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To link to this article: http://dx.doi.org/10.1080/00207179.2016.1201865

Accepted author version posted online: 29 Jun 2016.
Published online: 15 Jul 2016.

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Consensus of switched multi-agent systems with random networks

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ABSTRACT
This paper studies the consensus problem of the switched multi-agent system composed of continuous-time and discrete-time subsystems. Communication among agents is modelled as a random network where the existence of any information channel is probabilistic and independent of other channels. Then, some necessary and sufficient conditions are presented for solving average consensus of the switched multi-agent system under arbitrary switching. Furthermore, we show that the average consensus in different sense (mean square, almost surely and in probability, respectively) are equivalent. Finally, simulations are provided to illustrate the effectiveness of our theoretical results.

1. Introduction

Over the past several years, multi-agent systems have received considerable attention due to their potential applications in many areas including flocking in biology (Olfati-Saber, 2006), rendezvous problem of mobile autonomous robots (Lin, Morse, & Anderson, 2006), attitude alignment of satellite clusters (Lawton & Beard, 2002), etc. Therefore, the study of multi-agent coordination has been an important research field in the scientific community and attracted multi-disciplinary researchers' attention. Many results have been obtained about multi-agent coordination, such as consensus (Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Wang & Xiao, 2010; Zheng & Wang, 2012), containment control (Zheng & Wang, 2014), controllability (Ji, Lin, & Yu, 2015), rendezvous (Xiao, Wang, & Chen, 2012), and optimal control (Ma, Zheng, & Wang, 2015).

Consensus plays an important role in cooperative control of multi-agent systems, which aims to design appropriate distributed protocol that enables all agents to converge to a consistent state. As a fundamental research topic, consensus problem has a long history in the field of multi-agent systems. By graph theory, Jadbabaie, Lin, and Morse (2003) first gave theoretical analysis for the observed behaviour of the Vicsek model (Vicsek, Czirok, Jacob, Cohen, & Schochet, 1995). Olfati-Saber and Murray (2004) investigated the consensus for networks of dynamic agents with fixed and switching topologies. Some relaxed conditions were obtained for multi-agent consensus in Ren and Beard (2005). Furthermore, the second-order consensus protocols based on the absolute and relative velocity information were studied in Xie and Wang (2007) and Ren and Atkins (2007), respectively. Zheng, Zhu, and Wang (2011) studied the consensus of multi-agent systems with heterogeneous dynamics. With rapid development of this issue in the past decade, a tremendous amount of interesting results have been obtained under various circumstances, such as discrete-time consensus (Wang & Xiao, 2007), asynchronous consensus (Xiao & Wang, 2008), leader-following consensus (Hong, Hu, & Gao, 2006), group consensus (Zheng & Wang, 2015), etc. It should be noted that aforementioned works only analysed consensus problem of multi-agent systems under fixed and switching topologies. However, the realistic communication among agents may change with time due to link failures, packet drops, node failure, etc. Such variations in the network can happen randomly, which attracts researchers' great attention concerning random networks. The consensus problem of multi-agent systems under random networks was considered in Hatano and Mesbahi (2005). Porfiri and Stilwell (2007) extended the results in Hatano and Mesbahi (2005), and presented more relaxable conditions for solving consensus problem. Tahbaz-Salehi and Jadbabaie (2008, 2010) also gave some necessary and sufficient conditions for almost sure convergence to consensus. Other research topics of consensus with random networks have also been addressed (Kar & Moura, 2008; Lin, Hou, Yan, & Yu, 2015).
Switched system is a kind of system that consists of a series of subsystems, and these subsystems obey a logical rule to switch. For multi-agent systems, there have been numerous results related to the consensus with switching topologies (Olfati-Saber & Murray, 2004; Wang & Xiao, 2007; Zheng & Wang, 2012). By utilizing graph theory and Lyapunov theory, Olfati-Saber and Murray investigated the consensus of continuous-time multi-agent systems with switching topologies in Olfati-Saber and Murray (2004). Wang and Xiao investigated the consensus of discrete-time multi-agent systems with switching topologies in Wang and Xiao (2007). For such multi-agent systems, it is composed of only continuous-time subsystems or discrete-time subsystems. To the best of our knowledge, the multi-agent system which has been researched only consists of continuous-time subsystems or discrete-time subsystems. However, switching behaviour exists not only in the topologies but also on the dynamical behaviours of agents. In this paper, we considered a kind of switched multi-agent systems, in which dynamics of every agent switches between continuous-time dynamics and discrete-time dynamics. In practice, many applications contain such switched systems. For example, the system controlled either by a physically implemented regulator or by a digitally implemented one with a switching rule between them, i.e. the system is composed of both continuous-time and discrete-time subsystems (Zhai, Lin, Michel, & Yasuda, 2004). Zheng and Wang (2016) investigated the consensus problem of such switched multi-agent system which is composed of continuous-time and discrete-time subsystems under deterministic networks. By using graph theory and Lyapunov theory, they prove that the consensus problem can be solved if the graph is undirected connected or has a directed spanning tree. Moreover, containment control of such switched multi-agent system with fixed topology was investigated in Zhu, Zheng, and Wang (2015). Inspired by above works, we further investigate the consensus of the switched multi-agent system composed of continuous-time and discrete-time subsystems with random networks. Different from Zheng and Wang (2016), we mainly explore that the uncertainty is embedded in the network where the existence of any information channel is probabilistic and independent of other channels. Comparing with deterministic networks, it is more practical significance to study the random networks. Due to the random nature of networks, we need to solve the consensus problem under sense of probability. Owing to switching behaviour of dynamics and the random variation of the network, it is difficult to analyse the consensus by using classical methods. The main contribution of this paper is threefold. First, by using random graph theory and stochastic analysis tools, we prove that the consensus in different sense (mean square, almost surely and in probability, respectively) can be achieved if and only if the expected graph is connected. Second, the equivalent relation of the average consensus of switched multi-agent system in different sense (mean square, almost surely and in probability, respectively) is established under the connected expected graph. Finally, the per-step convergence factor in the mean square sense is given.

