International Journal of Systems Science

ISSN: 0020-7721 (Print) 1464-5319 (Online) Journal homepage: http://www.tandfonline.com/loi/tsys20

# Consensus of switched multi-agent systems under quantised measurements

Yunru Zhu, Yuanshi Zheng & Yongqiang Guan

**To cite this article:** Yunru Zhu, Yuanshi Zheng & Yongqiang Guan (2017) Consensus of switched multi-agent systems under quantised measurements, International Journal of Systems Science, 48:9, 1796-1804, DOI: 10.1080/00207721.2017.1290297

**To link to this article:** http://dx.doi.org/10.1080/00207721.2017.1290297

INTERNATIONAL JOURNAL OF SYSTEMS SCIENCE, 2017
VOL. 48, NO. 9, 1796–1804
http://dx.doi.org/10.1080/00207721.2017.1290297

# Consensus of switched multi-agent systems under quantised measurements

Yunru Zhu[a,b], Yuanshi Zheng[b] and Yongqiang Guan[b]

[a]Key Laboratory of Electronic Equipment Structure Design of Ministry of Education, School of Mechano-electronic Engineering, Xidian University, Xi'an, China; [b]Center for Complex Systems, School of Mechano-electronic Engineering, Xidian University, Xi'an, China

**ABSTRACT**

This paper deals with the quantised consensus problem for switched multi-agent system which is composed of continuous-time and discrete-time subsystems. We adopt the distributed consensus protocols based on the quantised relative state measurements of agents. By using the properties of Laplacian matrix, it is shown that the switched multi-agent system can reach consensus exponentially with logarithmic quantiser under arbitrary switching. It is also proved that the distance between the states of any pair of neighbouring agents just converges to a bounded set when uniform quantisers are utilised. Simulation examples are presented to illustrate the effectiveness of the theoretical results.

**ARTICLE HISTORY**
Received 22 August 2016
Accepted 13 January 2017

**KEYWORDS**
Consensus; switched multi-agent systems; logarithmic quantiser; uniform quantiser

## 1. Introduction

In recent years, much attention has been paid to distributed control of multi-agent systems due to its broad applications in vehicle systems, sensor networks, social networks and so on. Consensus problem is an important research topic in distributed multi-agent coordination, which aims to design a consensus protocol based on the local information of agents to make all agents reach an agreement on certain quantities of interest. Until now, many interesting results on the consensus problem have been obtained for multi-agent systems under different contexts, such as time delay, communication noises, switching topology, etc. (Cao, Xiao, & Wang, 2015; Cheng, Hou, Tan, & Wang, 2011; Huang & Manton, 2009; Li & Zhang, 2010; Lin & Jia, 2009; Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Wang, Cheng, Ren, Hou, & Tan, 2015; Xiao, Wang, & Chen, 2014; Zheng, Zhu, & Wang, 2014).

As digital sensors and wireless network are widely used in practical systems, the information available to each agent is not accurate and might have been quantised. Because of the quantisation, some undesirable system behaviour, e.g. oscillations, may happen even if the same system is stable without quantisation (Liu, Cao, & Persis, 2012). Thus, the quantisation effects have to be considered in multi-agent systems. Early works mainly focused on the quantised consensus of discrete-time (DT) multi-agent systems (Carli, Bullo, & Zampieri, 2010; Carli, Fagnani, Frasca, & Zampieri, 2010; Kashyap, Basar, & Srikant, 2007; Li, Fu, Xie, & Zhang, 2011; Li & Xie, 2012). In Kashyap et al. (2007), the authors introduced the notion of quantised consensus and studied the problem for DT multi-agent systems with uniform quantisers under undirected graphs, where each agent's state is always an integer. Carli et al. extended the results in Kashyap et al. (2007) to the case that each agent's initial state is real in Carli et al. (2010). In Carli et al. (2010), the coding/decoding scheme was introduced for multi-agent systems to transmit the logarithmic quantised information. Recently, increasing attention has also been focus on the quantised consensus of continuous-time (CT) multi-agent systems. By constructing proper Lyapunov functions, consensus problem involving quantised relative states was studied for CT multi-agent systems with first-order dynamics in Guo and Dimarogonas (2013), second-order dynamics in Liu et al. (2012), Guo and Dimarogonas (2013), general linear dynamics in Xu and Wang (2013), and nonlinear dynamics in Zhu, Zheng, & Wang (2015b). In Ceragioli, De Persis, and Frasca (2011), the consensus protocol was designed based on the relative quantised states, instead of quantised relative states. Ceragioli et al. investigated the consensus problem for first-order multi-agent systems with uniform quantisers and hysteretic quantisers under static communication topology. In Frasca (2012), the author extended the results in Ceragioli et al. (2011) to time varying communication topology.

For multi-agent systems, there have been many results on consensus under switching topologies. In this case, multi-agent systems often can be viewed as switched systems, which are composed of only CT subsystems or DT ones. However, in Zheng and Wang (2016), the authors proposed the switched multi-agent system which is composed of both CT and DT subsystems. In practice, it is very easy to find many applications involving switched systems, which are composed by both CT and DT subsystems. For example, CT plant is controlled either by a physically implemented regulator or by a digitally implemented one together with a switching rule between them (Zhai, Lin, Michel, & Yasuda, 2004; Zhai, Liu, & Imae, 2006). For a CT multi-agent system, if sometimes computers are used to activate all the agents in a DT manner, then the entire multi-agent system can be seen as a switched multi-agent system (Zheng & Wang, 2016). Based on graph theory and Lyapunov theory, some necessary and sufficient conditions were given for the switched multi-agent system to achieve consensus under arbitrary switching in Zheng and Wang (2016). In Lin and Zheng (2016), the authors

**CONTACT** Yuanshi Zheng zhengyuanshi2005@163.com

studied the finite-time consensus of switched multi-agent system. As an extension of consensus problem (Zheng & Wang, 2014), containment control problem was also investigated for switched multi-agent systems in Zhu, Zheng, and Wang (2015a). Inspired by the work above, we try to study the consensus of switched multi-agent system in the presence of quantised information. Distributed protocol using the quantised relative states was proposed for the switched multi-agent system. By utilising graph theory and non-smooth analysis, we give some sufficient conditions for the switched multi-agent system with logarithmic quantiser to exponentially achieve consensus under arbitrary switching in undirected and directed graphs, respectively. We also show that when uniform quantisers are used, all the states of agents in an undirected graph enter into a ball which is centred at the desired consensus value in finite time.

This paper is organised as follows. In Section 2, some mathematical preliminaries are presented. The quantised consensus of the switched multi-agent system is discussed in Section 3. In Section 4, the simulation results are given to show the effectiveness of the obtained results. Section 5 is a brief conclusion.

**Notation 1.1:** Let $R^n$ be the $n$-dimensional Euclidean space, $Z$ be the set of integer numbers, $I_n$ be the $n$-dimensional identity matrix, $1_n$ be the $n$-dimensional vector with each entry being 1. $\mathcal{I}_n = \{1, 2, \dots, n\}$. The superscript '$T$' represents the transpose. $\|\cdot\|$ denotes the 2-norm both for vectors and matrices. sign($\cdot$) represents the signum function. Let $B(x, \delta)$ be the open ball of radius $\delta$ centred at $x$, $\mu(S)$ be the Lebesgue measure of $S$ and $\bar{co}$ be the convex closure. Given a complex number $\lambda$, Re($\lambda$) and $|\lambda|$ are the real part and the modulus of $\lambda$, respectively.

