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Quantized consensus of second-order multi-agent systems via impulsive control☆

Yunru Zhu[a,b], Yuanshi Zheng[b,*], Yongqiang Guan[b]

[a] Key Laboratory of Electronic Equipment Structure Design of Ministry of Education, School of Mechano-Electronic Engineering, Xidian University, Xi'an 710071, China

[b] Center for Complex Systems, School of Mechano-Electronic Engineering, Xidian University, Xi'an 710071, China

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ABSTRACT

This paper studies the consensus of second-order continuous-time multi-agent systems with quantized interaction. For static consensus, a hybrid impulsive protocol is proposed using the quantized relative position information. We prove that for any quantizer accuracy, the multi-agent systems can reach consensus if impulsive intervals are less than a given value. For dynamical consensus, an impulsive protocol is presented using both the quantized relative position and velocity information. It is shown that for any given impulsive interval, the multi-agent system can achieve consensus by selecting appropriate control gains and sufficiently small quantizer accuracy. The simulation results are given to verify the effectiveness of the theoretical results.

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1. Introduction

Recently, considerable attention has been paid to multi-agent systems (MASs) and lots of work has been done, such as consensus, containment control, controllability, optimization, fault detection, oscillator synchronization problem. Consensus is a fundamental and important problem in distributed coordination control and has been widely studied due to its wide applications in many fields, such as distributed estimation, distributed time synchronization, etc [1–7]. In practice, as digital networks are widely used, we have to consider some network-induced problems, such as time-delay [2,8], packet loss [9], quantization [10,11]. Early works on quantized consensus mainly focuses on discrete-time (DT) MASs. The notion of quantized was introduced in [12]. Using encoding–decoding scheme, quantized consensus problem was studied for DT MASs under undirected graph in [10,13] and directed graph in [14]. In [15], the leader-following consensus based on encoding–decoding is investigated in directed graph. Recently, increasing attention has also been focus on the quantized consensus of continuous-time (CT) MASs. Because of the discontinuity of the quantized signals, the tools from nonsmooth analysis were often used for consensus analysis of CT MASs. Consensus with relative quantized state measurements were studied for first-order MASs under static topology in [16] and time-varying topology in [17]. Using incidence matrix, consensus with quantized relative state measurements was studied for MASs with first-order dynamics in [18,19] and second-order dynamics in [19]. By constructing a novel Lyapunov function, [20] improved the results in [18] for second-order MASs and proved that the consensus can been achieved for any quantizer accuracy under undirected and connected communication topology. In [21], the authors dealt with consensus problem for passive systems in the presence of quantized relative state measurements. Consensus of MASs with general linear dynamics under quantized relative states measurements was investigated in [22]. In [23], quantized consensus of heterogeneous MASs was investigated.

In practice, impulsive control strategy has many advantages, such as simple structure, small control cost, etc. There has been many works devoted to studying impulsive consensus problems. In [24], the author used a novel impulsive control method for MASs to reach consensus and the performance of the closed-loop system was improved. Impulsive consensus protocols were proposed for MASs with second-order dynamics using velocity information in [25] and without using any velocity information in [26,27]. In [28], the authors showed that by synthesizing the coupling weights and the average impulsive intermittence, MASs can achieve guaranteed performance.

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* Corresponding author.
E-mail addresses: yrzhu@xidian.edu.cn (Y. Zhu), zhengyuanshi2005@163.com (Y. Zheng), guan-jq@163.com (Y. Guan).

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In [18–23], the authors proved that the MASs can reach consensus asymptotically with the CT consensus protocol and quantized information. However, the Zeno behavior, i.e., the transmission times of quantized information accumulate in finite time, which is resulted from logarithmic quantized interactions between agents was not discussed in these papers. Motivated by impulsive control strategy, in this paper, we exclude the occurrence of Zeno behavior via impulsive control, which only need the quantized relative state information of agents at impulsive instance. First, a hybrid impulsive protocol is proposed for static consensus. By using stability theory of impulsive systems and properties of the Laplacian matrix, we prove that for any quantizer accuracy, the MASs can reach static consensus if impulsive intervals are less that a given value. Second, an impulsive protocol is presented and a sufficient condition is given to guarantee the achievement of dynamic consensus. Third, an intermittent consensus protocol is also proposed to avoid the abrupt change of states in impulsive protocol. Finally, the simulation results are given to verify the effectiveness of the theoretical results.

Notation: Let R^n be the set of n-dimensional real vectors and N be the set of nonnegative integers . A^T denotes the transpose of matrix A. $|\cdot|$ stands for the absolute value of a real number. 0_n is an $n \times 1$ column zero vector, I_n is an $n \times n$ identity matrix, $\mathrm{diag}\{a_1, \ldots, a_n\}$ is a diagonal matrix, $\mathcal{I}_n = \{1, 2, \ldots, n\}$. If the eigenvalues of a matrix $A \in R^{n \times n}$ are real, they are ordered $\lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_n(A)$ in this paper.