The structure of this paper is given as follows. In Section 2, we present some random graph theory concepts and some definitions. In Section 3, we present the main results. In Section 4, simulation examples are provided to illustrate the effectiveness of our theoretical results. Finally, we give a short conclusion in Section 5.

Throughout this paper, the following notations will be used: we denote by $I_n$ the column vector of all ones. $\mathbb{R}$ denotes the set of real number. $\mathbb{R}^m$ denotes the $n$ dimensional real vector space. $\| \cdot \|$ denotes the standard Euclidean norm. $I_n$ is the $n \times n$ identity matrix. $i.o.$ stands for infinitely often. For a given vector or matrix $A$, $A^T$ denotes its transpose. $\mathcal{I}_n = \{1, 2, \ldots, n\}$ and $\tilde{d} = \max_{i \in \mathcal{I}_n} \{d_{ii}\}$, $B = [b_{ij}] \in \mathbb{R}^{n \times n}$. $B \geq 0$ if all $b_{ij} \geq 0$, and $B > 0$ if $b_{ij} > 0$. If $B \geq 0$, we say that $B$ is a nonnegative matrix, and if $B > 0$, we say that $B$ is a positive matrix.

2. Preliminaries

2.1 Random Graphs

In this subsection, we present some basic concepts of algebraic graph theory which will be used in this paper.

The information flow among the nodes of an undirected random network can be described by a sequence of undirected random graphs $\mathcal{G}_t$. At each time $t$, the random graph is $\mathcal{G}_t = (\mathcal{V}, \mathcal{E}_t)$, where $\mathcal{V} = \{v_i, i = 1, \ldots, n\}$ is a determine vertex set and $\mathcal{E}_t = \{e_{ij}\} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges where $e_{ij}$ denotes that agents $i$ and $j$ can communicate with each other. In the random graph on $n$ vertices, we assume that the existence of $e_{ij} \in \mathcal{E}_t$ is determined randomly and independently of other edges with probability $p_{ij} \in [0, 1]$ for $i, j = 1, \ldots, n$, and $i \neq j$. In this paper, we do not consider self-loops and multiple edges. We define the edge probability matrix $P = P^T = [p_{ij}] \in \mathbb{R}^{n \times n}$, $0 \leq p_{ij} \leq 1$ and $p_{ii} = 0$. The adjacency matrix $\mathcal{A}_t = [a_{ij}] \in \mathbb{R}^{n \times n}$ of $\mathcal{G}_t$ can be defined as

$$a_{ij} = \begin{cases} 1, & \text{with probability } p_{ij}, \\ 0, & \text{with probability } 1 - p_{ij}, \end{cases}$$

where $i \neq j$, and $a_{ii} = 0$ for all $i$. The degree matrix $\mathcal{D}_t \in \mathbb{R}^{n \times n}$ is a diagonal matrix with $d_{ii} = \sum_{j \neq i} a_{ij}$ and $\max_{i \in \mathcal{I}_n} \{d_{ii}\}$ is maximum degree of agent under any random network. The Laplacian matrix is defined as $\mathcal{L}_t = [l_{ij}]_{n \times n} = \mathcal{D}_t - \mathcal{A}_t$. Due to the random nature of $\mathcal{A}_t$,
Laplacian matrix $\mathcal{L}_i$ is also random. It is easy to see that $\mathcal{L}_i 1 = 0$ and $1^T \mathcal{L}_i = 0$. Let $G$, $A$, and $L$ denote the sample spaces of all random graphs, all adjacency matrices, and all Laplacian matrices, respectively.