## 2. Preliminaries

In this section, we introduce some basic concepts and results which will be used in this paper (Filippov, 1988; Godsil & Royal, 2001; Paden & Sastry, 1987; Ren & Cao, 2010; Zeng, Wang, & Zheng, 2016).

### 2.1. Graph theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a weighted directed graph with vertex set $\mathcal{V} = \{s_1, s_2, \dots, s_n\}$ and edge set $\mathcal{E} = \{e_1, e_2, \dots, e_m\} \subseteq \mathcal{V} \times \mathcal{V}$. A directed path from $s_i$ to $s_j$ is a finite-ordered sequence of distinct edges $(s_i, s_{k_1}), (s_{k_1}, s_{k_2}), \dots, (s_{k_{h-1}}, s_j)$. The neighbouring set of $s_i$ is $\mathcal{N}_i = \{s_j : (s_j, s_i) \in \mathcal{E}\}$. A directed tree is a directed graph, where there exists a vertex called the root such that there exists a unique directed path from this vertex to every other vertex. A directed spanning tree is a directed tree, which consists of all the nodes and some edges in $\mathcal{G}$. In contrast to a directed graph, the pairs of nodes in an undirected graph are unordered. An edge $(s_i, s_j)$ in an undirected graph corresponds to the edges $(s_i, s_j)$ and $(s_j, s_i)$ in the directed graph. An undirected graph $\mathcal{G}$ is called connected if between any two distinct vertices, there is a path between them. The adjacency matrix $\mathcal{A} = (a_{ij}) \in R^{n\times n}$ of a directed graph $\mathcal{G}$ is defined such that adjacency element $a_{ij} > 0$ if $(s_j, s_i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. The adjacency matrix $\mathcal{A}$ of an undirected graph is define analogously except that $a_{ij} = a_{ji}$ for all $i \neq j$. The Laplacian matrix $L = (l_{ij}) \in R^{n \times n}$ of graph $\mathcal{G}$ is defined as $l_{ij} = -a_{ij}$, $i \neq j$, and $l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$. For every $e_k = (s_i, s_j) \in \mathcal{E}$ in a directed graph, $s_i$ is called the head of $e_k$, while $s_j$ is called the tail of $e_k$. The incidence matrix $B \in R^{n \times m}$ is defined as $[B]_{ik} = 1$, if $s_i$ is the head of $e_k$, $[B]_{ik} = -1$, if $s_i$ is the tail of $e_k$ and $[B]_{ik} = 0$, otherwise. The weighted in-directed matrix $B_\odot \in R^{n \times m}$ is defined as $[B_\odot]_{ik} = -1$, if $s_i$ is the tail of $e_k$, and $[B_\odot]_{ik} = 0$, otherwise. Define weighting matrix $W = \mathrm{diag}\{w_1, \dots, w_m\}$, where $w_k$ is the adjacency element associated with edge $e_k$. The edge Laplacian of directed graph $\mathcal{G}$ is defined as $L_e = B^T B_\odot W$.

**Lemma 2.1:** (*Ren & Cao, 2010*): *Let L be the Laplacian matrix associated with the directed graph $\mathcal{G}$ (respectively, the undirected graph $\mathcal{G}$). Then for the directed graph $\mathcal{G}$ (respectively, the undirected graph $\mathcal{G}$), L has at least one zero eigenvalue and all its nonzero eigenvalues have positive real parts (respectively, are positive real numbers). Furthermore, L has a simple zero eigenvalue and all other eigenvalues have positive real parts (respectively, are positive real numbers) if and only if $\mathcal{G}$ has a directed spanning tree (respectively, is connected).*

### 2.2. Differential inclusion

For the vector differential equation

$$\dot{x}(t) = f(x(t)),$$

where $x \in R^m$, $f$: $R^m \to R^m$ is measurable and locally essentially bounded. We say $x(t)$ is a Filippov solution to the differential equation on $[t_0, t_1]$ if $x(t)$ is absolutely continuous and satisfies $\dot{x} \in F[X](x)$ at almost every $t \in [t_0, t_1]$, where $F[X](x) = \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \bar{co}\{f(B(x, \delta))\backslash S\}$.

**Lemma 2.2:** (*Paden & Sastry, 1987*): *If $f_1, f_2$: $R^m \to R^n$ are locally bounded and $f_3$: $R^m \to R^{s \times n}$ be $C^0$, then $F[f_1](x) = \{f_1(x)\}$, $F[f_1 + f_2](x) \subseteq F[f_1](x) + F[f_2](x)$ and $F[f_3 f_1](x) = f_3(x) F[f_1](x)$, where $f_3 f_1(x) = f_3(x) f_1(x)$.*

## 3. Main results

Consider a switched multi-agent system which is composed of a CT subsystem

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{I}_n, \tag{1}$$

and a DT subsystem

$$x_i(t + h) = x_i(t) + h u_i(t), \quad i \in \mathcal{I}_n, \tag{2}$$

where $x_i, u_i \in R$ are the state and control input of agent $i$, respectively, and $h > 0$ is the sampling period. All results in this paper still hold for $x_i, u_i \in R^m$ by using the Kronecker product operations.

In this paper, we study the consensus problem for switched multi-agent system with quantised data. The uniform and logarithmic quantisers are adopted. The uniform quantiser is defined as $q_u(x) = \delta_u[\frac{x}{\delta_u}]$, where parameter $\delta_u > 0$, $[a]$, $a \in R$ denotes the nearest integer to $a$ and $[\frac{1}{2}] = 1$. From the definition, we have $q_u(-x) = -q_u(x)$, $x q_u(x) \geq 0$ and $q_u^2(x) \geq 2x q_u(x)$. The logarithmic quantiser is defined as $q_l(x) = \mathrm{sign}(x) e^{q_u(\ln |x|)}$, when $x \neq 0$ and $q_l(0) = 0$. Similarly, we have $x q_l(x) \geq 0$, $q_l(-x) = -q_l(x)$, $\frac{1}{1+\delta_l} x^2 \leq x q_l(x) \leq (1 + \delta_l) x^2$ and $\frac{1}{1+\delta_l} q_l^2(x) \leq x q_l(x) \leq (1 + \delta_l) q_l^2(x)$, where $\delta_l =$

$e^{\frac{\delta_u}{2}} - 1$. For a vector $x = (x_1, \ldots, x_n)^T \in R^n$, let $q(x) = (q(x_1), \ldots, q(x_n))^T$.

**Definition 3.1:** The switched multi-agent system (1)–(2) is said to reach consensus if there exists $x^*$ such that

$$\lim_{t\to+\infty} \| x_i(t) - x^* \| = 0, \quad i \in \mathcal{I}_n.$$

We adopt the following distributed protocol:

$$u_i(t) = k \sum_{j\in\mathcal{N}_i} a_{ij} q(x_j(t) - x_i(t)), \tag{3}$$

where $q(\cdot)$ is a logarithmic or uniform quantiser, $k > 0$ is the control gain.

First, we study the case when logarithmic quantisers are utilised. One can show that the switched multi-agent system can reach consensus under protocol (3).

