International Journal of Control

ISSN: 0020-7179 (Print) 1366-5820 (Online) Journal homepage: http://www.tandfonline.com/loi/tcon20

Consensus of switched multi-agent systems with random networks

Xue Lin, Yuanshi Zheng & Long Wang

To cite this article: Xue Lin, Yuanshi Zheng & Long Wang (2017) Consensus of switched multi-agent systems with random networks, International Journal of Control, 90:5, 1113-1122, DOI: 10.1080/00207179.2016.1201865

To link to this article: http://dx.doi.org/10.1080/00207179.2016.1201865

Accepted author version posted online: 29 Jun 2016.
Published online: 15 Jul 2016.

INTERNATIONAL JOURNAL OF CONTROL, 2017
VOL. 90, NO. 5, 1113–1122
https://doi.org/10.1080/00207179.2016.1201865

Taylor & Francis
Taylor & Francis Group

Consensus of switched multi-agent systems with random networks

Xue Lin[a,b], Yuanshi Zheng[a,b] and Long Wang[c]

[a]Key Laboratory of Electronic Equipment Structure Design of Ministry of Education, School of Mechano-electronic Engineering, Xidian University, Xi'an, 710071, China; [b]Center for Complex Systems, School of Mechano-electronic Engineering, Xidian University, Xi'an, 710071, China; [c]Center for Systems and Control, College of Engineering, Peking University, Beijing, 100871, China

ABSTRACT

This paper studies the consensus problem of the switched multi-agent system composed of continuous-time and discrete-time subsystems. Communication among agents is modelled as a random network where the existence of any information channel is probabilistic and independent of other channels. Then, some necessary and sufficient conditions are presented for solving average consensus of the switched multi-agent system under arbitrary switching. Furthermore, we show that the average consensus in different sense (mean square, almost surely and in probability, respectively) are equivalent. Finally, simulations are provided to illustrate the effectiveness of our theoretical results.

ARTICLE HISTORY
Received 11 January 2016
Accepted 12 June 2016

KEYWORDS
Consensus; switched multi-agent systems; random networks

1. Introduction

Over the past several years, multi-agent systems have received considerable attention due to their potential applications in many areas including flocking in biology (Olfati-Saber, 2006), rendezvous problem of mobile autonomous robots (Lin, Morse, & Anderson, 2006), attitude alignment of satellite clusters (Lawton & Beard, 2002), etc. Therefore, the study of multi-agent coordination has been an important research field in the scientific community and attracted multi-disciplinary researchers' attention. Many results have been obtained about multi-agent coordination, such as consensus (Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Wang & Xiao, 2010; Zheng & Wang, 2012), containment control (Zheng & Wang, 2014), controllability (Ji, Lin, & Yu, 2015), rendezvous (Xiao, Wang, & Chen, 2012), and optimal control (Ma, Zheng, & Wang, 2015).

Consensus plays an important role in cooperative control of multi-agent systems, which aims to design appropriate distributed protocol that enables all agents to converge to a consistent state. As a fundamental research topic, consensus problem has a long history in the field of multi-agent systems. By graph theory, Jadbabaie, Lin, and Morse (2003) first gave theoretical analysis for the observed behaviour of the Vicsek model (Vicsek, Czirok, Jacob, Cohen, & Schochet, 1995). Olfati-Saber and Murray (2004) investigated the consensus for networks of dynamic agents with fixed and switching topologies. Some relaxed conditions were obtained for multi-agent consensus in Ren and Beard (2005). Furthermore, the second-order consensus protocols based on the absolute and relative velocity information were studied in Xie and Wang (2007) and Ren and Atkins (2007), respectively. Zheng, Zhu, and Wang (2011) studied the consensus of multi-agent systems with heterogeneous dynamics. With rapid development of this issue in the past decade, a tremendous amount of interesting results have been obtained under various circumstances, such as discrete-time consensus (Wang & Xiao, 2007), asynchronous consensus (Xiao & Wang, 2008), leader-following consensus (Hong, Hu, & Gao, 2006), group consensus (Zheng & Wang, 2015), etc. It should be noted that aforementioned works only analysed consensus problem of multi-agent systems under fixed and switching topologies. However, the realistic communication among agents may change with time due to link failures, packet drops, node failure, etc. Such variations in the network can happen randomly, which attracts researchers' great attention concerning random networks. The consensus problem of multi-agent systems under random networks was considered in Hatano and Mesbahi (2005). Porfiri and Stilwell (2007) extended the results in Hatano and Mesbahi (2005), and presented more relaxable conditions for solving consensus problem. Tahbaz-Salehi and Jadbabaie (2008, 2010) gave some necessary and sufficient conditions for almost sure convergence to consensus. Other research topics of consensus with random networks have also been addressed (Kar & Moura, 2008; Lin, Hou, Yan, & Yu, 2015).

CONTACT Long Wang longwang@pku.edu.cn

Switched system is a kind of system that consists of a series of subsystems, and these subsystems obey a logical rule to switch. For multi-agent systems, there have been numerous results related to the consensus with switching topologies (Olfati-Saber & Murray, 2004; Wang & Xiao, 2007; Zheng & Wang, 2012). By utilising graph theory and Lyapunov theory, Olfati-Saber and Murray investigated the consensus of continuous-time multi-agent systems with switching topologies in Olfati-Saber and Murray (2004). Wang and Xiao investigated the consensus of discrete-time multi-agent systems with switching topologies in Wang and Xiao (2007). For such multi-agent systems, it is composed of only continuous-time subsystems or discrete-time subsystems. To the best of our knowledge, the multi-agent system which has been researched only consists of continuous-time subsystems or discrete-time subsystems. However, switching behaviour exists not only in the topologies but also on the dynamical behaviours of agents. In this paper, we considered a kind of switched multi-agent systems, in which dynamics of every agent switches between continuous-time dynamics and discrete-time dynamics. In practice, many applications contain such switched systems. For example, the system controlled either by a physically implemented regulator or by a digitally implemented one with a switching rule between them, i.e. the system is composed of both continuous-time and discrete-time subsystems (Zhai, Lin, Michel, & Yasuda, 2004). Zheng and Wang (2016) investigated the consensus problem of such switched multi-agent system which is composed of continuous-time and discrete-time subsystems under deterministic networks. By using graph theory and Lyapunov theory, they prove that the consensus problem can be solved if the graph is undirected connected or has a directed spanning tree. Moreover, containment control of such switched multi-agent system with fixed topology was investigated in Zhu, Zheng, and Wang (2015). Inspired by above works, we further investigate the consensus of the switched multi-agent system composed of continuous-time and discrete-time subsystems with random networks. Different from Zheng and Wang (2016), we mainly explore that the uncertainty is embedded in the network where the existence of any information channel is probabilistic and independent of other channels. Comparing with deterministic networks, it is more practical significance to study the random networks. Due to the random nature of networks, we need to solve the consensus problem under sense of probability. Owing to switching behaviour of dynamics and the random variation of the network, it is difficult to analyse the consensus by using classical methods. The main contribution of this paper is threefold. First, by using random graph theory and stochastic analysis tools, we prove that the consensus in different sense (mean square, almost surely and in probability, respectively) can be achieved if and only if the expected graph is connected. Second, the equivalent relation of the average consensus of switched multi-agent system in different sense (mean square, almost surely and in probability, respectively) is established under the connected expected graph. Finally, the per-step convergence factor in the mean square sense is given.

The structure of this paper is given as follows. In Section 2, we present some random graph theory concepts and some definitions. In Section 3, we present the main results. In Section 4, simulation examples are provided to illustrate the effectiveness of our theoretical results. Finally, we give a short conclusion in Section 5.