The matrices $\mathcal{A} = E[\phi(t)] = P$ and $\mathcal{L} = E[\mathcal{L}_i] = [\bar{L}]_{n \times n}$ denote the expected value of the adjacency matrix and Laplacian matrix, respectively. Then, it can be obtained

$$
\bar{L}_{ij} = \begin{cases} 
\sum_{j=1}^n p_{ij}, & \text{if } i = j, \\
-\bar{p}_{ij}, & \text{otherwise.}
\end{cases}
$$

(2)

Matrix $\bar{L}$ corresponds to graph $\mathcal{G}$ which does not necessarily belong to $G$. This graph $\mathcal{G}$ denotes the expected graph, i.e. the average graph over time. Expected graph $\mathcal{G}$ is connected if and only if $\lambda_2(\mathcal{L}) > 0$ (Kar & Moura, 2008, Lemma 1). Moreover, the eigenvalues of $\mathcal{L}_i$ can be denoted as $0 = \lambda_1(\mathcal{L}) \leq \lambda_2(\mathcal{L}) \leq \cdots \leq \lambda_n(\mathcal{L})$. Matrix $\mathcal{L}_i$ is positive semi-definite and has a simple zero eigenvalue when $\mathcal{G}_i$ is connected undirected graph.

2.2 System model

In this subsection, we consider a multi-agent system which consists of $n$ agents. Suppose that agent $i$ takes the switched dynamics, it switches between continuous-time dynamics and discrete-time dynamics. The continuous-time dynamics is

$$
\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{I}_n,
$$

(3)

and discrete-time dynamics is

$$
x_i(t + h) = x_i(t) + h u_i(t), \quad i \in \mathcal{I}_n,
$$

(4)

where $x_i(t) \in \mathcal{R}$ and $u_i(t) \in \mathcal{R}$ are the position and control input of agent $i$, respectively. $h > 0$ is the sampling period and $x_0 = [x_1(0), \ldots, x_n(0)]^T$ is the initial value. We apply the consensus protocol for switched multi-agent system (3–4) as follows

$$
u_i(t) = \sum_{j=1}^n a_{ij} (x_j(t) - x_i(t)).
$$

(5)

Suppose that the dynamics of each agent switches simultaneously from one to another and the agents with discrete-time dynamics finish integer sampling. Then, switched multi-agent system (3–4) with protocol (5) which is composed of continuous-time subsystem

$$
\dot{x}(t) = -\mathcal{L} x(t),
$$

(6a)

and discrete-time subsystem

$$
x(t + h) = (I_n - h \mathcal{L}) x(t),
$$

(6b)
i.e. the switched multi-agent system can be viewed as a system which is composed of (6a) and (6b) at different time interval.

Switched multi-agent system (6) switches arbitrarily in the random network. We assume that the network which consists of random edges is constant over each time interval $\Delta > 0$. The topology structure at point time $t_k$ is independent of topology structure at the previous time-intervals. By introducing the sequence of column vectors $\{x(t_k)\}$, the problem may be cast in a sample-data system setting. From subsystem (6a), we know that $x(t) = e^{\mathcal{L}_k(t-k)} x(t_k)$. Let the sequence of column vectors $\{x(t_k)\}$ be the subsequence of $\{x(t)\}$. Due to $e^{\mathcal{L}_k t} (I_n - h \mathcal{L}) = (I_n - h \mathcal{L}) e^{\mathcal{L}_k t}$, the trajectory $x(t_k)$ of switched multi-agent system (6) is expressed by

$$
x(t_{k+1}) = W_{t_k} x(t_k), \quad k = 0, 1, \ldots,
$$

(7)

where $W_{t_k}$ is the random state transition matrix defined by

$$
W_{t_k} = e^{-\mathcal{L}_k t_k} (I_n - h \mathcal{L}_k)^{t_k}.
$$

(8)

$\mathcal{L}_k$ is the Laplacian matrix of random graph $\mathcal{G}_k$ at time point $t_k$. Note that graph $\mathcal{G}_k$ is invariant during the time interval $\Delta$. Matrix $W_{t_k}$ at time point $t_k$ is independent of matrix $W_{t_{k-1}}$ at time point $t_{k-1}$. The total duration time on subsystem (6a) and the total sampling times on sub-system (6b) under time interval $t_{k+1} - t_k$ are denoted as $t_{c_k}$ and $d_k$, respectively. For the discrete-time subsystem, when the integer sampling of discrete-time subsystem are completed under time interval $\Delta$ or time interval $\Delta + h$, we make a sampling at time $t_k$. For the continuous-time subsystem, we can make arbitrary sampling at time $t_k$ under a network, i.e. before the network changes randomly at the transition instant. We know that state $x(t_k)$ of system (7) at each moment is sampled from state variables of system (6). Therefore, system (7) can be used to monitor the partial state of switched multi-agent system (6).

The objective of this paper is to make the switched multi-agent system to achieve average consensus in mean square (or almost surely) under arbitrary switching with random networks. The vector of averages $\bar{r}(x_0)$ is written as $\bar{r}(x_0) = \frac{1}{n} \sum_{i=1}^n x_i$. The following notions of consensus will be used.