When the communication topology $\mathcal{G}$ among agents is undirected and connected, we use $\lambda_2(L)$, $\lambda_n(L)$ to denote the minimum and maximum non-zero eigenvalue of Laplacian matrix $L$, respectively. Let $w_{\max} = \max_{(i,j)\in\mathcal{E}}\{a_{ij}\}$. Orient each edge in undirected graph $\mathcal{G}$ arbitrarily to make it have a head and tail. $B$ is the incidence matrix of the oriented graph. Let $x(t) = (x_1(t), \ldots, x_n(t))^T$ and $\hat{x}(t) = B^T x(t)$. Then, the switched multi-agent system (1)–(2) with protocol (3) is composed of a CT subsystem

$$\dot{x}(t) = -kBWq(\hat{x}(t)), \tag{4}$$

and a DT subsystem

$$x(t+h) = x(t) - khBWq(\hat{x}(t)). \tag{5}$$

Because of the discontinuity of the quantised signals, we consider the solution of CT subsystem in Filippov sense, which is defined as an absolutely continuous solution of the differential inclusion

$$\dot{x}(t) \in F\left[-kBWq\right](\hat{x}(t)) = -kBWF\left[q\right](\hat{x}(t)), \tag{6}$$

where the final equality follows from Lemma 2.2.

**Lemma 3.1:** *Suppose that the communication topology $\mathcal{G}$ is undirected and connected. Let $x(t)$ be a solution of switched multi-agent system (1)-(2) with protocol (3) under arbitrary switching. Then $x(t)$ satisfies $c(t) = \frac{1}{n}1_n^T x(t) = c(0)$ for all $t \geq 0$.*

**Proof:** According to Equation (6), we have

$$\dot{c}(t) \in -\frac{k}{n} 1_n^T BWF\left[q\right](\hat{x}(t)) = \{0\},$$

where the final equality follows from $1_n^T B = 0$.

By Equation (5), we obtain that

$$\begin{aligned} c(t+h) &= \frac{1}{n}1_n^T x(t+h) = \frac{1}{n}1_n^T (x(t) - khBWq(\hat{x}(t))) \\ &= \frac{1}{n}1_n^T x(t) = c(t). \end{aligned}$$

Therefore, $c(t)$ is constant. ■

**Theorem 3.1:** *Suppose that the communication topology $\mathcal{G}$ is undirected and connected. Then switched multi-agent system (1)–(2) with protocol (3) can solve the consensus problem under arbitrary switching if $k < \frac{2}{(1+\delta_l)h\lambda_n(L)}$.*

**Proof:** Let $y(t) = x(t) - c(t)1_n$, from Equations (5) and (6), we have

$$\dot{y}(t) = \dot{x}(t) \in -kBWF\left[q_l\right](\hat{x}(t)),$$

$$\begin{aligned} y(t+h) &= x(t+h) - c(t+h)1_n \\ &= x(t) - khBWq_l(\hat{x}(t)) - c(t)1_n \\ &= y(t) - khBWq_l(\hat{x}(t)). \end{aligned}$$

Take $V(t) = \frac{1}{2}y^T(t)y(t)$. When CT subsystem is activated, let $\upsilon(t) \in F[q_l](\hat{x}(t))$ and we obtain

$$\begin{aligned} \nabla V(y)\dot{y}(t) &= -ky^T(t)BW\upsilon(t) \\ &= -k\left(x(t) - c(t)1_n\right)^T BW\upsilon(t) \\ &= -kx^T(t)BW\upsilon(t) \\ &= -k\hat{x}^T(t)W\upsilon(t). \end{aligned}$$

From the definition of the Filippov set-valued map, we have, if $a \geq 0$, $F[q_l](a) = q_l(a)$, when $x \neq e^{(k-\frac{1}{2})\delta_u}$, $k \in Z$ and $F[q_l](a) = \left[e^{(k-1)\delta_u}, e^{k\delta_u}\right]$, otherwise. Moreover, $F[q_l](-a) = -F[q_l](a)$. Therefore, it has $aF[q_l](a) \geq \frac{1}{1+\delta_l}a^2$. Hence,

$$\begin{aligned} \nabla V(y)\dot{y}(t) &\leq -\frac{k}{1+\delta_l}\hat{x}^T(t)W\hat{x}(t) \\ &= -\frac{k}{1+\delta_l}x^T(t)BWB^Tx(t) \\ &= -\frac{k}{1+\delta_l}x^T(t)Lx(t). \end{aligned}$$

It follows from $L1_n = 0$ that

$$y^T(t)Ly(t) = (x(t) - c(t)1_n)^T L\,(x(t) - c(t)1_n) = x^T(t)Lx(t).$$

Owing to $\min_{\varepsilon\neq 0, 1_n^T\varepsilon=0} \frac{\varepsilon^t L\varepsilon}{\varepsilon^T\varepsilon} = \lambda_2$, one obtains

$$\begin{aligned} \nabla V(y)\dot{y}(t) &\leq -\frac{k}{1+\delta_l}y^T(t)Ly(t) \\ &\leq -\frac{k\lambda_2(L)}{1+\delta_l}y^T(t)y(t) \\ &= -\frac{2k\lambda_2(L)}{1+\delta_l}V(t). \end{aligned}$$

When DT subsystem is activated, we have

$$\begin{aligned} V(t+h) &= \frac{1}{2}(y(t) - khBWq_l(\hat{x}(t)))^T(y(t) - khBWq_l(\hat{x}(t))) \\ &= \frac{1}{2}y^T(t)y(t) - khy^T(t)BWq_l(\hat{x}(t)) \\ &\quad + \frac{1}{2}k^2h^2q_l^T(\hat{x}(t))WB^TBWq_l(\hat{x}(t)). \end{aligned}$$

Note that

$$\begin{aligned} y^T(t)BWq_l(\hat{x}(t)) &= (x(t)-c(t)1_n)^T BWq_l(\hat{x}(t)) \\ &= x^T(t)BWq_l(\hat{x}(t))) \\ &= \hat{x}^T(t)Wq_l(\hat{x}(t)), \end{aligned}$$

and

$$\begin{aligned} q_l^T(\hat{x}(t))WB^TBWq_l(\hat{x}(t)) &\le \lambda_n(L)q_l^T(\hat{x}(t))Wq_l(\hat{x}(t)) \\ &\le (1+\delta_l)\lambda_n(L)\hat{x}^T(t)Wq_l(\hat{x}(t)). \end{aligned}$$

Hence, we have

$$\begin{aligned} V(t+h) &\le \frac{1}{2}y^T(t)y(t) \\ &\quad + \frac{1}{2}\left((1+\delta_l)\lambda_n(L)k^2h^2 - 2kh\right)\hat{x}^T(t)Wq_l(\hat{x}(t)). \end{aligned}$$

Note that

$$\begin{aligned} \hat{x}^T(t)Wq_l(\hat{x}(t)) &\ge \frac{1}{1+\delta_l}\hat{x}^T(t)W\hat{x}(t) \\ &= \frac{1}{1+\delta_l}x^T(t)Lx(t) \\ &= \frac{1}{1+\delta_l}y^T(t)Ly(t). \end{aligned}$$

Since $k < \frac{2}{(1+\delta_l)h\lambda_n(L)}$, we have $(1+\delta_l)\lambda_n(L)k^2h^2 - 2kh < 0$. Thus,

$$\begin{aligned} V(t+h) &\le \frac{1}{2}y^T(t)\left(I + \lambda_n(L)k^2h^2L - \frac{2kh}{1+\delta_l}L\right)y(t) \\ &\le \frac{1}{2}\left(1 + \lambda_n(L)\lambda_2(L)k^2h^2 - \frac{2\lambda_2(L)kh}{1+\delta_l}\right)y^T(t)y(t), \\ &= \alpha V(t), \end{aligned}$$

where $\alpha = 1 + \lambda_n(L)\lambda_2(L)k^2h^2 - \frac{2\lambda_2(L)kh}{1+\delta_l}$. From $0 < k < \frac{2}{(1+\delta_l)h\lambda_n(L)}$, it is easy to obtain that $0 < \alpha < 1$.

