}\mathbf{e}_j = \mathbf{1}_n^T(\tilde{L}_j)^{-1}\mathbf{e}_i$. Hence, by Lemma 11, we get $u_{ij} = \frac{1}{2}$ for all $i, j \in V$, i.e., $U = \frac{1}{2}\mathbf{1}_n\mathbf{1}_n^T$. Therefore, we can deduce that the strategy set is $S^* = S$. In another words, the graph $\tilde{G}(\mathbf{e}_i, \mathbf{e}_j)$ is the equilibrium topology for every $(\mathbf{e}_i, \mathbf{e}_j) \in S \times S$.

Next, we will give the proof of (2). For simplicity, we may take $i^c = n$. Owing to $\mathcal{N}_n = \{1, 2, \ldots, n-1\}$, one has $\mathcal{N}_j \setminus \{n\} \subseteq \mathcal{N}_n \setminus \{j\}$ for all $j \in \{1, 2, \ldots, n-1\}$. By Theorem 15, we have $u_{nj} \le \frac{1}{2}$

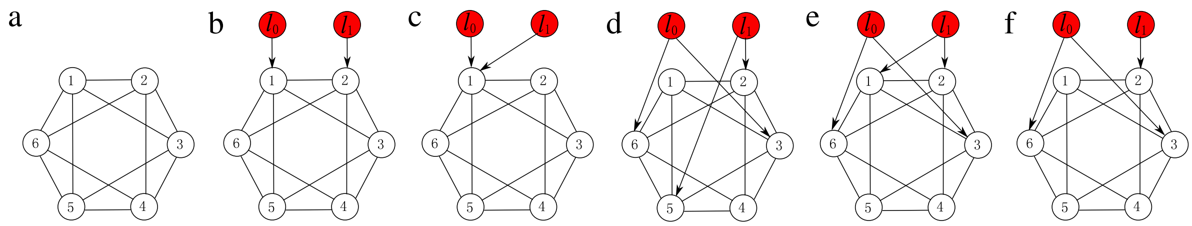
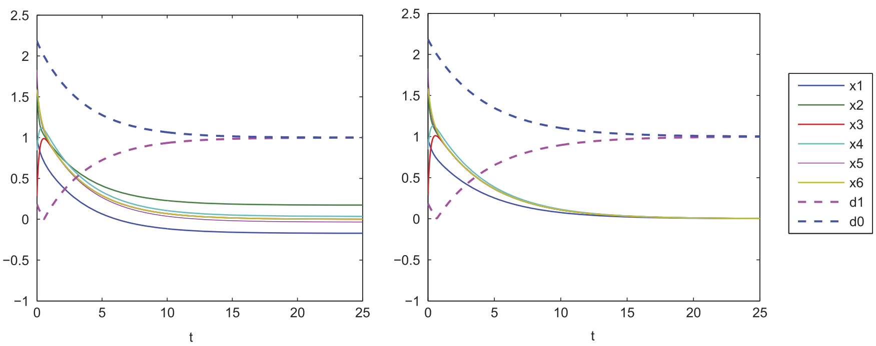


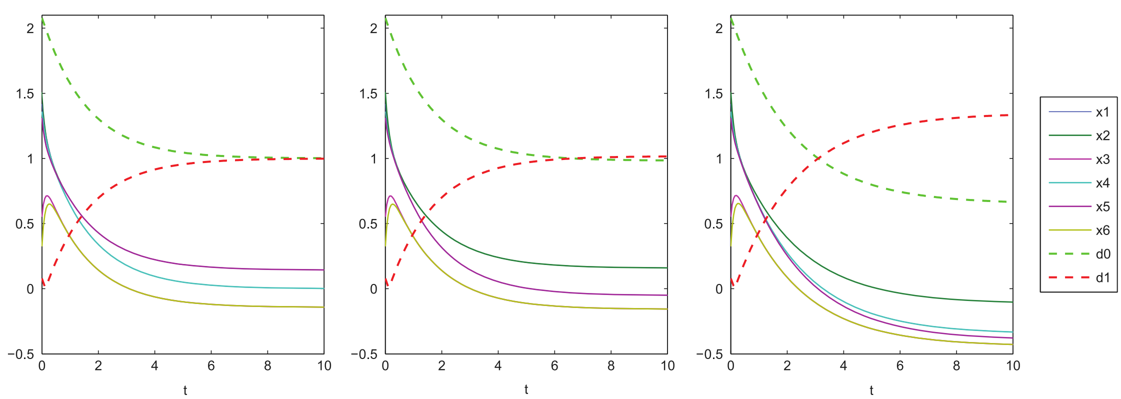
Fig. 2. The interaction graph.