Definition 2.1: We say that switched multi-agent system (6) converges to average consensus
(a) in mean square if for any $x_0 \in \mathbb{R}^n$ it holds that
\[ \lim_{t \to \infty} E \left[ \| x(t) - v(x_0) \|^2 \right] = 0; \tag{9} \]
(b) almost surely if for any $x_0 \in \mathbb{R}^n$ it holds that
\[ P \left\{ \lim_{t \to \infty} \| x(t) - v(x_0) \| = 0 \right\} = 1; \tag{10} \]
(c) in probability if for any $x_0 \in \mathbb{R}^n$ it holds that
\[ \lim_{t \to \infty} P \{ \| x(t) - v(x_0) \| > \varepsilon \} = 0; \tag{11} \]
(d) in mean if for any $x_0 \in \mathbb{R}^n$ it holds that
\[ \lim_{t \to \infty} E \{ x(t) \} = v(x_0). \tag{12} \]

**Definition 2.2:** The per-step convergence factor, in the mean square sense, for system (7) is defined as
\[ \gamma_s = \sup_{\xi(t_k) \neq 0, \ k \in \mathbb{N}} \left( \frac{E[\| \xi(t_{k+1}) \|^2]}{\| \xi(t_k) \|^2} \right), \tag{13} \]
where $\xi(t_k) = x(t_k) - v(x_0)$.

Next, we will give some properties of state transition matrix of system (7) for solving the consensus problem.

**Lemma 2.1:** Matrix $W_{tk}$ is doubly stochastic matrix if the sampling period $0 < h < \frac{1}{d}$.

**Proof:** Due to
\[ W_{tk} 1_n = e^{-L_{tk}^T t_k} (I_n - h L_{tk}^T) 1_n = \left( \sum_{j=0}^{\infty} \frac{(-t_k/h)^j}{j!} L_{tk}^T \right) (I_n - h L_{tk}^T) 1_n = 1_n, \]
and $L_{tk}^T = L_{tk}$, it is obvious that $1_n^T W_{tk} = 1_n^T$. Laplacian matrix $L_{tk}$ can be rewritten as $L_{tk} = dI_n - \tilde{A}_k$. Because $0 < h < \frac{1}{d}$ and $\tilde{A}_k$ is nonnegative matrix, it is clear that $e^{-d t_k / h} \geq 0, e^{\tilde{A}_k t_k} \geq 0$, and $(I_n - h \tilde{A}_k) \geq 0$.

Note that
\[ W_{tk} = e^{-d t_k / h} e^{\tilde{A}_k t_k} (I_n - h \tilde{A}_k) \]
thus, matrix $W_{tk}$ is nonnegative. Hence, $W_{tk}$ is a doubly stochastic matrix.

**Lemma 2.2:** The $i$-th eigenvalue of matrix $W_{tk}$ is $\lambda_i = e^{-\lambda_i(L_{tk}) t_k} (1 - h \lambda_i(L_{tk}))^d$, $i = 2, \ldots, n$.

**Proof:** Since $L_{tk}$ is symmetric matrix, there exists a unitary matrix $U$ such that $L_{tk} = U^T \Lambda U$, where $\Lambda$ is the diagonal matrix consisting of $\lambda_i(L_{tk})$, $i = 1, \ldots, n$. Hence, $W_{tk} = e^{U^T \Lambda U t_k} (I_n - h U^T \Lambda U)^d = U^T e^{\Lambda t_k (I_n - h \Lambda)^d} U$, and $\lambda_i(W_{tk}) = e^{-\lambda_i(L_{tk}) t_k} (1 - h \lambda_i(L_{tk}))^d$, $i = 1, \ldots, n$.

According to Lemma 2.1, it follows that $W_{tk} \frac{1+\gamma_s}{n} = \frac{1}{n} W_{tk}$. By virtue of Theorem 4.5.15 in Horn and Johnson (2012), it can be obtained $\frac{1+\gamma_s}{n} W_{tk} = \tilde{U}^T \Xi_1 \tilde{U}$ and $W_{tk} = \tilde{U}^T \Xi_2 \tilde{U}$ where $\tilde{U} = [\tilde{u}_1, \ldots, \tilde{u}_n]$. $\Xi_1$ and $\Xi_2$ are the diagonal matrices consisting of $\lambda_i(L_{tk})$ and $\lambda_i(W_{tk})$, respectively. Obviously, $W_{tk} \frac{1+\gamma_s}{n}$ is irreducible and nonnegative. Note that there exists the normalised eigenvector of $\tilde{u}_i$ corresponding to $\lambda_1(W_{tk}) = 1$ and $\lambda_1(L_{tk}) = 1$ where $\tilde{u}_1 = \frac{1}{\sqrt{n}} 1$. Hence, $\lambda_i = e^{-\lambda_i(L_{tk}) t_k} (1 - h \lambda_i(L_{tk}))^d$, $i = 2, \ldots, n$.

**Remark 2.1:** When $t_k = 0$ or $d_k = 0$, the $i$-th eigenvalue of $W_{tk} \frac{1+\gamma_s}{n}$ are $\lambda_{i\geq 2} = e^{-\lambda_i(L_{tk}) t_k}$ and $\lambda_{i\geq 2} = (1 - h \lambda_i(L_{tk}))^d$, respectively.

**Lemma 2.3:** Expected matrix $E[W_{tk}]$ is irreducible and nonnegative if $\lambda_2(L_{tk}) > 0$ and the sampling period $0 < h < \frac{1}{d}$.