For any time $t > 0$, we can divide the time interval $[0, t]$ as $t = t_c + rh$, where $t_c \in R \ge 0$ is the total duration time on CT subsystem and $rh$, $r \in Z \ge 0$ is the total duration time on DT subsystem. Then, one has

$$V(t) \le e^{-\frac{2k\lambda_2(L)}{1+\delta_l}t_c}\alpha^r V(0) \le e^{-2\beta t}V(0),$$

where $\beta = \min\{\frac{k\lambda_2(L)}{1+\delta_l}, -\frac{\ln\alpha}{2h}\}$. Therefore, $\|y(t)\| \le e^{-\beta t}\|y(0)\|$, i.e. $\|x(t) - c(0)1_n\| \le e^{-\beta t}\|y(0)\|$. Thus, the switched multi-agent system can reach consensus exponentially under arbitrary switching. ■

If the communication topology among agents is switching among different undirected connected topologies, to describe the switching topology $\mathcal{G}(t)$, we define a piece-wise constant switching function $\sigma(t)$: $[0, +\infty) \to P = \{1, 2, \ldots, M\}$, where $M$ denotes the total number of all possible connected graphs. Let $\{t_s\}_{s=1}^{\infty}$ denote the set of switching instants, which satisfies $0 = t_0 < t_1 < \cdots < t_s < t_{s+1} < \cdots$, $\lim_{s\to+\infty} t_s = +\infty$. To avoid infinitely frequent switching, we define a strictly positive dwell time $\tau > 0$ such that $t_{s+1} - t_s > \tau$. Then following the proof of Theorem 3.3, we can easily obtain the following convergence results on switching topology.

**Corollary 3.1:** *Suppose that the communication topology $\mathcal{G}(t)$ remains undirected and connected for all intervals $[t_s, t_{s+1})$. Then switched multi-agent system (1)–(2) with protocol (3) can solve the consensus problem under arbitrary switching if $k < \min_{s\in P} \frac{2}{(1+\delta_l)h\lambda_n(L_s)}$.*

**Remark 3.1:** If $t_d = 0$, the switched multi-agent system is actually the CT multi-agent system considered in Guo and Dimarogonas (2013), where the communication topology is switching among different undirected tree topologies. Compared with it, the switching graphs are undirected and connected in this paper.

When the communication topology $\mathcal{G}$ among agents is directed, some techniques used in Zeng et al. (2016) are borrowed here to solve the quantised consensus problem. For a directed graph $\mathcal{G}$ containing a spanning tree, the graph $\mathcal{G}$ can be rewritten as $\mathcal{G} = \mathcal{G}_\tau \bigcup \mathcal{G}_c$, where $\mathcal{G}_\tau = (\mathcal{V}, \mathcal{E}_1)$ denotes the directed spanning tree and $\mathcal{G}_c = (\mathcal{V}, \mathcal{E}-\mathcal{E}_1)$. Thus, the incidence matrix $B$ can be rewritten as $B = (B_\tau, B_c)$ and there exists a linear transformation such that $B_\tau T = B_c$. Let $R = (I, T)$, and then we have $B = B_\tau R$. Define $\hat{x}_\tau(t) = B_\tau^T x(t)$, $\hat{x}_c(t) = B_c^T x(t) = T^T\hat{x}_\tau(t)$ and $\hat{x}(t) = (\hat{x}_\tau^T(t), \hat{x}_c^T(t))^T = B^T x(t) = R^T\hat{x}_\tau(t)$, the switched multi-agent system (1)–(2) with protocol (3) can be written as

$$\dot{\hat{x}}(t) = -kL_e q(\hat{x}(t)), \tag{7}$$

and

$$\hat{x}(t+h) = \hat{x}(t) - khL_e q(\hat{x}(t)). \tag{8}$$

According to Zeng et al. (2016), there exists transformation matrix $S_e$, such that $S_e^{-1}\hat{x} = \binom{\hat{x}_\tau}{0}$, $S_e^{-1}L_e = \binom{L_c}{0}$ and $S_e^{-1}L_eS_e = \begin{pmatrix} \hat{L}_e & * \\ 0 & 0 \end{pmatrix}$, where $L_c = B_\tau^T B_\odot W$ and $\hat{L}_e = L_cR^T$. Using the transformation, we can obtain a reduced model of switched multi-agent system (7)–(8) as

$$\dot{\hat{x}}_\tau(t) = -kL_c q(\hat{x}(t)), \tag{9}$$

and

$$\hat{x}_\tau(t+h) = \hat{x}_\tau(t) - khL_c q(\hat{x}(t)). \tag{10}$$

For Equation (9), we consider the absolutely continuous solution of the differential inclusion

$$\dot{\hat{x}}_\tau(t) \in -kL_cF\left[q\right](\hat{x}(t)). \tag{11}$$

**Lemma 3.2:** (*Zeng et al., 2016*): *For a graph $\mathcal{G}$ containing directed spanning tree, the essential edge Laplacian $\hat{L}_e$ contains exactly all the non-zero eigenvalues of Laplacian $L$.*

From the above Lemma, we can get that the matrix $I - kh\hat{L}_e$ is Schur stable if and only if $kh < \min_{i=2,\ldots,n} \frac{2\mathrm{Re}(\lambda_i)}{|\lambda_i|^2}$, where $\lambda_i$ is the eigenvalue of Laplacian $L$. Construct the following Lyapunov equations as

$$\hat{L}_e^T P + P\hat{L}_e = Q, \tag{12}$$

$$Q - (I - kh\hat{L}_e)^T Q(I - kh\hat{L}_e) = H, \tag{13}$$

where $H$ is an arbitrary positive definite matrix and $P$, $Q$ are the unique positive definite solutions. Let $M = P - (I - kh\hat{L}_e)^T P(I - kh\hat{L}_e)$ and we can get $M > 0$ by substituting $Q$ in Equation (13) into Equation (12). Define $d_1 =\| (I - kh\hat{L}_e)^T \|\| P \|\| L_c \|\| R^T \|$ and $d_2 = \|P\|\|L_c\|^2\|R^T\|^2$.

**Theorem 3.2:** *Suppose that the communication topology $\mathcal{G}$ contains a directed spanning tree. Select arbitrary $kh < \min_{i=2,\dots,n} \frac{2Re(\lambda_i)}{|\lambda_i|^2}$, then switched multi-agent system (1)–(2) with protocol (3) can solve the consensus problem under arbitrary switching if $\delta_l < \min\{\frac{\lambda_{min}(Q)}{2\|P\|\|L_c\|\|R^T\|}, \frac{\lambda_{min}(M)}{kh(d_1+\sqrt{d_1^2+d_2\lambda_{min}(M)})}\}$.*

**Proof:** Take $V(t) = \hat{x}_\tau^T(t)P\hat{x}_\tau(t)$. When CT subsystem is activated, let $\upsilon(t) \in F[q_l](\hat{x}(t))$ and we obtain

$$\begin{aligned}\nabla V(\hat{x}_\tau)\dot{x}_\tau(t) &\le -k\hat{x}_\tau^T(t)(L_c^T P + PL_c)\upsilon(t)\\ &= -k\hat{x}_\tau^T(t)(L_c^T P + PL_c)\hat{x}(t) - 2k\hat{x}_\tau^T(t)PL_c e(t),\end{aligned}$$

where $e(t) = \upsilon(t) - \hat{x}(t)$.