Throughout this paper, the following notations will be used: we denote by $\mathbf{1}_n$ the column vector of all ones. $\mathscr{R}$ denotes the set of real number. $\mathscr{R}^n$ denotes the n dimensional real vector space. $\|\cdot\|$ denotes the standard Euclidean norm. I_n is the $n \times n$ identity matrix. *i.o.* stands for infinitely often. For a given vector or matrix A, A^T denotes its transpose. $\mathcal{I}_n = \{1, 2, \ldots, n\}$ and $\bar{d} = \max_{i \in \mathcal{I}_n}\{d_{ii}\}$. $B = [b_{ij}] \in \mathscr{R}^{n\times n}$, $B \geq 0$ if all $b_{ij} \geq 0$, and $B > 0$ if $b_{ij} > 0$. If $B \geq 0$, we say that B is a nonnegative matrix, and if $B > 0$, we say that B is a positive matrix.

2. Preliminaries

2.1 Random Graphs

In this subsection, we present some basic concepts of algebraic graph theory which will be used in this paper.

The information flow among the nodes of an undirected random network can be described by a sequence of undirected random graphs $\mathscr{G}_i$. At each time i, the random graph is $\mathscr{G}_i = (\mathscr{V}, \mathscr{E}_i)$, where $\mathscr{V} = \{v_i, i = 1, \ldots, n\}$ is a determinate vertex set and $\mathscr{E}_i = \{e_{ij}\} \subseteq \mathscr{V} \times \mathscr{V}$ is the set of edges where e_{ij} denotes that agents i and j can communicate with each other. In the random graph on n vertices, we assume that the existence of $e_{ij} \in \mathscr{E}_i$ is determined randomly and independently of other edges with probability $p_{ij} \in [0, 1]$ for $i, j = 1, \ldots, n$, and $j \neq i$. In this paper, we do not consider self-loops and multiple edges. We define the edge probability matrix $P = P^T = [p_{ij}] \in \mathscr{R}^{n\times n}$, $0 \leq p_{ij} \leq 1$ and $p_{ii} = 0$. The adjacency matrix $\mathscr{A}_i = [a_{ij}] \in \mathscr{R}^{n\times n}$ of $\mathscr{G}_i$ can be defined as

$$a_{ij} = \begin{cases} 1, & \text{with probability } p_{ij}, \\ 0, & \text{with probability } 1 - p_{ij}, \end{cases} \tag{1}$$

where $i \neq j$, and $a_{ii} = 0$ for all i. The degree matrix $\mathscr{D}_i \in \mathscr{R}^{n\times n}$ is a diagonal matrix with $d_{ii} = \sum_{j\in N_i} a_{ij}$, and $\max_{i\in\mathcal{I}_n}\{d_{ii}\}$ is maximum degree of agent under any random network. The Laplacian matrix is defined as $\mathscr{L}_i = [l_{ij}]_{n\times n} = \mathscr{D}_i - \mathscr{A}_i$. Due to the random nature of $\mathscr{A}_i$, Laplacian matrix $\mathscr{L}_i$ is also random. It is easy to see that $\mathscr{L}_i\mathbf{1} = 0$ and $\mathbf{1}^T\mathscr{L}_i = 0$. Let G, A, and L denote the sample spaces of all random graphs, all adjacency matrices, and all Laplacian matrices, respectively.

The matrices $\bar{\mathscr{A}} = E[\mathscr{A}_i] = P$ and $\bar{\mathscr{L}} = E[\mathscr{L}_i] = [\bar{l}_{ij}]_{n\times n}$ denote the expected value of the adjacency matrix and Laplacian matrix, respectively. Then, it can be obtained

$$\bar{l}_{ij} = \begin{cases} \sum_{j=1}^{n} p_{ij}, & \text{if } i = j, \\ -p_{ij}, & \text{otherwise.} \end{cases} \tag{2}$$

Matrix $\bar{\mathscr{L}}$ corresponds to graph $\bar{\mathscr{G}}$ which does not necessarily belong to G. This graph $\bar{\mathscr{G}}$ denotes the expected graph, i.e. the average graph over time. Expected graph $\bar{\mathscr{G}}$ is connected if and only if $\lambda_2(\bar{\mathscr{L}}) > 0$ (Kar & Moura, 2008, Lemma 1). Moreover, the eigenvalues of $\mathscr{L}_i$ can be denoted as $0 = \lambda_1(\mathscr{L}_i) \leq \lambda_2(\mathscr{L}_i) \leq \cdots \leq \lambda_n(\mathscr{L}_i)$. Matrix $\mathscr{L}_i$ is positive semi-definite and has a simple zero eigenvalue when $\mathscr{G}_i$ is connected undirected graph.

2.2 System model

In this subsection, we consider a multi-agent system which consists of n agents. Suppose that agent i takes the switched dynamics, it switches between continuous-time dynamics and discrete-time dynamics. The continuous-time dynamics is

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{I}_n, \tag{3}$$

and discrete-time dynamics is

$$x_i(t+h) = x_i(t) + hu_i(t), \quad i \in \mathcal{I}_n, \tag{4}$$

where $x_i(t) \in \mathscr{R}$ and $u_i(t) \in \mathscr{R}$ are the position and control input of agent i, respectively. $h > 0$ is the sampling period and $x_0 = [x_1(0), \dots, x_n(0)]^T$ is the initial value. We apply the consensus protocol for switched multi-agent system $(3-4)$ as follows

$$u_i(t) = \sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t)). \tag{5}$$

Suppose that the dynamics of each agent switches simultaneously from one to another and the agents with discrete-time dynamics finish integer sampling. Then, switched multi-agent system $(3-4)$ with protocol (5) which is composed of continuous-time subsystem

$$\dot{x}(t) = -\mathscr{L}x(t), \tag{6a}$$

and discrete-time subsystem

$$x(t+h) = (I_n - h\mathscr{L})x(t), \tag{6b}$$

i.e. the switched multi-agent system can be viewed as a system which is composed of (6a) and (6b) at different time interval.

Switched multi-agent system (6) switches arbitrarily in the random network. We assume that the network which consists of random edges is constant over each time interval $\Delta > 0$. The topology structure at time point t_k is independent of topology structure at the previous time-intervals. By introducing the sequence of column vectors $\{x(t_k)\}$, the problem may be cast in a sample-data system setting. From subsystem (6a), we know that $x(t) = e^{\mathscr{L}(t-t_0)}x(t_0)$. Let the sequence of column vectors $\{x(t_k)\}$ be the subsequence of $\{x(t)\}$. Due to $e^{\mathscr{L}t}(I_n - h\mathscr{L}) = (I_n - h\mathscr{L})e^{\mathscr{L}t}$, the trajectory $x(t_k)$ of switched multi-agent system (6) is expressed by

$$x(t_{k+1}) = W_{t_k}x(t_k), \quad k = 0, 1, \dots, \tag{7}$$

where W_{t_k} is the random state transition matrix defined by

$$W_{t_k} = e^{-\mathscr{L}_{t_k}t_{c_k}}(I - h\mathscr{L}_{t_k})^{d_k}. \tag{8}$$

$\mathscr{L}_{t_k}$ is the Laplacian matrix of random graph $\mathscr{G}_{t_k}$ at time point t_k. Note that graph $\mathscr{G}_{t_k}$ is invariant during the time interval Δ. Matrix W_{t_k} at time point t_k is independent of matrix Wt_{k-1} at time point t_{k-1}. The total duration time on subsystem (6a) and the total sampling times on subsystem (6b) under time interval $t_{k+1} - t_k$ are denoted as t_{c_k} and d_k, respectively. For the discrete-time subsystem, when the integer sampling of discrete-time subsystem are completed under time interval Δ or time interval $\Delta + h$, we make a sampling at time t_k. For the continuous-time subsystem, we can make arbitrary sampling at time t_k under a network, i.e. before the network changes randomly at the transition instant. We know that state $x(t_k)$ of system (7) at each moment is sampled from state variables of system (6). Therefore, system (7) can be used to monitor the partial state of switched multi-agent system (6).