(a) (e_1,e_2) .

(b) (e_1,e_1) .

Fig. 3. The followers' states and average distances under two strategy pairs for the game $\mathcal{G}^1(G_F)$.



(a) (s_1,s_2) .

(b) (s_1,s_3) .

(c) (s_1,s_4) .

Fig. 4. The followers' states and average distances under three strategy pairs for the game $\mathcal{G}^2(G_F)$.

for all $j \in \{1, 2, \ldots, n-1\}$. Hence, $\overline{V}(U) = \underline{V}(U) = \frac{1}{2}$ and $\mathbf{e}_n \in S^*$. By Theorem 9, it follows that the strategy pair $(\mathbf{e}_n, \mathbf{e}_n)$ is a Nash equilibrium point. Thus, we can conclude that $\tilde{G}(\mathbf{e}_n, \mathbf{e}_n)$ is the equilibrium topology. ■

Remark 6. The results of Theorem 16 may be used to shed light on some phenomena in reality. Firstly, if the interaction graph of the followers is a circulant graph, then it implies that every follower's influence is equal. Consequently, everyone is the optimal strategy to both two leaders and every strategy pair is a Nash equilibrium strategy pair. Secondly, for a 'center' follower who can influence all the others in G_F, its influence power is biggest among all followers. As a result, both two leaders will choose connecting with it to minimize their payoff. As a result, the corresponding strategy pair is a Nash equilibrium.

4. Simulations

Suppose that there are 6 followers labeled as 1–6 and two leaders labeled as l_0 and l_1. The followers' interaction graph G_F is a circulant graph depicted in Fig. 2(a). Let the leaders' initial states be -1 and 1, respectively. Let $d_0(t) = \frac{1}{6}\sum_{i=1}^{6}|x_i + 1|$ and $d_1(t) = \frac{1}{6}\sum_{i=1}^{6}|x_i - 1|$ be the average distances function of l_0 and l_1, respectively. Obviously, we have $\lim_{t\to\infty} d_r(t) = U_r, r = 0, 1$.

Example 1. Consider game $\mathcal{G}^1(G_F)$. It is easy to obtain that the outcome matrix is $U = \frac{1}{2}\mathbf{1}_6\mathbf{1}_6^T$, which implies that all 36 strategy pairs are Nash equilibrium points. This result illustrates the effectiveness of theoretical results in Theorem 16. Consider two strategy pairs $(\mathbf{e}_1, \mathbf{e}_2)$ and $(\mathbf{e}_1, \mathbf{e}_1)$. The corresponded interaction graphs are described in Fig. 2(b) and (c), respectively. Fig. 3(a) and (b) show the followers' states, $d_0(t)$ and $d_1(t)$ under those two strategy pairs, respectively, where $d_0(t)$ and $d_1(t)$ are depicted by dashed lines in Fig. 3. Since $(\mathbf{e}_1, \mathbf{e}_2)$ and $(\mathbf{e}_1, \mathbf{e}_1)$ are Nash strategy pairs, $\lim_{t\to\infty} d_0(t) = \lim_{t\to\infty} d_1(t) = 1$. For $(\mathbf{e}_1, \mathbf{e}_1)$, we can find that all followers converge to the middle point of two leaders' initial states, which is consistent with the result of Property 8.