Note that $\hat{x}(t) = R^T\hat{x}_\tau(t)$ and $\| e(t) \|\le \delta_l \| \hat{x}(t) \|\le \delta_l \| R^T \|\| x_\tau(t) \|$, and thus it obtains that

$$\begin{aligned}\nabla V(\hat{x}_\tau)\dot{x}_\tau(t) &= -k\hat{x}_\tau^T(t)(\hat{L}_e^T P + P\hat{L}_e)\hat{x}_\tau(t) - 2k\hat{x}_\tau^T(t)PL_c e(t)\\ &\le -k(\lambda_{\min}(Q)\\ &\quad - 2\delta_l \| P \|\| L_c \|\| R^T \|) \| x_\tau(t) \|^2\\ &= -\alpha \| x_\tau(t) \|^2 .\end{aligned}$$

where $\alpha = k(\lambda_{\min}(Q) - 2\delta_l\|P\|\|L_c\|\|R^T\|)$. Moreover, one can obtain that $\nabla V(\hat{x}_\tau)\dot{x}_\tau(t) \le -\frac{\alpha}{\lambda_{\max}(P)}V(t)$.

For DT subsystem, we can rewrite Equation (10) as

$$\hat{x}_\tau(t+h) = (I - kh\hat{L}_e)\hat{x}_\tau(t) + khL_c w(t),$$

where $w(t) = \hat{x}(t) - q(\hat{x}(t))$. Then, when DT subsystem is activated, it obtains that

$$\begin{aligned}V(t+h) - V(t) &= -\hat{x}_\tau^T(t)M\hat{x}_\tau(t)\\ &\quad + 2kh\hat{x}_\tau^T(t)(I - kh\hat{L}_e)^T PL_c w(t)\\ &\quad + k^2h^2 w^T(t)L_c^T PL_c w(t).\end{aligned}$$

Since $\| w(t) \|\le \delta_l \| \hat{x}(t) \|\le \delta_l \| R^T \|\| x_\tau(t) \|$, we have

$$\begin{aligned}&V(t+h) - V(t)\\ &\le -(\lambda_{\min}(M) - 2khd_1\delta_l - k^2h^2d_2\delta_l^2) \| x_\tau(t) \|^2 .\end{aligned}$$

Let $\beta = \lambda_{\min}(M) - 2khd_1\delta_l - k^2h^2d_2\delta_l^2$, and it obtains that $V(t+h) \le (1 - \frac{\beta}{\lambda_{\max}(P)})V(t)$. It follows from $0 < \delta < \frac{\lambda_{\min}(M)}{kh(d_1+\sqrt{d_1^2+d_2\lambda_{\min}(M)})}$ that $0 < \beta < \lambda_{\min}(M)$. Notice $\lambda_{\min}(M) < \lambda_{\max}(P)$ from the definition of $M$. Then, $0 < 1 - \frac{\beta}{\lambda_{\max}(P)} < 1$.

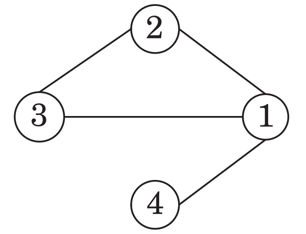



**Figure 1.** An undirected graph.

Thus, similar to the proof in Theorem 3.3, we can get that the switched multi-agent system can reach consensus exponentially under arbitrary switching. ■

Next, we discuss the consensus problem when uniform quantisers are used.

**Theorem 3.3:** *Suppose that the communication topology $\mathcal{G}$ is undirected and connected. Let $x(t)$ be a solution of switched multi-agent system (1)–(2) with protocol (3) under arbitrary switching. Then $x(t)$ converges to the set $\{x \mid |x_i - x_j| \le \frac{\delta_u}{2}, (i, j) \in \mathcal{E}\}$ if $k < \frac{1}{h\lambda_n(L)}$.*

**Proof:** Similar to the proof of Theorem 3.3, we take $V(t) = \frac{1}{2}y^T(t)y(t)$. When CT subsystem is activated, let $\upsilon(t) \in F[q_u](\hat{x}(t))$ and we can get

$$\nabla V(y)\dot{y}(t) = -ky^T(t)BW\upsilon(t) = -k\hat{x}^T(t)W\upsilon(t) \le 0,$$

where the equality holds only when $q_u(\hat{x}(t)) = 0$, which implies $|x_i - x_j| < \frac{\delta_u}{2}$, $(i, j) \in \mathcal{E}$.

When DT subsystem is activated, one has

$$\begin{aligned}V(t+h) - V(t) &= -khy^T(t)BWq_u(\hat{x}(t))\\ &\quad + \frac{1}{2}k^2h^2q_u^T(\hat{x}(t))WB^TBWq_u(\hat{x}(t))\\ &\le -kh\hat{x}^T(t)Wq_u(\hat{x}(t))\\ &\quad + \lambda_n(L)k^2h^2\hat{x}^T(t)Wq_u(\hat{x}(t)).\end{aligned}$$

From $k < \frac{1}{h\lambda_n(L)}$, it obtains that

$$V(t+h) - V(t) \le \big(\lambda_n(L)k^2h^2 - kh\big)\,\hat{x}^T(t)Wq_u(\hat{x}(t)) \le 0.$$

where the equality holds only when $q_u(\hat{x}(t)) = 0$, which implies $|x_i - x_j| < \frac{\delta_u}{2}$, $(i, j) \in \mathcal{E}$.

Based on the Lasalle's invariance principle, the solutions of switched multi-agent system (1)–(2) with protocol (3) converge to a set $\{x \mid |x_i - x_j| \le \frac{\delta_u}{2}, (i, j) \in \mathcal{E}\}$. ■

**Corollary 3.2:** *Suppose that the communication topology $\mathcal{G}$ is undirected and connected. Let $x(t)$ be a solution of switched multi-agent system (1)–(2) with protocol (3) under arbitrary switching. Then for any $0 < \epsilon < 1$, $x(t)$ converges to the set $\{x \mid \|x(t) - \frac{1}{n}\sum_{i=1}^n x_i(0)1_n\| \le \frac{\delta_u\sqrt{m\lambda_n(L)w_{max}}}{2(1-\varepsilon)\lambda_2(L)}\}$ in finite time if $k < \frac{1}{h\lambda_n(L)}$.*