The objective of this paper is to make the switched multi-agent system to achieve average consensus in mean square (or almost surely) under arbitrary switching with random networks. The vector of averages $v(x_0)$ is written as $v(x_0) = \frac{\mathbf{1}^T x_0}{n}\mathbf{1}$. The following notions of consensus will be used.

Definition 2.1: We say that switched multi-agent system (6) converges to average consensus

(a) in mean square if for any $x_0 \in \mathscr{R}^n$ it holds that

$$\lim_{t\to\infty} E\left[\|x(t) - v(x_0)\|^2\right] = 0; \qquad (9)$$

(b) almost surely if for any $x_0 \in \mathscr{R}^n$ it holds that

$$P\left\{\lim_{t\to\infty} \|x(t) - v(x_0)\| = 0\right\} = 1; \qquad (10)$$

(c) in probability if $\forall \epsilon > 0$ and any $x_0 \in \mathscr{R}^n$ it holds that

$$\lim_{t\to\infty} P\{\|x(t) - v(x_0)\| > \varepsilon\} = 0; \qquad (11)$$

(d) in mean if for any $x_0 \in \mathscr{R}^n$ it holds that

$$\lim_{t\to\infty} E[x(t)] = v(x_0). \qquad (12)$$

Definition 2.2: The per-step convergence factor, in the mean square sense, for system (7) is defined as

$$\gamma_s = \sup_{\xi(t_k)\neq 0,\ k\in N} \left(\frac{E[\|\xi(t_{k+1})\|^2 | \xi(t_k)]}{\|\xi(t_k)\|^2} \right), \qquad (13)$$

where $\xi(t_k) = x(t_k) - v(x_0)$.

Next, we will give some properties of state transition matrix of system (7) for solving the consensus problem.

Lemma 2.1: *Matrix W_{t_k} is doubly stochastic matrix if the sampling period $0 < h < \frac{1}{\bar{d}}$.*

Proof: Due to

$$\begin{aligned} W_{t_k}\mathbf{1}_n &= e^{-\mathscr{L}_{t_k} t_{c_k}} (I_n - h\mathscr{L}_{t_k})^{d_k} \mathbf{1}_n \\ &= \left(\sum_{j=0}^{\infty} \frac{(-t_{c_k})^j}{j!} \mathscr{L}_{t_k}^j \right) \left(I_n - h\mathscr{L}_{t_k}\right)^{d_k} \mathbf{1}_n \\ &= \mathbf{1}_n, \end{aligned}$$

and $\mathscr{L}_{t_k}^T = \mathscr{L}_{t_k}$, it is obvious that $\mathbf{1}_n^T W_{t_k} = \mathbf{1}_n^T$. Laplacian matrix $\mathscr{L}_{t_k}$ can be rewritten as $\mathscr{L}_{t_k} = \bar{d}I_n - \tilde{A}_k$. Because $0 < h < \frac{1}{\bar{d}}$ and $\tilde{A}_k$ is nonnegative matrix, it is clear that $e^{-\bar{d}I_n t_{c_k}} \geq 0$, $e^{\tilde{A}_k t_{c_k}} \geq 0$, and $(I_n - h\bar{d}I_n + h\tilde{A}_k) \geq 0$.

Note that

$$\begin{aligned} W_{t_k} &= e^{(-\bar{d}I_n + \tilde{A}_k)t_{c_k}} (I_n - h\bar{d}I_n + h\tilde{A}_k)^{d_k} \\ &= e^{-\bar{d}I_n t_{c_k}} e^{\tilde{A}_k t_{c_k}} (I_n - h\bar{d}I_n + h\tilde{A}_k)^{d_k}, \end{aligned}$$

thus, matrix W_{t_k} is nonnegative. Hence, W_{t_k} is a doubly stochastic matrix. ■

Lemma 2.2: *The i-th eigenvalue of matrix $W_{t_k} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}$ is $\lambda_i = e^{-\lambda_i(\mathscr{L}_{t_k})t_{c_k}} (1 - h\lambda_i(\mathscr{L}_{t_k}))^{d_k}$, $i = 2, \ldots, n$.*

Proof: Since $\mathscr{L}_{t_k}$ is symmetric matrix, there exists a unitary matrix U such that $\mathscr{L}_{t_k} = U^T \Lambda U$, where Λ is the diagonal matrix consisting of $\lambda_i(\mathscr{L}_{t_k})$, $i = 1, \ldots, n$. Hence, $W_{t_k} = e^{-U^T \Lambda U t_{c_k}} (I_n - hU^T\Lambda U)^{d_k} = U^T e^{-\Lambda t_{c_k}} (I_n - h\Lambda)^{d_k} U$, and $\lambda_i(W_{t_k}) = e^{-\lambda_i(\mathscr{L}_{t_k})t_{c_k}} (1 - h\lambda_i(\mathscr{L}_{t_k}))^{d_k}$, $i = 1, \ldots, n$.

According to Lemma 2.1, it follows that $W_{t_k}\frac{\mathbf{1}_n\mathbf{1}_n^T}{n} = \frac{\mathbf{1}_n\mathbf{1}_n^T}{n} W_{t_k}$. By virtue of Theorem 4.5.15 in Horn and Johnson (2012), it can be obtained $\frac{\mathbf{1}_n\mathbf{1}_n^T}{n} = \bar{U}^T \Sigma_1 \bar{U}$ and $W_{t_k} = \bar{U}^T \Sigma_2 \bar{U}$ where $\bar{U} = [\bar{u}_1, \ldots, \bar{u}_n]$. Σ_1 and Σ_2 are the diagonal matrices consisting of $\lambda_i(\frac{\mathbf{1}_n\mathbf{1}_n^T}{n})$ and $\lambda_i(W_{t_k})$, respectively. Obviously, $W_{t_k} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n} = \bar{U}^T(e^{-\Lambda t_{c_k}}(I_n - h\Lambda)^{d_k} - \Sigma_1)\bar{U}$. Note that there exists the normalised eigenvector of $\bar{u}_1$ corresponding to $\lambda_1(W_{t_k}) = 1$ and $\lambda_1(\frac{\mathbf{1}_n\mathbf{1}_n^T}{n}) = 1$ where $\bar{u}_1 = \frac{1}{\sqrt{n}}\mathbf{1}$. Hence, $\lambda_i = e^{-\lambda_i(\mathscr{L}_{t_k})t_{c_k}} (1 - h\lambda_i(\mathscr{L}_{t_k}))^{d_k}$, $i = 2, \ldots, n$. ■

Remark 2.1: When $t_{c_k} = 0$ or $d_k = 0$, the i-th eigenvalue of matrix of $W_{t_k} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}$ are $\lambda_{i\geq 2} = e^{-\lambda_i(\mathscr{L}_{t_k})t_{c_k}}$ and $\lambda_{i\geq 2} = (1 - h\lambda_i(\mathscr{L}_{t_k}))^{k_d}$, respectively.