**Proof:** Matrix $\tilde{\Omega}$ is irreducible due to $\lambda_2(L_{tk}) > 0$. The irreducibility of $\tilde{\Omega}$ shows that there exists the graph with non-zero probability for which $\lambda_2(L_{tk}) > 0$. The adjacency matrix $A_k$ corresponding to $\lambda_2(L_{tk}) > 0$ is irreducible and nonnegative. By Lemma 2.1, it can be obtained
\[ W_{tk} = e^{-d t_k / h} e^{\tilde{A}_k t_k} (I_n - h \tilde{A}_k) \]
\[ = e^{-d t_k / h} (I_n + \tilde{A}_k t_k + \frac{1}{2!} (\tilde{A}_k t_k)^2 + \cdots) (K + h \tilde{A}_k) \]
where $\tilde{A}_k = A_k + M \geq 0$, and $K = I_n - h \tilde{A}_k > 0$. It is obvious that $W_{tk}$ is irreducible and nonnegative when $t_k = 0$ or $d_k = 0$. Hence,
\[ W_{tk} = e^{-d t_k / h} (K + h K d_k - 1 \tilde{A}_k + \cdots) \]

Thus, $W_{tk}$ is irreducible and nonnegative. Consequently, $E[W_{tk}]$ is irreducible and nonnegative.

**Lemma 2.4:** For system (7), we have $\| x(t_{k+1}) - v(x_0) \|^2 \leq \rho^2 (W_{tk} \frac{1+\gamma_s}{n}) \| x(t_k) - v(x_0) \|^2$. 

Based on Lemma 2.1 and $1_n^T x(k+1) = 1_n^T x_0$, we can straightly obtain the following result.
3. Main results

In this section, the average consensus problem of the switched multi-agent system will be investigated. Moreover, we further derive the per-step (mean square) convergence factor for system (7).

**Theorem 3.1:** Assume that the sampling period $0 < h \leq \frac{1}{2d}$. Then, switched multi-agent system (6) reaches average consensus in mean square (or almost surely) if and only if system (7) reaches average consensus in the same sense.

**Proof:** (Sufficiency) We first analyse the state of continuous-time subsystem. The state transition matrix of the continuous-time subsystem matrix is $e^{-\lambda t}$. By Lemma 2.2 and $0 < h \leq \frac{1}{2d}$, we have $\rho^2(e^{-\lambda t} - \frac{11}{n} I) \leq 1$. By Lemma 2.4, it can easily be verified that the state at any moment such that $\|x(t_{k+1}) - v(x_0)\|^2 \leq \|x(t_k) - v(x_0)\|^2$ where $t_k$ is any sampling time for continuous-time subsystem. The state of discrete-time subsystem also possesses the same property and the proof is omitted. This implies that the sequence $\|x(t) - v(x_0)\|^2$ is monotonic. Therefore, the convex combination $E[\|x(t) - v(x_0)\|^2]$ is bounded monotonic sequence. It contains a convergent subsequence $E[\|x(t) - v(x_0)\|^2]$ such that $\lim_{k \to \infty} E[\|x(t) - v(x_0)\|^2] = 0$. It follows that

$$\lim_{k \to \infty} E[\|x(t) - v(x_0)\|^2] = 0.$$ 

Therefore, switched multi-agent system (6) reaches average consensus in mean square.

(Necessity) Since the sequence $\{x(t_k)\}$ is a subsequence of $\{x(t)\}$, it is obvious that $\lim_{k \to \infty} E[\|x(t_k) - v(x_0)\|^2] = 0$ if $\lim_{k \to \infty} E[\|x(t_k) - v(x_0)\|^2] = 0$.

We only show the equivalence of average consensus in mean square. The equivalence of average consensus almost surely can be proved similarly. Thus, it is omitted here. \[\square\]

According to Theorem 3.1, it is shown that we analyse the average consensus of system (6) is equivalent to analyse the average consensus of system (7). Next, we mainly discuss the average consensus of system (7).

**Theorem 3.2:** Assume that the sampling period $0 < h \leq \frac{1}{2d}$. Then, system (7) can solve average consensus in mean square under arbitrary switching if and only if the expected graph $\mathcal{G}$ is connected.

**Proof:** (Sufficiency) By Lemma 2.4, we know that

$$\|x(t_{k+1}) - v(x_0)\|^2 \leq \rho^2 \left( W_h - \frac{11}{n} I \right) \|x(t_k) - v(x_0)\|^2,$$

which implies that

$$\|x(t_k) - v(x_0)\|^2 \leq \rho^2 \left( W_h - \frac{11}{n} I \right) \|x(t_{k-1}) - v(x_0)\|^2.$$  \hspace{1cm} (14)

Taking expectation on both sides of (15) and using the independent property of the random matrix $W_h$, we have

$$E[\|x(t_k) - v(x_0)\|^2] \leq E\left[ \rho^2 \left( W_h - \frac{11}{n} I \right) \right] \cdots \times E\left[ \rho^2 \left( W_{h_i} - \frac{11}{n} I \right) \right] \|x(0) - v(x_0)\|^2 \leq \max_{i=0, \ldots, k-1} E\left[ \rho^2 \left( W_{h_i} - \frac{11}{n} I \right) \right]^k \|x(0) - v(x_0)\|^2.$$ \hspace{1cm} (16)