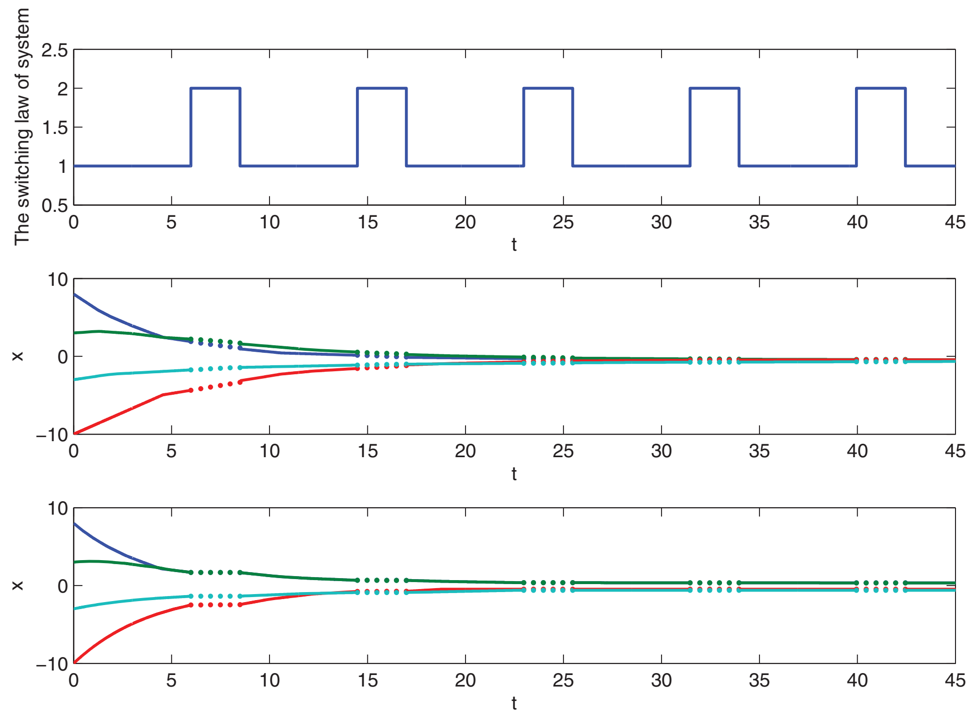



**Figure 2.** (Top) Switching law of system (1)–(2), the (middle) state trajectories of all the agents under logarithmic quantisers and the (bottom) state trajectories of all the agents under uniform quantisers.

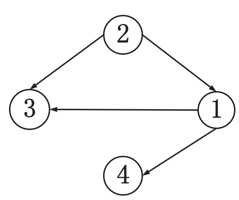



**Figure 3.** A directed graph.

**Proof:** Take $V(t) = \frac{1}{2}y^T(t)y(t)$. Note that if $v(t) \in F[q_u](\hat{x}(t))$, then $\|v(t) - \hat{x}(t)\| \leq \frac{\delta_u}{2}\sqrt{m}$. When CT subsystem is activated, then

$$\begin{aligned}
\nabla V(y)\dot{y}(t) &= -ky^T(t)BWv(t) \\
&= -ky^T(t)BW\left(\hat{x}(t) + v(t) - \hat{x}(t)\right) \\
&= -ky^T(t)BW\hat{x}(t) - ky^T(t)BW(v(t) - \hat{x}(t)) \\
&= -kx(t)^T BW\hat{x}(t) - ky^T(t)BW(v(t) - \hat{x}(t)) \\
&= -kx(t)^T Lx(t) - ky^T(t)BW(v(t) - \hat{x}(t)) \\
&= -ky^T(t)Ly(t) - ky^T(t)BW(v(t) - \hat{x}(t)) \\
&\leq -k\lambda_2(L)\|y(t)\|^2 \\
&\quad + k\|y(t)\|\|BW^{\frac{1}{2}}\|\|W^{\frac{1}{2}}\|\|v(t) - \hat{x}(t)\| \\
&\leq -k\lambda_2(L)\|y(t)\|^2 + k\|y(t)\|\sqrt{\lambda_n(L)w_{\max}}\frac{\delta_u}{2}\sqrt{m} \\
&= -k\lambda_2(L)\|y(t)\|\left(\varepsilon\|y(t)\| + (1-\varepsilon)\|y(t)\|\right. \\
&\quad \left. - \frac{\delta_u\sqrt{m\lambda_n(L)w_{\max}}}{2\lambda_2(L)}\right),
\end{aligned}$$

where $\epsilon > 0$ is a constant which can be chosen as small as possible.

Thus, we see that, if $\|y(t)\| > \frac{\delta_u\sqrt{m\lambda_n(L)w_{\max}}}{2(1-\varepsilon)\lambda_2(L)}$,

$$\nabla V(y)\dot{y}(t) \leq -\varepsilon k\lambda_2(L)\|y(t)\|^2 = -2\varepsilon k\lambda_2(L)V(t).$$

When DT subsystem is activated, similar to the proof of Theorem 3.7, we obtain

$$\begin{aligned}
V(t+h) - V(t) &\leq -k_1\hat{x}^T(t)Wq_u(\hat{x}(t)) \\
&= -k_1x^T(t)BWq_u(\hat{x}(t)) \\
&= -k_1y^T(t)BWq_u(\hat{x}(t)),
\end{aligned}$$

where $k_1 = kh - \lambda_n(L)k^2h^2$. From $0 < k < \frac{1}{h\lambda_n(L)}$, we have $0 < k_1 < \frac{1}{4\lambda_n(L)}$. Then, as before,

$$\begin{aligned}
V(t+h) - V(t) \leq -k_1\lambda_2(L)\|y(t)\|\Bigg(&\varepsilon\|y(t)\| \\
&+ (1-\varepsilon)\|y(t)\| - \frac{\delta_u\sqrt{m\lambda_n(L)w_{\max}}}{2\lambda_2(L)}\Bigg).
\end{aligned}$$

Thus, if $\|y(t)\| > \frac{\delta_u\sqrt{m\lambda_n(L)w_{\max}}}{2(1-\varepsilon)\lambda_2(L)}$,

$$V(t+h) \leq (1 - 2\varepsilon k_1\lambda_2(L))V(t),$$

where $0 < 1 - 2\epsilon k_1\lambda_2(L) < 1$ since $0 < k_1 < \frac{1}{4\lambda_n(L)}$.

If $\|y(0)\| > \frac{\delta_u\sqrt{m\lambda_n(L)w_{\max}}}{2(1-\varepsilon)\lambda_2(L)}$, there exists $t = t_c + rh$ such that $V(t) \leq e^{-2\varepsilon k\lambda_2(L)t_c}(1 - 2\varepsilon k_1\lambda_2(L))^r V(0) \leq e^{-2\beta t}V(0)$, where

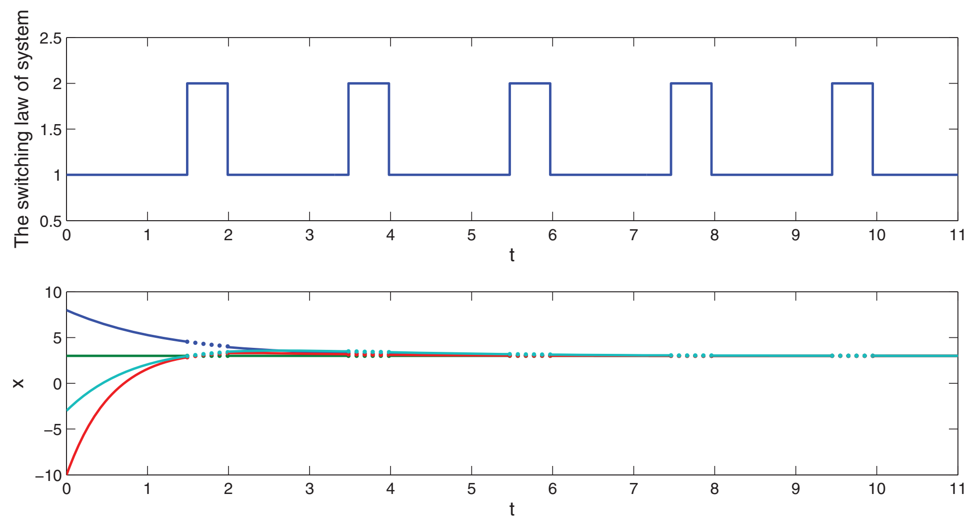



Figure 4. (Top) Switching law of system (1)–(2) and the (bottom) state trajectories of all the agents under logarithmic quantisers.