Lemma 2.3: *Expected matrix $E[W_{t_k}]$ is irreducible and nonnegative if $\lambda_2(\bar{\mathscr{L}}) > 0$ and the sampling period $0 < h < \frac{1}{\bar{d}}$.*

Proof: Matrix $\bar{\mathscr{A}}$ is irreducible due to $\lambda_2(\bar{\mathscr{L}}) > 0$. The irreducibility of $\bar{\mathscr{A}}$ shows that there exists the graph with non-zero probability for which $\lambda_2(\mathscr{L}_{t_k}) > 0$. The adjacency matrix A_{t_k} corresponding to $\lambda_2(\mathscr{L}_{t_k}) > 0$ is irreducible and nonnegative. By Lemma 2.1, it can be obtained

$$\begin{aligned} W_{t_k} &= e^{-\bar{d}I_n t_{c_k}} e^{\tilde{A}_k t_{c_k}} (I_n - h\bar{d}I_n + h\tilde{A}_k)^{d_k} \\ &= e^{-\bar{d}I_n t_{c_k}} (I_n + \tilde{A}_k t_{c_k} + \tfrac{1}{2!}(\tilde{A}_k t_{c_k})^2 + \cdots)(K + h\tilde{A}_k)^{d_k}, \end{aligned}$$

where $\tilde{A}_{t_k} = A_{t_k} + M \geq 0$, and $K = I_n - h\bar{d}I_n > 0$. It is obvious that W_{t_k} is irreducible and nonnegative when $t_{c_k} = 0$ or $d_k = 0$. Hence,

$$\begin{aligned} W_{t_k} &= e^{-\bar{d}I_n t_{c_k}} (I_n + \tilde{A}_k t_{c_k} + \tfrac{1}{2!}(\tilde{A}_k t_{c_k})^2 + \cdots)(K + h\tilde{A}_k)^{d_k} \\ &= e^{-\bar{d}I_n t_{c_k}} (K^{d_k} + hK^{d_k - 1}\tilde{A}_k + \cdots). \end{aligned}$$

Thus, W_{t_k} is irreducible and nonnegative. Consequently, $E[W_{t_k}]$ is irreducible and nonnegative. ■

Based on Lemma 2.1 and $\mathbf{1}_n^T x(k+1) = \mathbf{1}_n^T x_0$, we can straightly obtain the following result.

Lemma 2.4: *For system (7), we have $\|x(t_{k+1}) - v(x_0)\|^2 \leq \rho^2(W_{t_k} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n})\|x(t_k) - v(x_0)\|^2$.*

3. Main results

In this section, the average consensus problem of the switched multi-agent system will be investigated. Moreover, we further derive the per-step (mean square) convergence factor for system (7).

Theorem 3.1: *Assume that the sampling period $0 < h \leq \frac{1}{2\bar{d}}$. Then, switched multi-agent system (6) reaches average consensus in mean square (or almost surely) if and only if system (7) reaches average consensus in the same sense.*

Proof: (**Sufficiency**) We firstly analyse the state of continuous-time subsystem. The state transition matrix of the continuous-time subsystem matrix is $e^{-\mathscr{L}t}$. By Lemma 2.2 and $0 < h \leq \frac{1}{2\bar{d}}$, we have $\rho^2(e^{-\mathscr{L}t} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}) \leq 1$. By Lemma 2.4, it can easily be verified that the state at any moment such that $\|x(t_{k_{i+1}}) - \upsilon(x_0)\|^2 \leq \|x(t_{k_i}) - \upsilon(x_0)\|^2$ where t_{k_i} is any sampling time for continuous-time subsystem. The state of discrete-time subsystem also possesses the same property and the proof is omitted. This implies that the sequence $\|x(t) - v(x_0)\|$ is monotonic. Therefore, the convex combination $E[\|x(t) - v(x_0)\|^2]$ is bounded monotonic sequence. It contains a convergent subsequence $E[\|x(t_k) - v(x_0)\|^2]$ such that $\lim_{t_k\to\infty} E[\|x(t_k) - \upsilon(x_0)\|^2] = 0$. It follows that

$$\lim_{t\to\infty} E[\|x(t) - \upsilon(x_0)\|^2] = 0.$$

Therefore, switched multi-agent system (6) reaches average consensus in mean square.

(**Necessity**) Since the sequence $\{x(t_k)\}$ is a subsequence of $\{x(t)\}$, it is obvious that $\lim_{t_k\to\infty} E[\|x(t_k) - \upsilon(x_0)\|^2] = 0$ if $\lim_{t\to\infty} E[\|x(t) - v(x_0)\|^2] = 0$.

We only show the equivalence of average consensus in mean square. The equivalence of average consensus almost surely can be proved similarly. Thus, it is omitted here. ■

According to Theorem 3.1, it is shown that we analyse the average consensus of system (6) is equivalent to analyse the average consensus of system (7). Next, we mainly discuss the average consensus of system (7).

Theorem 3.2: *Assume that the sampling period $0 < h \leq \frac{1}{2\bar{d}}$. Then, system (7) can solve average consensus in mean square under arbitrary switching if and only if the expected graph $\bar{\mathscr{G}}$ is connected.*

Proof: (**Sufficiency**) By Lemma 2.4, we know that

$$\|x(t_{k+1}) - \upsilon(x_0)\|^2 \leq \rho^2\left(W_{t_k} - \frac{\mathbf{1}\mathbf{1}^T}{n}\right)\|x(t_k) - \upsilon(x_0)\|^2, \tag{14}$$

which implies that

$$\|x(t_k) - \upsilon(x_0)\|^2 \leq \rho^2\left(W_{t_0} - \frac{\mathbf{1}\mathbf{1}^T}{n}\right)\cdots\rho^2 \times \left(W_{t_{k-1}} - \frac{\mathbf{1}\mathbf{1}^T}{n}\right)\|x(0) - \upsilon(x_0)\|^2. \tag{15}$$

Taking expectation on both sides of (15) and using the independent property of the random matrix W_{t_k}, we have

$$\begin{aligned} E[\|x(t_k) - \upsilon(x_0)\|^2] &\leq E\left[\rho^2\left(W_{t_0} - \frac{\mathbf{1}\mathbf{1}^T}{n}\right)\right]\cdots \\ &\quad\times E\left[\rho^2\left(W_{t_{k-1}} - \frac{\mathbf{1}\mathbf{1}^T}{n}\right)\right]\|x(0) - \upsilon(x_0)\|^2 \\ &\leq \max_{i=0,\ldots,k-1} E\left[\rho^2\left(W_{t_i} - \frac{\mathbf{1}\mathbf{1}^T}{n}\right)\right]^k\|x(0) - \upsilon(x_0)\|^2. \end{aligned} \tag{16}$$

In order to prove that the state vector sequence of system (7) converges in mean square, we only need to prove that $E[\rho^2(W_{t_i} - \frac{\mathbf{1}\mathbf{1}^T}{n})] < 1$ for $i = 0, 1, \ldots$. By Lemma 2.2, we have the eigenvalues $\lambda_i(W_{t_i} - \frac{\mathbf{1}\mathbf{1}^T}{n}) = e^{-\lambda_i(\mathscr{L}_{t_i})t_{c_i}}(1 - h\lambda_i(\mathscr{L}_{t_i}))^{d_i}$, $i \geq 2$. Based on Geršgorin Disc theorem, we have the sampling period $0 < h \leq \frac{1}{2\bar{d}} \leq \frac{1}{\bar{\lambda}_n}$ with non-zero probability where $\bar{\lambda}_n = \max_{i=0,\ldots,k-1}\{\lambda_n(\mathscr{L}_{t_i})\}$. This also implies that $e^{-\lambda_i(\mathscr{L}_{t_i})t_{c_i}}(1 - h\lambda_i(\mathscr{L}_{t_i}))^{d_i} \geq 0$ and $\rho^2(W_{t_i} - \frac{\mathbf{1}\mathbf{1}^T}{n}) = e^{-2\lambda_2(\mathscr{L}_{t_i})t_{c_i}}(1 - h\lambda_2(\mathscr{L}_{t_i}))^{2d_i}) \leq 1$. Indeed, the expected graph $\bar{\mathscr{G}}$ which is connected shows that there is at least one graph $\mathscr{G}$ with non-zero probability for $\lambda_2(\mathscr{L}) > 0$. It is obvious that $(e^{-2\lambda_i(\mathscr{L})t_c}(1 - h\lambda_i(\mathscr{L}))^{2d}) < 1$, i.e., $E[\rho^2(W_{t_i} - \frac{\mathbf{1}\mathbf{1}^T}{n})] < 1$. It is shown that

$$\lim_{t_k\to\infty} E[\|x(t_k) - \upsilon(x_0)\|^2] = 0. \tag{17}$$

(**Necessity**) If the expected graph is $\bar{\mathscr{G}}$ which is not connected, there exist at least two components with zero probability of communication between each other. This implies that there is no path between two components. Hence, the information of these two components cannot reach consensus for any initial condition (Ren & Beard, 2005). ■