In order to prove that the state vector sequence of system (7) converges in mean square, we only need to prove that $E[\rho^2 (W_{h_i} - \frac{11}{n} I)] < 1$ for $i = 0, 1, \ldots$. By Lemma 2.2, we have the eigenvalues $\lambda_i (W_{h_i} - \frac{11}{n} I) = e^{-\lambda_i (\mathcal{G}_i) h_i (1 - h \lambda_i (\mathcal{L}_i))^d_i}$, $i \geq 2$. Based on Gersgorin Disc theorem, we have the sampling period $0 < h \leq \frac{1}{2d} \leq \frac{1}{\lambda_{\max}}$ with non-zero probability where $\lambda_{\max} = \max_{i=0, \ldots, k-1} \lambda_i (\mathcal{L}_i)$. This also implies that $e^{-\lambda_i (\mathcal{G}_i) h_i (1 - h \lambda_i (\mathcal{L}_i))^d_i} \geq 0$ and $\rho^2 (W_{h_i} - \frac{11}{n} I) = e^{-2\lambda_i (\mathcal{G}_i) h_i (1 - h \lambda_i (\mathcal{L}_i))^d_i} \leq 1$. Indeed, the expected graph $\mathcal{G}$ which is connected shows that there is at least one graph $\mathcal{G}$ with non-zero probability for $\lambda_2 (\mathcal{L}) > 0$. It is obvious that $e^{-2\lambda_2 (\mathcal{L}) h (1 - h \lambda_2 (\mathcal{L}))^d} < 1$, i.e., $E[\rho^2 (W_{h_i} - \frac{11}{n} I)] < 1$. It is shown that

$$\lim_{h_i \to \infty} E[\|x(t) - v(x_0)\|^2] = 0.$$ \hspace{1cm} (17)

(Necessity) If the expected graph is $\mathcal{G}$ which is not connected, there exist at least two components with zero probability of communication between each other. This implies that there is no path between two components. Hence, the information of these two components cannot reach consensus for any initial condition (Ren & Beard, 2005).

**Remark 3.1:** In fact, the expected graph $\mathcal{G}$ is connected which implies $\lambda_2 (\mathcal{L}) > 0$. Under sampling period $0 < h \leq \frac{1}{2d}$, switched multi-agent system (6) can solve average consensus in mean square if $\lambda_2 (\mathcal{L}) > 0$. For $\lambda_2 (\mathcal{L})$, we just need to know the edge probability matrix $P$ to calculate it.
Next, we show that both average consensus almost surely and average consensus in mean can be achieved.

**Theorem 3.3:** Assume that the sampling period \( 0 < h \leq \frac{1}{2d} \). Then, system (7) can solve average consensus almost surely under arbitrary switching if and only if the expected graph \( \mathcal{G} \) is connected.

**Proof:** (Sufficiency) As a result of Markovpo’s inequality (Sheldon, 1976), for any \( a > 0 \),

\[
P\{\|x(t_k) - v(x_0)\|^2 \geq a^2\} \leq \frac{E[\|x(t_k) - v(x_0)\|^2]}{a^2}.
\]

(18)

Because \( \|x(t_k) - v(x_0)\|^2 \geq a^2 \) is equivalent to \( \|x(t_k) - v(x_0)\| \geq a \), inequality (18) can be written as

\[
P\{\|x(t_k) - v(x_0)\| \geq a\} \leq \frac{E[\|x(t_k) - v(x_0)\|^2]}{a^2}.
\]

(19)

By Theorem 3.2, it is easy to know that \( \|x(t_{k+1}) - v(x_0)\|^2 \leq \rho^2(W_{t_k} - \frac{11T}{n})\|x(t_k) - v(x_0)\|^2 \). Therefore,

\[
\sum_{k=0}^{\infty} P\{\|x(t_k) - v(x_0)\| \geq a\} \leq \frac{E[\|x_0 - v(x_0)\|^2]}{a^2(1 - \beta)} < \infty,
\]

where \( \beta = \max_{k=0,1,...}\{E[\rho^2(W_{t_k} - \frac{11T}{n})]\} \).

Using the Borel–Cantelli Lemma (Durrett, 2010) leads to

\[
P\{\|x(t_k) - v(x_0)\| \geq a \text{ i.o.}\} = 0.
\]

(21)

Thus, we conclude that the agents reach average consensus almost surely.

(Necessity) The proof is similar to the argument used in Theorem 3.2. ■

**Corollary 3.1:** Assume that the sampling period \( 0 < h \leq \frac{1}{2d} \) and the expected graph \( \mathcal{G} \) is connected. Then, the following statements are equivalent.

1. switched multi-agent system (6) reaches average consensus in mean square;
2. switched multi-agent system (6) reaches average consensus almost surely;
3. switched multi-agent system (6) reaches average consensus in probability.