$\beta = \min\{\varepsilon k \lambda_2(L), -\frac{\ln(1-2\varepsilon k_1\lambda_2(L))}{2h}\}$. Therefore, one can easily obtain that there is $T > 0$ dependent on $y(0)$ and $\epsilon$ such that $\|y(t)\| \le \frac{\delta_u \sqrt{m\lambda_n(L)} w_{\max}}{2(1-\varepsilon)\lambda_2(L)}$ for all $t \ge T$. ■

**Remark 3.2:** If $t_d = 0$ and $w_{\max} = 1$ in switched multi-agent system (4)–(5), we can easily get the, $x(t)$ converges to the set $\{x \mid \|x(t) - \frac{1}{n}\sum_{i=1}^{n} x_i(0)1_n\| \le \frac{\delta_u\sqrt{m\lambda_n(L)}}{2(1-\varepsilon)\lambda_2(L)}\}$ with uniform quantisers in finite time. In Xu and Wang (2013), the bound is $\frac{\delta_u \lambda_n(L)\sqrt{m}}{2(1-\varepsilon)\lambda_2(L)\sqrt{\lambda_2(L)}}$.

## 4. Simulations

In this section, we will provide computer simulations to demonstrate the effectiveness of the theoretical results.

**Example 4.1:** Consider a group of agents with connected communication topology $\mathcal{G}$ shown in Figure 1. For simpleness, each edge weight is assumed to be 1. We can easily calculate the max non-zero eigenvalue of Laplacian matrix is $\lambda_4(L) = 4$. Set sampling period $h = 1$, $\delta_u = 1$ and $\delta_l = e^{0.5} - 1$. Take control gain $k = 0.1$, which satisfies the condition in Theorem 3.3 and Theorem 3.8. The switching law is shown in the top panel in Figure 2. State trajectories of agents of switched multi-agent system under logarithmic and uniform quantisers are shown in the middle and bottom planes in Figure 2, respectively. We can see that when logarithmic quantisers are used, the switched multi-agent system can reach consensus, while when uniform quantisers are used, due to the constraint of uniform quantisation, state trajectories of all the agents just converge to a bound set.

**Example 4.2:** Suppose that the communication topology contains a directed spanning tree depicted in Figure 3. Each edge weight is also assumed to be 1. We choose $h = 0.1$ and $k = 0.8$ to satisfy $kh < \min_{i=2,\ldots,n} \frac{2\mathrm{Re}(\lambda_i)}{|\lambda_i|^2}$. Take $H = I_3$. By calculation, we choose $\delta_l = 0.04$ to satisfy the conditions in Theorem 3.7. We can see from Figure 4 that all agents can reach consensus.

**Example 4.3:** Suppose that the communication topology is switching among the three undirected and connected graphs depicted in Figure 5, where each edge weight is assumed to be 1. By calculation, we choose $k = 0.1$ to satisfy the condition in Corollary 3.4. The switching law of communication topology is shown in the top panel in Figure 6. The switching law of switched multi-agent system is depicted in the middle panel in Figure 6. From the state trajectories of agents under logarithmic quantisers, shown in the bottom plane in Figure 6, we can see the consensus is achieved as desired.

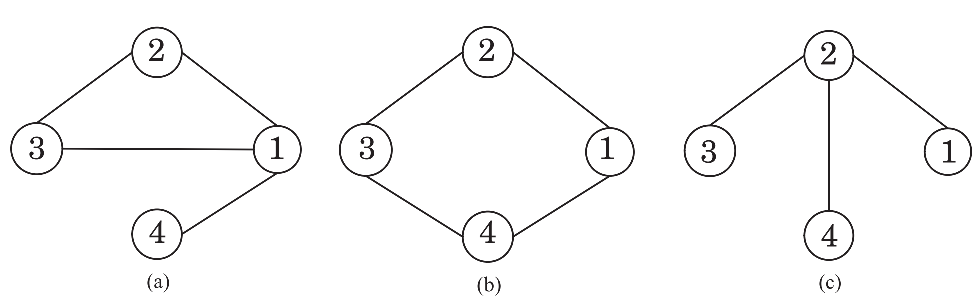



Figure 5. Three undirected connected graphs.

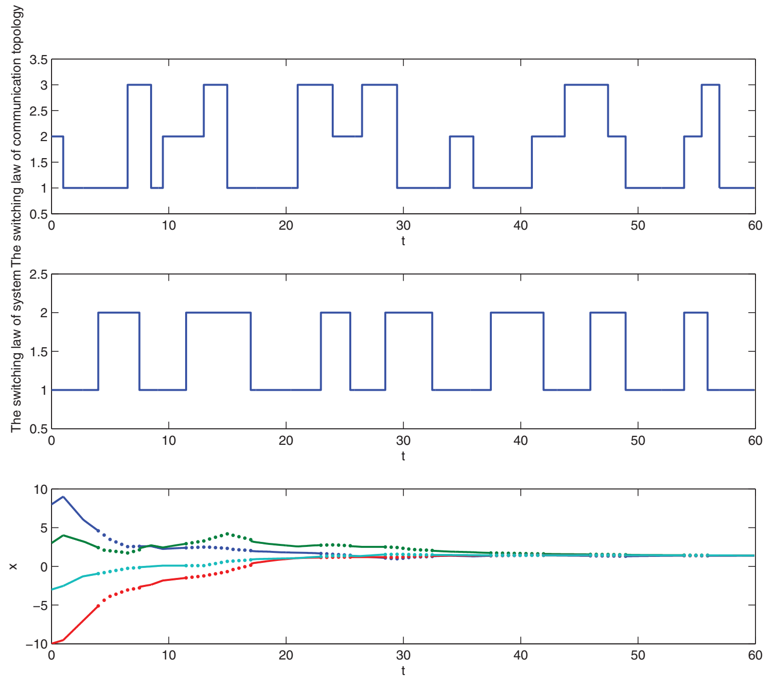



Figure 6. (Top) Switching law of the communication topology depicted in Figure 3 , the (middle) switching law of system (1)–(2) and the (bottom) state trajectories of all the agents under logarithmic quantisers.

## 5. Conclusion

In this paper, we have considered the consensus problem for switched multi-agent system with quantised information communication. It has been shown that when logarithmic quantisers are used, the switched multi-agent system can reach consensus exponentially under static and time-varying undirected communication topologies by choosing proper control gain. It has also been pointed out that the switched multi-agent system with logarithmic quantiser can reach consensus exponentially under arbitrary switching if the communication topology contains a directed spanning tree and $\delta_l$ is small enough. In addition, we have shown that when uniform quantisers are used under static undirected communication topologies, if $k < \frac{1}{h\lambda_n(L)}$, all the states of agents enter into a ball which is centred at the desired consensus value in finite time. The future work will focus on the consensus of switched multi-agent system with time delays and communication noise.