Remark 3.1: In fact, the expected graph $\bar{\mathscr{G}}$ is connected which implies $\lambda_2(\bar{\mathscr{L}}) > 0$. Under sampling period $0 < h \leq \frac{1}{2\bar{d}}$, switched multi-agent system (6) can solve average consensus in mean square if $\lambda_2(\bar{\mathscr{L}}) > 0$. For $\lambda_2(\bar{\mathscr{L}})$, we just need to know the edge probability matrix P to calculate it.

Next, we show that both average consensus almost surely and average consensus in mean can be achieved.

Theorem 3.3: *Assume that the sampling period $0 < h \leq \frac{1}{2d}$. Then, system (7) can solve average consensus almost surely under arbitrary switching if and only if the expected graph $\bar{\mathscr{G}}$ is connected.*

Proof: (**Sufficiency**) As a result of Markovapo's inequality (Sheldon, 1976), for any $a > 0$,

$$P\left\{\|x(t_k) - \upsilon(x_0)\|^2 \geq a^2\right\} \leq \frac{E[\|x(t_k) - \upsilon(x_0)\|^2]}{a^2}. \tag{18}$$

Because $\|x(t_k) - v(x_0)\|^2 \geq a^2$ is equivalent to $\|x(t_k) - v(x_0)\| \geq a$, inequality (18) can be written as

$$P\{\|x(t_k) - \upsilon(x_0)\| \geq a\} \leq \frac{E[\|x(t_k) - \upsilon(x_0)\|^2]}{a^2}. \tag{19}$$

By Theorem 3.2, it is easy to know that $\|x(t_{k+1}) - \upsilon(x_0)\|^2 \leq \rho^2(W_{t_k} - \frac{\mathbf{1}\mathbf{1}^T}{n})\|x(t_k) - \upsilon(x_0)\|^2$. Therefore,

$$\sum_{k=0}^{\infty} P\{\|x(t_k) - \upsilon(x_0)\| \geq a\} \leq \frac{E[\|x_0 - \upsilon(x_0)\|^2]}{a^2(1-\beta)} < \infty, \tag{20}$$

where $\beta = \max_{k=0,1,\ldots}\{E[\rho^2(W_{t_k} - \frac{\mathbf{1}\mathbf{1}^T}{n})]\}$.

Using the Borel–Cantelli Lemma (Durrett, 2010) leads to

$$P\left(\|x(t_k) - \upsilon(x_0)\| \geq a \ i.o.\right) = 0. \tag{21}$$

Thus, we conclude that the agents reach average consensus almost surely.

(**Necessity**) The proof is similar to the argument used in Theorem 3.2. ■

Corollary 3.1: *Assume that the sampling period $0 < h \leq \frac{1}{2d}$ and the expected graph $\bar{\mathscr{G}}$ is connected. Then, the following statements are equivalent.*

(1) *switched multi-agent system (6) reaches average consensus in mean square;*

(2) *switched multi-agent system (6) reaches average consensus almost surely;*

(3) *switched multi-agent system (6) reaches average consensus in probability.*

Proof: Based on Theorems 3.2 and 3.3, it is obvious that (1)⇔(2). It is well known that convergence in probability can be obtained by using convergence almost surely, i.e., (2)⇒(3). From Theorem 3.1, we know that the sequence $\{\|x(t) - v(x_0)\|\}$ is bounded monotonic. Thus, it follows from the dominated convergence theorem (Durrett, 2010) that (3)⇒(1). ■

Corollary 3.2: *Assume that the sampling period $0 < h \leq \frac{1}{2d}$. Then, switched multi-agent system (6) can solve average consensus in mean under arbitrary switching if the expected graph $\bar{\mathscr{G}}$ is connected.*

Proof: Let $\Xi = \sum_{j=1}^{n}(x_j(t))$. Then we take expectation on both sides of it, it follows that $E[\Xi] = E[\mathbf{1}^T x(t)]$. We have

$$\lim_{t\to\infty} E[\mathbf{1}^T x(t)] = \sum_{j=1}^{n}(x_j(0)).$$

Moreover, by Theorem 3.3, we know that switched multi-agent system (6) reaches consensus almost surely. By using the property of expectation $E[X + Y] = E[X] + E[Y]$ and the Lebesgue's Dominated Convergence Theorem (Rudin, 1987), we can get

$$\lim_{t\to\infty} E[x_j(t)] = \frac{1}{n}\sum_{j=1}^{n}(x_j(0)).$$

■

In the following, we consider switched multi-agent system (6) with switching topologies $\{\mathscr{G}_s : s \in \zeta_0\}$, where ζ_0 is a finite index set, the topologies are fixed, and switching is probabilistic. The topology switches randomly and independently with probability $0 \leq p_i \leq 1$.

Theorem 3.4: *Assume that the sampling period $0 < h \leq \min_{s\in\zeta_0} \frac{1}{2d_s}$. Then, system (7) can solve average consensus in mean square under arbitrary switching if the expected graph $\bar{\mathscr{G}}$ is connected.*

Proof: Let $\xi(t_k) = x(t_k) - v(x_0)$. By Lemma 2.1, it can be obtained that

$$\begin{aligned}
&E[\xi^T(t_k)\xi(t_k)] \\
&= E\left[\xi^T(t_{k-1})\left(W_{t_{k-1}}^T W_{t_{k-1}} - \frac{\mathbf{1}\mathbf{1}^T}{n}\right)\xi(t_{k-1})\right] \\
&= E\left[\xi^T(t_{k-1})E\left[W_{t_{k-1}}^T W_{t_{k-1}} - \frac{\mathbf{1}\mathbf{1}^T}{n}\right]\xi(t_{k-1})|\xi(t_{k-1})\right] \\
&\leq \rho\left(E\left[W_{t_{k-1}}^T W_{t_{k-1}} - \frac{\mathbf{1}\mathbf{1}^T}{n}\right]\right)E[\xi^T(t_{k-1})\xi(t_{k-1})].
\end{aligned} \tag{22}$$

Recursive application of (22) yields

$$E[\|\xi(t_k)\|^2] \leq \max_{i=0,\ldots,k-1} \rho^k\left(E\left[W_{t_i}^T W_{t_i} - \frac{\mathbf{1}\mathbf{1}^T}{n}\right]\right)\|\xi(0)\|^2. \tag{23}$$