**Proof:** Based on Theorems 3.2 and 3.3, it is obvious that (1)⇒(2). It is well known that convergence in probability can be obtained by using convergence almost surely, i.e., (2)⇒(3). From Theorem 3.1, we know that the sequence \( \{\|x(t) - v(x_0)\|\} \) is bounded monotonic. Thus, it follows from the dominated convergence theorem (Durrett, 2010) that (3)⇒(1). ■

**Corollary 3.2:** Assume that the sampling period \( 0 < h \leq \frac{1}{2d} \). Then, switched multi-agent system (6) can solve average consensus in mean under arbitrary switching if the expected graph \( \mathcal{G} \) is connected.

**Proof:** Let \( \Xi = \sum_{j=1}^{n} (x_j(t)) \). Then we take expectation on both sides of it, it follows that \( E[\Xi] = E[1^T x(t)] \). We have

\[
\lim_{t \to \infty} E[1^T x(t)] = \frac{1}{n} \sum_{j=1}^{n} (x_j(0)).
\]

Moreover, by Theorem 3.3, we know that switched multi-agent system (6) reaches consensus almost surely. By using the property of expectation \( E[X + Y] = E[X] + E[Y] \) and the Lebesgue’s Dominated Convergence Theorem (Rudin, 1987), we can get

\[
\lim_{t \to \infty} E[x_j(t)] = \frac{1}{n} \sum_{j=1}^{n} (x_j(0)).
\]

In the following, we consider switched multi-agent system (6) with switching topologies \( \{\mathcal{G}_s : s \in \zeta_0\} \), where \( \zeta_0 \) is a finite index set, the topologies are fixed, and switching is probabilistic. The topology switches randomly and independently with probability \( 0 \leq p_i \leq 1 \).

**Theorem 3.4:** Assume that the sampling period \( 0 < h \leq \frac{1}{2d} \). Then, system (7) can solve average consensus in mean square under arbitrary switching if the expected graph \( \mathcal{G} \) is connected.

**Proof:** Let \( \xi(t_k) = x(t_k) - v(x_0) \). By Lemma 2.1, it can be obtained that

\[
E[\xi^T (t_k) \xi (t_k)] = E \left[ \xi^T (t_{k-1}) \left( W_{t_{k-1}} - \frac{11T}{n} \right) \xi (t_{k-1}) \right]
\]

\[
= E \left[ \xi^T (t_{k-1}) E \left[ W_{t_{k-1}} - \frac{11T}{n} \right] \xi (t_{k-1}) \right] \xi (t_{k-1})
\]

\[
\leq \rho \left( E \left[ W_{t_{k-1}}^T W_{t_{k-1}} - \frac{11T}{n} \right] \right) E[\xi^T (t_{k-1})] \xi (t_{k-1})
\]

(22)

Recursive application of (22) yields

\[
E[\|\xi (t_k)\|^2] \leq \max_{i=0,...,k-1} \rho^k \left( E \left[ W_{t_i}^T W_{t_i} - \frac{11T}{n} \right] \right) \|\xi (0)\|^2.
\]

(23)
By Lemma 2.3, we can see that

\[
W_t \bar{W}_t = e^{-2d_k t} \left( I_n + 2\bar{A}_k t + \frac{1}{2!}(2\bar{A}_k t)^2 + \cdots \right) \\
\times (K + h\bar{A}_k)^{2d_k} \\
= e^{-2d_k t} (K^{2d_k} + hK^{2d_k-1}\bar{A}_k + \cdots) \\
\geq \Pi\bar{A}_k,
\]

where \( \Pi > 0 \) is a diagonal matrix. Because the expected graph \( \bar{G} \) is connected, we can have that the union of the graphs \( \bar{G}_i \) is connected. Hence, the union of the graphs of \( \bar{A}_k \) is connected. Consequently, we can obtain that matrix \( E[W_t^T W_t] \) is irreducible. It can also be verified that the matrix \( E[W_t^T W_t] \) is stochastic. By virtue of \( E[W_t^T W_t]I = \Pi \) and Horn and Johnson (2012, Theorem 8.4.4), we have \( |\lambda_i(E[W_t^T W_t])| < 1, i = 2, \ldots, n \). By Lemma 2.2, we know that \( \lambda_i(E[W_t^T W_t - \bar{W}_t]) = \lambda_i(E[W_t^T W_t]), i = 2, \ldots, n \). Thus, it is obvious that \( \rho(E[W_t^T W_t - \bar{W}_t]) < 1 \). Hence, it follows that \( \lim_{t_k \to \infty} E[\|e(t_k) - \nu(x_0)\|^2] = 0 \).

**Remark 3.2:** Note that switched multi-agent system (6) constructs a unified form by combining continuous-time multi-agent system (3) and discrete-time multi-agent system (4). Switched multi-agent system (6) can be a continuous-time multi-agent system or discrete-time multi-agent system when \( t_k = 0 \) and \( d_k = 0 \), respectively. The above results remain valid for multi-agent system (6) when \( t_k = 0 \) or \( d_k = 0 \).

In the following, we give the convergence factor of system (7). We focus on per-step convergence factor in the mean square sense.