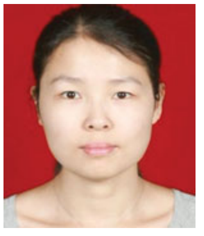
## Disclosure statement

No potential conflict of interest was reported by the authors.


## Funding

This work was supported by National Natural Science Foundation of China (NSFC) [grant number 61375120], [grant number 61304160], [grant number 61603288]; Fundamental Research Funds for the Central Universities [grant number JB160419]; Shaanxi Provincial Natural Science Foundation of China [grant number 2014JQ-5029]; Young Talent Fund of University Association for Science and Technology in Shaanxi of China [grant number 20160208].


## Notes on contributors



*Yunru Zhu* was born in Xi'an, Shaanxi Province. She received the B.S. degree in Automatic from Xidian University in 2003, the M.S. degree from Huazhong University of Science and Technology in 2006, and Ph.D. degree from Xidian University in 2015, respectively. Since 2006, she has been worked at the School of Mechano-electronic Engineering, Xidian University. Her current research interests are in the fields of coordination of multi-agent systems, quantized control and impulsive control.

*Yuanshi Zheng* was born in Jiangshan, Zhejiang Province, China. He received his Bachelor, Master, and Doctorate degrees from Ningxia University and Xidian University in 2006, 2009, and 2012, respectively. He is currently an assocaiate professor of Xidian University. His research interests are in the fields of coordination of multi-agent systems , game theory and collective intelligence.

*Yongqiang Guan* received his Ph.D. degree in complex systems at Beihang University. He is currently a lecturer at School of Mechano-Electronic Engineering, Xidian University. His current research interests are in the fields of coordination of multiagent systems and complex networks.

## References


Cao, M., Xiao, F., & Wang, L. (2015). Event-based second-order consensus control for multi-agent systems via synchronous periodic event detection. *IEEE Transactions on Automatic Control*, *60*, 2452–2457.

Carli, R., Bullo, F., & Zampieri, S. (2010). Quantized average consensus via dynamic coding/decoding schemes. *International Journal of Robust and Nonlinear Control*, *20*, 156–175.

Carli, R., Fagnani, F., Frasca, P., & Zampieri, S. (2010). Gossip consensus algorithms via quantized communication. *Automatica*, *46*, 70–80.

Ceragioli, F., De Persis, C., & Frasca, P. (2011). Discontinuities and hysteresis in quantized average consensus. *Automatica*, *47*, 1916–1928.

Cheng, L., Hou, Z.G., Tan, M., & Wang, X. (2011). Necessary and sufficient conditions for consensus of double-integrator multi-agent systems with measurement noises. *IEEE Transactions on Automatic Control*, *56*(8), 1958–1963.

Filippov, A.F. (1988). *Differential equations with discontinuous righthand sides*. Amsterdam: Kluwer Academic.

Frasca, P. (2012). Continuous-time quantized consensus: Convergence of Krasovskii solutions. *Systems & Control Letters*, *61*, 273–278.

Godsil, C., & Royal, G. (2001). *Algebraic graph theory*. New York, NY: Springer-Verlag.

Guo, M., & Dimarogonas, D.V. (2013). Consensus with quantized relative state measurements. *Automatica*, *49*, 2531–2537.

Huang, M., & Manton, J.H. (2009). Coordination and consensus of networked agents with noisy measurements: Stochastic algorithms and asymptotic behavior. *SIAM Journal on Control and Optimization*, *48*(1), 134–161.

Kashyap, A., Basar, T., & Srikant, R. (2007). Quantized consensus. *Automatica*, *43*, 1192–1203.

Li, T., Fu, M., Xie, L., & Zhang, J. (2011). Distributed consensus with limited communication data rate. *IEEE Transactions on Automatic Control*, *56*, 279–291.

Li, T., & Xie, L. (2012). Distributed coordination of multi-agent systems with quantized-observer based encoding-decoding. *IEEE Transactions on Automatic Control*, *57*, 3023–3037.

Li, T., & Zhang, J. (2010). Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises. *IEEE Transactions on Automatic Control*, *55*, 2043–2057.

Lin, P., & Jia, Y. (2009). Consensus of second-order discrete-time multiagent systems with nonuniform time-delays and dynamically changing topologies. *Automatica*, *45*, 2154–2158.

Lin, X., & Zheng, Y. (2016). Finite-time consensus of switched multi-agent systems. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*.Advance online publication. doi:10.1109/TSMC.2016.2631659

Liu, H., Cao, M., & Persis, C.D. (2012). Quantization effects on synchronized motion of teams of mobile agents with second-order dynamics. *Systems & Control Letters*, *61*, 1157–1167.

Olfati-Saber, R., & Murray, R. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, *49*, 1520–1533.

Paden, B., & Sastry, S. (1987). A calculus for computing filippov's differential inclusion with application to the variable structure control of robot manipulators. *IEEE Transactions on Circuits and Systems*, *34*(1), 73–82.

Ren, W., & Beard, R.W. (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, *50*, 655–661.

Ren, W., & Cao, Y. (2010). *Distributed coordination of multi-agent networks: Emergent problems, models, and issues*, London: Springer Science & Business Media.

Wang, Y., Cheng, L., Ren, W., Hou, Z.G., & Tan, M. (2015). Seeking consensus in networks of linear agents: Communication noises and Markovian switching topologies. *IEEE Transactions on Automatic Control*, *60*(5), 1374–1379.

Xiao, F., Wang, L., & Chen, T. (2014). Finite-time consensus in networks of integrator-like dynamic agents with directional link failure. *IEEE Transactions on Automatic Control*, *59*, 756–762.

Xu, Y., & Wang, J. (2013). The synchronization of linear systems under quantized measurements. *Systems & Control Letters*, *62*, 972–980.

Zeng, Z., Wang, X., & Zheng, Z. (2016). Edge agreement of second-order multi-agent system with dynamic quantization via directed edge Laplacian. *IET Control Theory & Applications*, *13*(10), 1583–1589.

Zhai, G., Lin, H., Michel, A.N., & Yasuda, K. (2004). Stability analysis for switched with continuous-time and discrete-time subsystems. In *Proceedings of the American Control Conference* (pp. 4555–4560). Boston, MA.

Zhai, G., Liu, D., & Imae, J. (2006). Lie algebraic stability analysis for switched systems with continuous-time and discrete-time subsystems. *IEEE Transactions on Circuits and Systems II: Express Briefs*, *53*(2), 152–156.

Zheng, Y., & Wang, L. (2014). Containment control of heterogeneous multi-agent systems. *International Journal of Control*, *87*, 1–8.

Zheng, Y., & Wang, L. (2016). Consensus of switched multi-agent systems. *IEEE Transactions on Circuits and Systems II: Express Briefs*, *63*, 314–318.

Zheng, Y., Zhu, Y., & Wang, L. (2014). Finite-time consensus of multiple second-order dynamic agents without velocity measurements. *International Journal of Systems Science*, *45*, 579–588.

Zhu, Y., Zheng, Y., & Wang, L. (2015a). Containment control of switched multi-agent systems. *International Journal of Control*, *88*, 2570–2577.

Zhu, Y., Zheng, Y., & Wang, L. (2015b). Quantized consensus of multi-agent systems with nonlinear dynamics. *International Journal of Systems Science*, *46*, 2061–2071.