By Lemma 2.3, we can see that

$$\begin{aligned} W_{t_k}^T W_{t_k} &= e^{-2\bar{d}I_n t_{c_k}} \left(I_n + 2\tilde{A}_k t_{c_k} + \frac{1}{2!}(2\tilde{A}_k t_{c_k})^2 + \cdots \right) \\ &\quad \times (K + h\tilde{A}_k)^{2d_k} \\ &= e^{-2\bar{d}I_n t_{c_k}} (K^{2d_k} + hK^{2d_k-1}\tilde{A}_k + \cdots) \\ &\geq \Pi \tilde{A}_k, \end{aligned}$$

where $\Pi > 0$ is a diagonal matrix. Because the expected graph $\bar{\mathscr{G}}$ is connected, we can have that the union of the graphs $\mathscr{G}_s$ is connected. Hence, the union of the graphs of $\tilde{A}_k$ is connected. Consequently, we can obtain that matrix $E[W_{t_i}^T W_{t_i}]$ is irreducible. It can also be verified that the matrix $E[W_{t_i}^T W_{t_i}]$ is stochastic. By virtue of $E[W_{t_i}^T W_{t_i}]\mathbf{1} = \mathbf{1}\mathbf{1}$ and Horn and Johnson (2012, Theorem 8.4.4), we have $|\lambda_i(E[W_{t_i}^T W_{t_i}])| < 1$, $i = 2, \ldots, n$. By Lemma 2.2, we know that $\lambda_i(E[W_{t_i}^T W_{t_i} - \frac{\mathbf{1}\mathbf{1}^T}{n}]) = \lambda_i(E[W_{t_i}^T W_{t_i}])$, $i = 2, \ldots, n$. Thus, it is obvious that $\rho(E[W_{t_i}^T W_{t_i} - \frac{\mathbf{1}\mathbf{1}^T}{n}]) < 1$. Hence, it follows that $\lim_{t_k \to \infty} E[\|x(t_k) - \upsilon(x_0)\|^2] = 0$. ∎

Remark 3.2: Note that switched multi-agent system (6) constructs a unified form by combining continuous-time multi-agent system (3) and discrete-time multi-agent system (4). Switched multi-agent system (6) can be a continuous-time multi-agent system or discrete-time multi-agent system when $t_c = 0$ and $d_k = 0$, respectively. The above results remain valid for multi-agent system (6) when $t_c = 0$ or $d_k = 0$.

In the following, we give the convergence factor of system (7). We focus on per-step convergence factor in the mean square sense.

Theorem 3.5: *Assume that the sampling period $0 < h \leq \frac{1}{2\bar{d}}$. Then, the per-step (mean square) convergence factor for system (7) under the connected expected graph $\bar{\mathscr{G}}$ is*

$$\gamma_s = \beta_s = E\left[\rho^2\left(W_{t_k} - \frac{\mathbf{1}\mathbf{1}^T}{n}\right)\right]. \tag{24}$$

Proof: From Theorem 3.2 and (14), we get $E[\|\xi(t_{k+1})\|^2 | \xi(t_k)] \leq E[\rho^2(W_{t_k} - \frac{\mathbf{1}\mathbf{1}^T}{n})]\|\xi(t_k)\|^2$. Hence, we have $\gamma_s \leq \beta_s$ where γ_s defined in Definition 2.2. Note that all eigenvalues of matrix $(W_{t_k} - \frac{\mathbf{1}\mathbf{1}^T}{n})$ is nonnegative and thus $\rho(W_{t_k} - \frac{\mathbf{1}\mathbf{1}^T}{n})$ is the largest eigenvalue. Hence, there exists a eigenvector ϑ corresponding to $\rho(W_{t_k} - \frac{\mathbf{1}\mathbf{1}^T}{n})$. It is obtained that $E[\|\xi(t_{k+1})\|^2 | \xi(t_k)] = E[\rho^2(W_{t_k} - \frac{\mathbf{1}\mathbf{1}^T}{n})]\|\xi(t_k)\|^2$ and $\beta_s \leq \gamma_s$. Obviously, we have $\beta_s = \gamma_s$. ∎

Remark 3.3: Note that from Theorem 3.2, we can conclude that the per-step convergence factor is corresponding to $E(e^{-2\lambda_2(\mathscr{L}_{t_k})t_{c_k}}(1 - h\lambda_2(\mathscr{L}_{t_k}))^{2d_k})$. It is obvious that the convergence speed of the average consensus is influenced by the second eigenvalue of the Laplacian matrix $\mathscr{L}_{t_k}$. We know that β_s is a decreasing function of $\lambda_2(\mathscr{L}_{t_k})$ for fixed p.

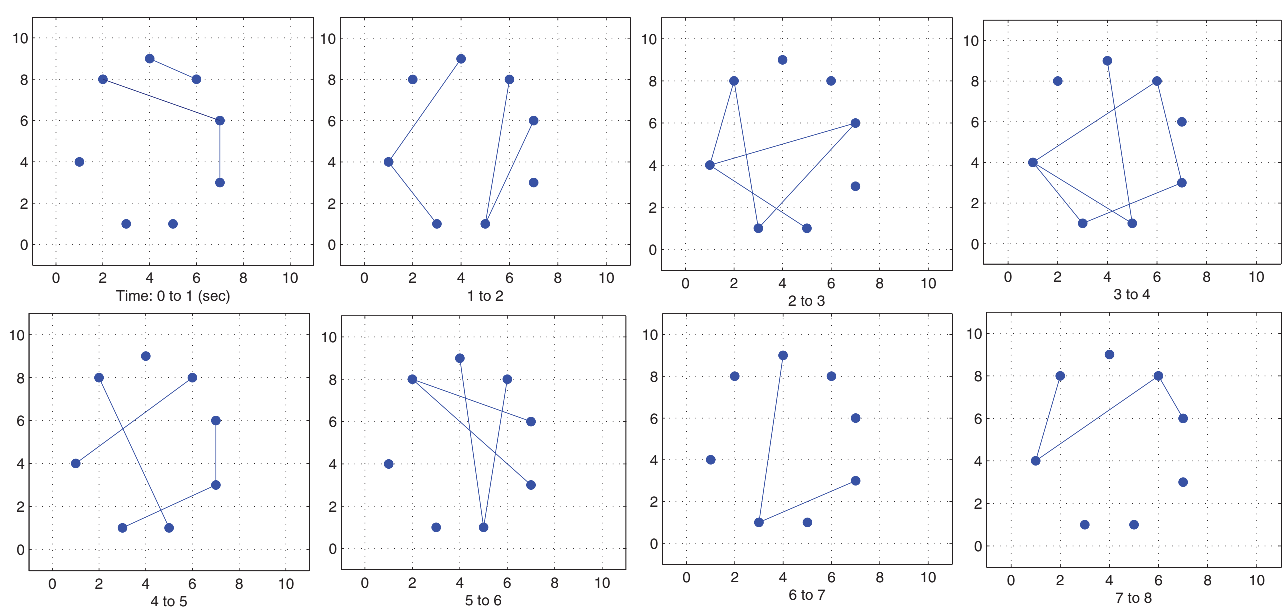

Figure 1. Dynamics of random graphs $\mathscr{G}_i$ with $i = 1, \ldots, 8$.