Example 2. Consider game $\mathcal{G}^2(G_F)$. Then, it follows from $N = \sum_{m=1}^{2} C_6^m = 21$ that the outcome matrix U is 21×21-dimensional. By computing U, we can obtain $\overline{V}(U) = \frac{1}{2}$ and the set of security strategies $S^* = \{\mathbf{e}_1 + \mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_5, \mathbf{e}_3 + \mathbf{e}_6\}$. It is shown that every security strategy has exactly 2 followers which is consistent with the result of Theorem 10. By Theorem 9, we have a strategy pair (s_i, s_j) is a Nash equilibrium if and only if $(s_i, s_j) \in S^* \times S^*$. Consider three strategy pairs $(s_1, s_2) = (\mathbf{e}_3 + \mathbf{e}_6, \mathbf{e}_2 + \mathbf{e}_5)$, $(s_1, s_3) = (\mathbf{e}_3 + \mathbf{e}_6, \mathbf{e}_1 + \mathbf{e}_2)$, and $(s_1, s_4) = (\mathbf{e}_3 + \mathbf{e}_6, \mathbf{e}_2)$. The corresponded interaction graphs are described in Fig. 2(d), (e) and (f), respectively. Fig. 4(a), (b) and (c) show the followers' states, $d_0(t)$ and $d_1(t)$ under those three strategy pairs, respectively. It is easy to show that $\lim_{t\to\infty} d_0(t) = \lim_{t\to\infty} d_1(t) = 1$ since (s_1, s_2) is a Nash equilibrium. One can observe that $\lim_{t\to\infty} d_0(t) < \lim_{t\to\infty} d_1(t)$ because s_1 is a security strategy and s_3 and s_4 are not security strategies.

5. Conclusions

In this paper, we made use of game theory to tackle the containment control problem with conflicting leaders. We formulated a standard two-player zero-sum game denoted as $\mathcal{G}^k(G_F)$. For $\mathcal{G}^1(G_F)$, we presented a necessary and sufficient condition for an interaction graph to be the equilibrium topology. Moreover, if G_F was a circulant graph or a graph with a center vertex, then the equilibrium topology can be obtained directly. This work puts containment control in a game theoretical framework. This perspective will foster the understanding of the interactions between leaders. In the future, we may consider this game for the case where G_F is a directed graph or for MASs with constraints, such as MASs with switching topologies, MASs under measurement noises, etc.

References

Altafini, C., & Lini, G. (2015). Predictable dynamics of opinion forming for networks with antagonistic interactions. *IEEE Transactions on Automatic Control*, *60*(2), 342–357.

Basar, T., & Olsder, G. J. (1995). *Dynamic noncooperative game theory*. London: Academic Press.

Bauso, D., Giarré, L., & Pesenti, R. (2006). Non-linear protocols for optimal distributed consensus in networks of dynamic agents. *Systems & Control Letters*, *55*(11), 918–928.

Gharesifard, B., & Cortés, J. (2013). Distributed convergence to Nash equilibria in two-network zero-sum games. *Automatica*, *49*(6), 1683–1692.

Godsil, C., & Royal, G. (2001). *Algebraic graph theory*. New York: Springer-Verlag.

Gu, D. (2008). A differential game approach to formation control. *IEEE Transactions on Control Systems Technology*, *16*(1), 85–93.

Guan, Y., Ji, Z., Zhang, L., & Wang, L. (2013). Decentralized stabilizability of multi-agent systems under fixed and switching topologies. *Systems & Control Letters*, *62*(5), 438–446.

Hegselmann, R., & Krause, U. (2002). Opinion dynamics and bounded confidence models, analysis, and simulation. *Journal of Artificial Societies and Social Simulation*, *5*(3), 1–33.

Ji, M., Ferrari-Trecate, G., Egerstedt, M., & Buffa, A. (2008). Containment control in mobile networks. *IEEE Transactions on Automatic Control*, *53*(8), 1972–1975.

Jing, G., Zheng, Y., & Wang, L. (2014). Flocking of multi-agent systems with multiple groups. *International Journal of Control*, *87*(12), 2573–2582.

Liu, H., Xie, G., & Wang, L. (2012). Necessary and sufficient conditions for containment control of networked multi-agent systems. *Automatica*, *48*(7), 1415–1422.

Ma, J., Zheng, Y., & Wang, L. (2015). LQR-based optimal topology of leader-following consensus. *International Journal of Robust and Nonlinear Control*, *25*(17), 3404–3421.

Ni, W., & Cheng, D. (2010). Leader-following consensus of multi-agent systems under fixed and switching topologies. *Systems & Control Letters*, *59*(3), 209–217.

Notarstefano, G., Egerstedt, M., & Haque, M. (2011). Containment in leader-follower networks with switching communication topologies. *Automatica*, *47*(5), 1035–1040.

Olfati-Saber, R., & Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, *49*(9), 1520–1533.