**Theorem 3.5:** Assume that the sampling period \( 0 < h \leq \frac{1}{2d} \). Then, the per-step (mean square) convergence factor for system (7) under the connected expected graph \( \bar{G} \) is

\[
\gamma_s = \beta_s = E \left[ \rho^2 \left( W_n - \frac{11^T}{n} \right) \right].
\]

**Proof:** From Theorem 3.2 and (14), we get

\[
E[\|e(t_{k+1})\|^2] \leq E[\rho^2 (W_n - \frac{11^T}{n}) \|e(t_k)\|^2].
\]

Hence, we have \( \gamma_s \leq \beta_s \), where \( \gamma_s \) defined in Definition 2.2. Note that all eigenvalues of matrix \( (W_n - \frac{11^T}{n}) \) is nonnegative and thus \( \rho(W_n - \frac{11^T}{n}) \) is the largest eigenvalue. Hence, there exists an eigenvector \( \tilde{\xi} \) corresponding to \( \rho(W_n - \frac{11^T}{n}) \). It is obtained that

\[
E[\|\tilde{\xi}(t_{k+1})\|^2] = E[\rho^2 (W_n - \frac{11^T}{n}) \|\tilde{\xi}(t_k)\|^2]
\]

and \( \beta_s \leq \gamma_s \). Obviously, we have \( \beta_s = \gamma_s \).

**Remark 3.3:** Note that from Theorem 3.2, we can conclude that the per-step convergence factor is corresponding to \( E(e^{-2\lambda_2(L^\infty)}h) < 1 - h\lambda_2(L^\infty)^{2d_k} \). It is obvious that the convergence speed of the average consensus is influenced by the second eigenvalue of the Laplacian matrix \( L^\infty \). We know that \( \beta_s \) is a decreasing function of \( \lambda_2(L^\infty) \) for fixed \( p \).

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**Figure 1.** Dynamics of random graphs \( G_i \) with \( i = 1, \ldots, 8 \).
4. Simulations

In this section, we provide two numerical simulations to demonstrate the effectiveness of the theoretical results in this paper.

**Example 4.1:** Case I: We consider the random network with edge probability matrix $P$ where $p_{ii} = 0$ and $p_{ij} = 0.2$ for $i, j = 1, \ldots, 8$. The expected network is connected under edge probability matrix $P$. By calculation, we can get the sampling period $0 < h \leq \frac{1}{14}$. We choose $h = 0.01$. The evolution of network from 0 second to 8 second is shown in Figure 1. The switching law of switched multi-agent system (6) is shown in Figure 2(a). The state trajectories of all the agents are shown in Figure 2(b). We can see that switched multi-agent system (6) reaches consensus.

Case II: We consider the unconnected expected network with edges probability matrix $P$ where $p_{ij} = p_{ji} = \ldots$
\[ p_{j2} = p_{2j} = 0 \text{ for } j = 3, \ldots, 8, \quad p_{ij} = 0, \text{ and } p_{ij} = 0.2 \text{ for } i, j = 1, \ldots, 8. \] By calculation, the multiplicity of \( \lambda_1(\tilde{L}) = 0 \) is 2 and the sampling period \( 0 < h \leq \frac{1}{14} \). The sampling period \( h \) and switching law are the same as Case I. Under unconnected expected network, the state trajectories of all the agents are shown in Figure 2(c). From Figure 2(c), we can see that switched multi-agent system (6) cannot reach consensus.

**Example 4.2:** We consider that the communication network is chosen as in Figure 3. Note that the union of switching topologies is connected. The network switches randomly where \( p = 0.25 \) and \( \Delta = 1 \). By calculation, we can get the sampling period \( 0 < h \leq 0.125 \). We choose \( h = 0.1 \). The switching law of switched multi-agent system (6) is depicted in Figure 4(a). The state trajectories of all the agents are shown in the Figure 4(b). We can see the state trajectories of all the agents reach consensus which is consistent with the sufficiency of Theorem 3.4.

### 5. Conclusions

This paper investigated the consensus problem of the switched multi-agent system which is composed of continuous-time and discrete-time subsystems with random networks. An equivalent discrete-time system was proposed to solve the average consensus problem of the switched multi-agent system. Then, we derived some necessary and sufficient conditions for solving average consensus in mean square (or almost surely) under arbitrary switching. Moreover, we showed that the average consensus in different sense (mean square, almost surely and in probability, respectively) are equivalent if the sampling period satisfies certain condition and the expected graph is connected. Finally, we analysed the per-step convergence factor in the mean square sense. The future work will focus on the containment control problem of switched multi-agent systems with random networks, the consensus problem of switched multi-agent systems with disturbances under random networks, etc.

### Acknowledgment

The authors would like to thank the anonymous reviewers and the associate editors for their helpful remarks that improved the presentation of the paper.

### Disclosure statement

No potential conflict of interest was reported by the authors.

### Funding

This work was supported by National Natural Science Foundation of China [grant number 61375120, [grant number 61304160], [grant number 61533001]; the Fundamental Research Funds for the Central Universities [grant number JB160419].
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