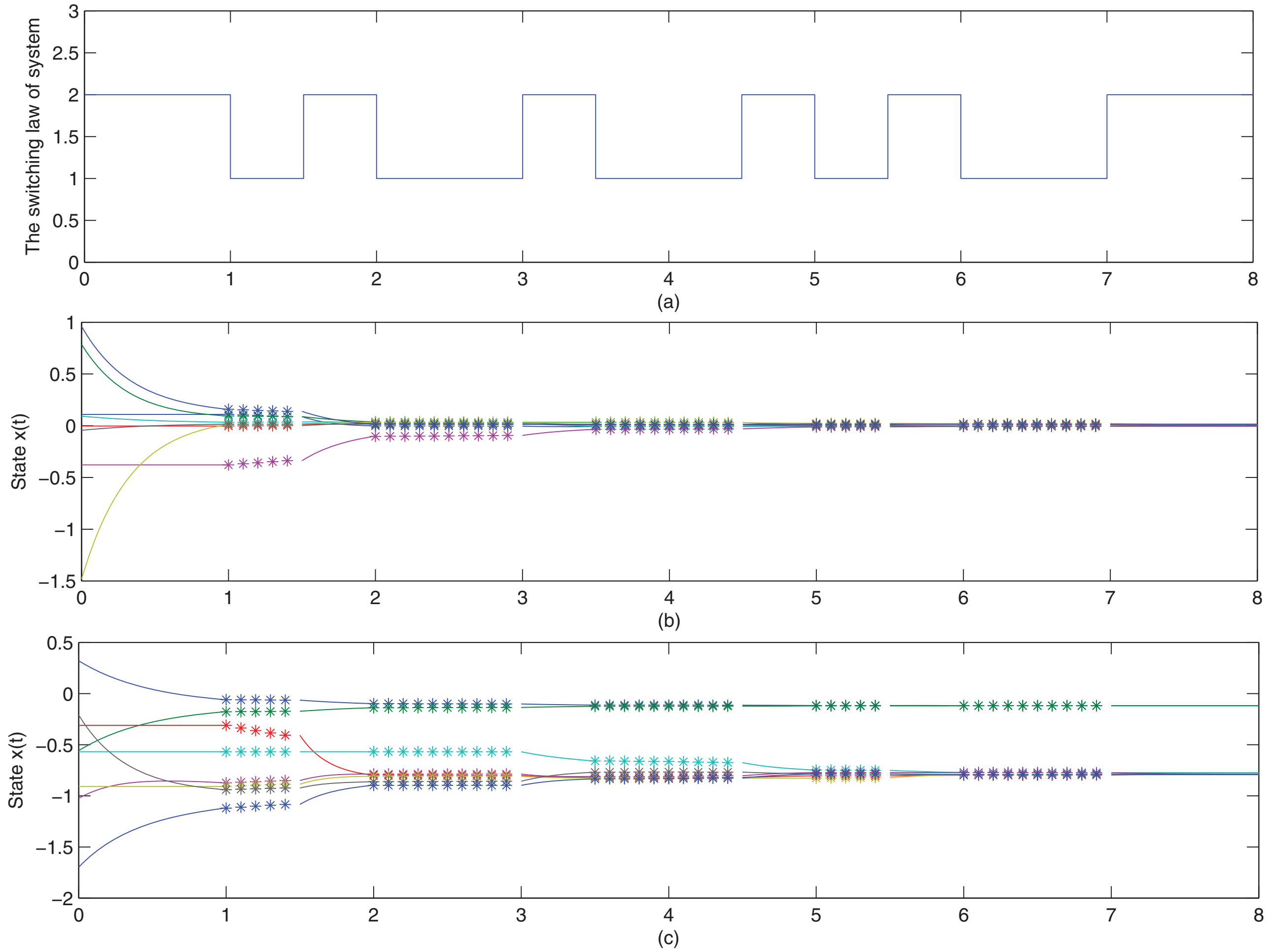

Figure 2. The switching law of system (6) and the state trajectories of all the agents.

4. Simulations

In this section, we provide two numerical simulations to demonstrate the effectiveness of the theoretical results in this paper.

Example 4.1: *Case I*: We consider the random network with edge probability matrix P where $p_{ii} = 0$ and $p_{ij} = 0.2$ for $i, j = 1, \ldots, 8$. The expected network is connected under edge probability matrix P. By calculation, we can get the sampling period $0 < h \leq \frac{1}{14}$. We choose $h = 0.01$. The evolution of network from 0 second to 8 second is shown in Figure 1. The switching law of switched multi-agent system (6) is shown in Figure 2(a). The state trajectories of all the agents are shown in Figure 2(b). We can see that switched multi-agent system (6) reaches consensus.

Case II: We consider the unconnected expected network with edges probability matrix P where $p_{j1} = p_{1j} =$

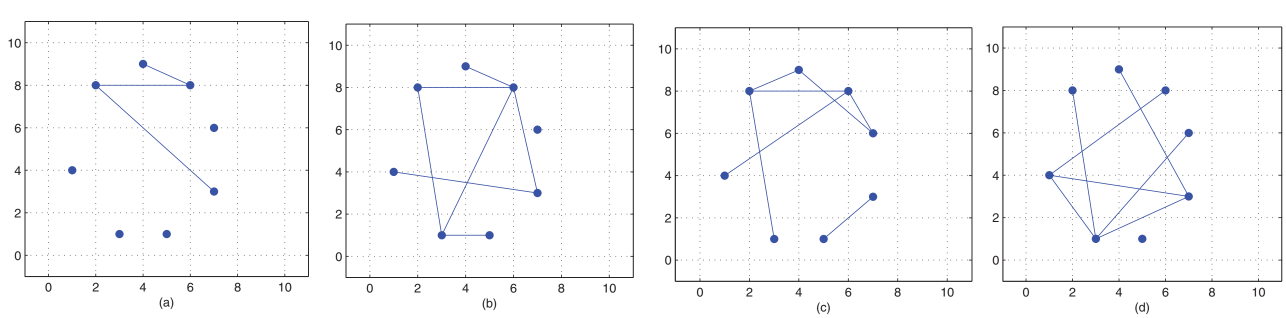

Figure 3. Four undirected graphs.

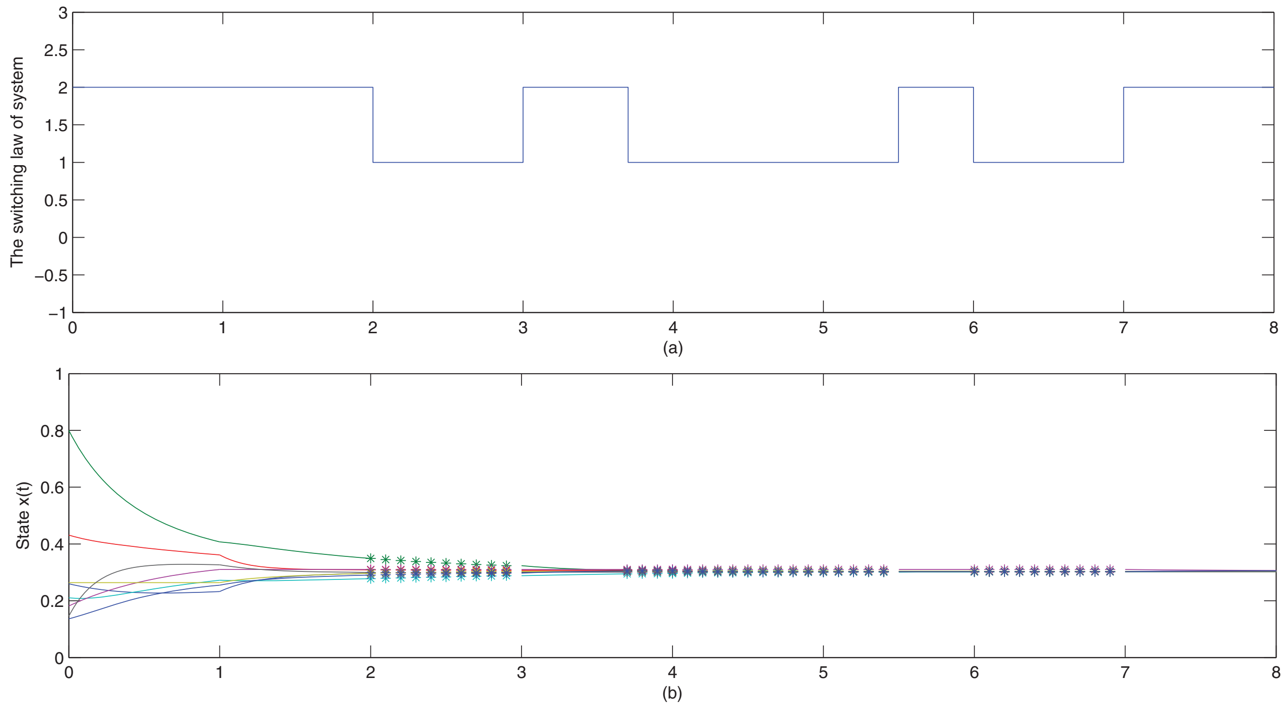

Figure 4. The switching law of system (6) and the state trajectories of all the agents.