Rahmani, A., Ji, M., Mesbahi, M., & Egerstedt, M. (2009). Controllability of multi-agent systems from a graph-theoretic perspective. *SIAM Journal on Control and Optimization*, *48*(1), 162–186.

Semsar-Kazerooni, E., & Khorasani, K. (2009). Multi-agent team cooperation: A game theory approach. *Automatica*, *45*(10), 2205–2213.

Vamvoudakis, K. G., Lewis, F. L., & Hudas, G. R. (2012). Multi-agent differential graphical games: Online adaptive learning solution for synchronization with optimality. *Automatica*, *48*(8), 1598–1611.

Wang, L., Jiang, F., Xie, G., & Ji, Z. (2009). Controllability of multi-agent systems based on agreement protocols. *Science in China Series F: Information Sciences*, *52*(11), 2074–2088.

Wang, L., & Xiao, F. (2010). Finite-time consensus problems for networks of dynamic agents. *IEEE Transactions on Automatic Control*, *55*(4), 950–955.

Xiao, F., Wang, L., & Chen, T. (2012). Connectivity preservation for multi-agent rendezvous with link failure. *Automatica*, *48*(1), 25–35.

Xie, G., & Wang, L. (2007). Consensus control for a class of networks of dynamic agents. *International Journal of Robust and Nonlinear Control*, *17*(10–11), 941–959.

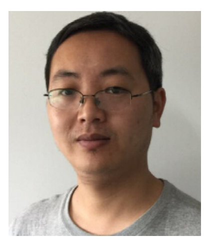
Zheng, Y., & Wang, L. (2012). Finite-time consensus of heterogeneous multi-agent systems with and without velocity measurements. *Systems & Control Letters*, *61*(8), 871–878.

Zheng, Y., & Wang, L. (2014). Containment control of heterogeneous multi-agent systems. *International Journal of Control*, *87*(1), 1–8.

Zheng, Y., & Wang, L. (2016). Consensus of switched multi-agent systems. *IEEE Transactions on Circuits and Systems II*, *63*(3), 314–318.

Zheng, Y., Zhu, Y., & Wang, L. (2011). Consensus of heterogeneous multi-agent systems. *IET Control Theory & Applications*, *5*(16), 1881–1888.

Jingying Ma was born in Ningxia, China. She received her Bachelor and Master degrees in applied mathematics from Tongji University in 2002 and 2006, respectively. Since 2007, she has been a teacher at School of Mathematics and Statistics, Ningxia University. She is working towards the Ph.D. degree in complex systems at Xidian University. Her research interests are in the fields of coordination of multi-agent systems, game theory and optimal control theory.



Yuanshi Zheng was born in Jiangshan, Zhejiang Province, China. He received his Bachelor, Master, and Doctorate degrees from Ningxia University and Xidian University in 2006, 2009, and 2012, respectively. He is currently an associate professor of Xidian University. His research interests are in the fields of coordination of multi-agent systems, consensus problems, containment control and coverage control.

Bin Wu is currently an assistant professor in the department of mathematics, Beijing University of Posts and Telecommunications. He received his bachelor degree in mathematics in Inner Mongolia University with honours in 2006. He received the Ph.D. in dynamics and control with Prof. Long Wang in Peking University in 2012. He was a postdoctoral fellow in Max-Planck Institute for Evolutionary Biology from 2012 to 2015. He was a visiting scholar in Harvard University, USA in 2014 and in Monash University, Australia in 2015. His research interests include evolutionary game theory, multi-agent systems and epidemics control.

Long Wang was born in Xi'an, China on February 13, 1964. He received his Ph.D. degree in Dynamics and Control from Peking University in 1992. He has held research positions at the University of Toronto, Canada, and the German Aerospace Center, Munich, Germany. He is currently Cheung Kong Chair Professor of Dynamics and Control, Director of Center for Systems and Control of Peking University. He is also Guest Professor of Wuhan University and Beihang University. He serves as the Chairman of Chinese Intelligent Networked Things Committee, and a member of IFAC Technical Committee on Networked Systems. He is in the editorial boards of Science in China, Journal of Intelligent Systems, Journal of Control Theory and Applications, PLoS ONE, Journal of Intelligent and Robotic Systems, IEEE Trans. on Industrial Electronics, etc. His research interests are in the fields of complex networked systems, collective intelligence, and biomimetic robotics. Email: longwang@pku.edu.cn