$p_{j2} = p_{2j} = 0$ for $j = 3, \ldots, 8$, $p_{ii} = 0$, and $p_{ij} = 0.2$ for i, $j = 1, \ldots, 8$. By calculation, the multiplicity of $\lambda_1(\bar{\mathscr{L}}) = 0$ is 2 and the sampling period $0 < h \leq \frac{1}{14}$. The sampling period h and switching law are the same as Case I. Under unconnected expected network, the state trajectories of all the agents are shown in Figure 2(c). From Figure 2(c), we can see that switched multi-agent system (6) cannot reach consensus.

Example 4.2: We consider that the communication network is chosen as in Figure 3. Note that the union of switching topologies is connected. The network switches randomly where $p = 0.25$ and $\Delta = 1$. By calculation, we can get the sampling period $0 < h \leq 0.125$. We choose $h = 0.1$. The switching law of switched multi-agent system (6) is depicted in Figure 4(a). The state trajectories of all the agents are shown in the Figure 4(b). We can see the state trajectories of all the agents reach consensus which is consistent with the sufficiency of Theorem 3.4.

5. Conclusions

This paper investigated the consensus problem of the switched multi-agent system which is composed of continuous-time and discrete-time subsystems with random networks. An equivalent discrete-time system was proposed to solve the average consensus problem of the switched multi-agent system. Then, we derived some necessary and sufficient conditions for solving average consensus in mean square (or almost surely) under arbitrary switching. Moreover, we showed that the average consensus in different sense (mean square, almost surely and in probability, respectively) are equivalent if the sampling period satisfies certain condition and the expected graph is connected. Finally, we analysed the per-step convergence factor in the mean square sense. The future work will focus on the containment control problem of switched multi-agent systems with random networks, the consensus problem of switched multi-agent systems with disturbances under random networks, etc.

Acknowledgment

The authors would like to thank the anonymous reviewers and the associate editors for their helpful remarks that improved the presentation of the paper.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by National Natural Science Foundation of China [grant number 61375120, [grant number 61304160], [grant number 61533001]; the Fundamental Research Funds for the Central Universities [grant number JB160419).

References

Durrett, R. (2010). *Probability: Theory and examples*. Cambridge: Cambridge university press.

Hatano, Y., & Mesbahi, M. (2005). Agreement over random networks. *IEEE Transactions on Automatic Control, 50*(11), 1867–1872.

Hong, Y., Hu, J., & Gao, L. (2006). Tracking control for multi-agent consensus with an active leader and variable topology. *Automatica, 42*(7), 1177–1182.

Horn, R.A., & Johnson, C.R. (2012). *Matrix analysis*. Cambridge: Cambridge University press.

Jadbabaie, A., Lin, J., & Morse, A.S. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control, 48*(6), 988–1001.

Ji, Z., Lin, H., & Yu, H. (2015). Protocols design and uncontrollable topologies construction for multi-agent networks. *IEEE Transactions on Automatic Control, 60*(3), 781–786.

Kar, S., & Moura, José M.F. (2008). Sensor networks with random links: Topology design for distributed consensus. *IEEE Transactions on Signal Processing, 56*(7), 3315–3326.

Lawton, J.R., & Beard, R.W. (2002). Synchronized multiple spacecraft rotations. *Automatica, 38*(8), 1359–1364.

Lin, J., Morse, A.S., & Anderson, B.D.O. (2003). The multi-agent rendezvous problem. In *Proceedings of 42nd IEEE Conference on Decision and Control* (Vol. 2, no. 9, pp. 1508–1513), Maui, Hawaii USA.

Lin, Z., Hou, J., Yan, G., & Yu, C. (2015). Reach almost sure consensus with only group information. *Automatica, 52*, 283–289.

Ma, J., Zheng, Y., & Wang, L. (2015). LQR-based optimal topology of leader-following consensus. *International Journal of Robust and Nonlinear Control, 25*(17), 3404–3421.

Olfati-Saber, R. (2006). Flocking for multi-agent dynamics systems: Algorithms and theory. *IEEE Transactions on Automatic Control, 51*(3), 401–420.

Olfati-Saber, R., & Murray, R.M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Autommatic Control, 49*(9), 1520–1533.

Porfiri, M., & Stilwell, D.J. (2007). Consensus seeking over random weighted directed graphs. *IEEE Transactions on Automatic Control, 52*(9), 1767–1773.

Ren, W., & Beard, R.W. (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control, 50*(5), 655–661.

Ren, W., & Atkins, E. (2007). Distributed multi-vehicle coordinated control via local information exchange. *International Journal of Robust and Nonlinear Control, 17*(10–11), 1002–1033.

Rudin, W. (1987). *Real and complex analysis*. New York: Tata McGraw-Hill Education.

Sheldon, R. (1976). *A first course in probability*.London: Collier Macmillan Publishers.

Tahbaz-Salehi, A., & Jadbabaie, A. (2008). A necessary and sufficient condition for consensus over random networks. *IEEE Transactions on Automatic Control, 53*(3), 791–795.

Tahbaz-Salehi, A., & Jadbabaie, A. (2010). Consensus over ergodic stationary graph processes. *IEEE Transactions on Automatic Control, 55*(1), 225–230.

Vicsek, T., Czirok, A., Jacob, E.B., Cohen, I., & Schochet, O. (1995). Novel type of phase transition in a system of self-driven particles. *Physical Review Letters, 75*(6), 1226–1229.

Wang, L., & Xiao, F. (2007). A new approach to consensus problems in discrete-time multiagent systems with time-delays. *Science in China Series F: Information Sciences, 50*(4), 625–635.

Wang, L., & Xiao, F. (2010). Finite-time consensus problems for networks of dynamic agents. *IEEE Transactions on Automatic Control, 55*(4), 950–955.

Xiao, F., & Wang, L. (2008). Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays. *IEEE Transactions on Automatic Control, 53*(8), 1804–1816.

Xiao, F., Wang, L., & Chen, T. (2012). Connectivity preservation for multi-agent rendezvous with link failure. *Automatica, 48*(1), 25–35.

Xie, G., & Wang, L. (2007). Consensus control for a class of networks of dynamic agents. *International Journal of Robust and Nonlinear Control, 17*(10–11), 941–959.

Zhai, G., Lin, H., Michel, A.N., & Yasuda, K. (2004). Stability analysis for switched systems with continuous-time and discrete-time subsystems. In *Proceedings of the American Control Conference* (pp. 4555–4560), Boston, USA.

Zheng, Y., & Wang, L. (2012). Distributed consensus of heterogeneous multi-agent systems with fixed and switching topologies. *International Journal of Control, 85*(12), 1967–1976.

Zheng, Y., & Wang, L. (2014). Containment control of heterogeneous multi-agent systems. *International Journal of Control, 87*(1), 1–8.

Zheng, Y., & Wang, L. (2015). A novel group consensus protocol for heterogeneous multi-agent systems. *International Journal of Control, 88*(11), 2347–2353.

Zheng, Y., & Wang, L. (2016). Consensus of switched multi-agent systems. *IEEE Transactions on Circuits and Systems II: Express Briefs, 63*(3), 314–318.

Zheng, Y., Zhu, Y., & Wang, L. (2011). Consensus of heterogeneous multi-agent systems. *IET Control Theory and Applications, 5*(16), 1881–1888.

Zhu, Y., Zheng, Y., &Wang, L. (2015). Containment control of switched multi-agent systems. *International Journal of Control, 88*(12), 2570–2577.