2. Preliminaries

2.1. Graph theory

Let $\mathcal{G}(\mathcal{A}) = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted undirected graph, where $\mathcal{V} = \{s_1, s_2, \ldots, s_n\}$ is a vertex set, $\mathcal{E} = \{e_1, e_2, \ldots, e_m\} \subseteq \mathcal{V} \times \mathcal{V}$ is an edge set, and $\mathcal{A} = (a_{ij})_{n\times n}$ is an adjacency matrix with $a_{ij} > 0$ if $(s_j, s_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. We assume $a_{ii} = 0$. The graph $\mathcal{G}$ is called connected if for any two distinct vertices s_i and s_j, there exists a sequence of distinct edges $(s_i, s_{k_1}), (s_{k_1}, s_{k_2}), \ldots, (s_{k_{h-1}}, s_j)$ between them. Define $D = \mathrm{diag}\{d_1, d_2, \ldots, d_n\}$ with $d_i = \sum_{j=1, j\neq i}^{n} a_{ij}$ and the Laplacian matrix $L = D - \mathcal{A}$. Orientate each edge in graph $\mathcal{G}$ arbitrarily to make it have a head and tail. The incidence matrix $B = (b_{ij})_{n\times m}$ is defined as $b_{ij} = 1$, if s_i is the head of e_j, $b_{ij} = -1$, if s_i is the tail of e_j, and $b_{ij} = 0$, otherwise. Let $W = \mathrm{diag}\{w_1, \ldots, w_m\}$, where w_i denotes the weight of e_i. Then, $L = BWB^T$.

2.2. Quantizer

Define the set of quantization levels as

$$U = \left\{ \pm u_i, u_i = \left(\frac{1-\delta}{1+\delta}\right)^i u_0, i = \pm 1, \pm 2, \ldots \right\} \cup \{\pm u_0\} \cup \{0\},$$

where $u_0 > 0$ and accuracy parameter $0 < \delta < 1$.

A logarithmic quantizer q: $R \to R$ is a map defined as [29]:

$$q(a) = \begin{cases} u_i & \text{if } \frac{1}{1+\delta} u_i < a \le \frac{1}{1-\delta} u_i, a > 0, \\ 0 & \text{if } a = 0, \\ -q(-a) & \text{if } a < 0. \end{cases}$$

From the definition, it can be easily derive that $|a - q(a)| \le \delta|a|$, $\forall a \in R$. For a vector $x = [x_1, \ldots, x_d]^T \in R^d$, define $q(x) = [q(x_1), \ldots, q(x_d)]^T$.

3. Main results

Consider a MAS which consists of n second-order agents as

$$\dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = u_i(t), \tag{1}$$

where $x_i \in R, v_i \in R$ are the ith agent's position and velocity, $i \in \mathcal{I}_n$.

Definition 1. Consensus in the MAS (1) is said to be achieved if the states of agents satisfy

$$\lim_{t\to+\infty} |x_i(t) - x_j(t)| = 0, \quad \lim_{t\to+\infty} |v_i(t) - v_j(t)| = 0, \quad \forall i, j \in \mathcal{I}_n.$$

3.1. Static consensus

When logarithmic quantizers are used, we adopt the following hybrid impulsive protocol:

$$u_i(t) = -\alpha v_i(t) + k \sum_{l=0}^{\infty} \left[\sum_{j=1}^{n} a_{ij} q(x_j(t) - x_i(t)) \right] \delta(t - t_l), \tag{2}$$

where the control gains $\alpha > 0$, $k > 0$, $\delta(\cdot)$ is the Dirac function. Impulsive instants satisfy $0 = t_0 < t_1 < \cdots < t_l < \cdots$, $\lim_{l\to+\infty} t_l = +\infty$. We define $\tau > 0$ such that the sampling period $h_l = t_{l+1} - t_l > \tau$.

Lemma 1 [30]. *If G, H are Hermitian matrices and $1 \le i_1 < i_2 < \cdots < i_k \le n$, then*

$$\sum_{t=1}^{k} \lambda_{i_t}(G + H) \ge \sum_{t=1}^{k} \lambda_{i_t}(G) + \sum_{t=1}^{k} \lambda_t(H).$$

Lemma 2. *Let $L_1 = BW_1B^T$ and $L_2 = BW_2B^T$, where $B \in R^{n\times m}$ is an incidence matrix of graph $\mathcal{G}$, $W_1, W_2 \in R^{m\times m}$ are nonnegative diagonal matrices. Then $\lambda_i(L_1 + L_2) \ge \lambda_i(L_1), i \in \mathcal{I}_n$.*

Proof. From Lemma 1, we have $\lambda_i(L_1 + L_2) \ge \lambda_i(L_1) + \lambda_1(L_2)$. From $\lambda_1(L_2) = 0$, we get $\lambda_i(L_1 + L_2) \ge \lambda_i(L_1)$. □

Theorem 1. *Suppose that the communication graph $\mathcal{G}$ is connected. System (1) achieves consensus under protocol (2) if $k(1+\delta)\lambda_n(L) < \alpha$ and $h_l < \frac{2}{\alpha} \ln \frac{\alpha+(1-\delta)k\lambda_2(L)}{\alpha-(1-\delta)k\lambda_2(L)}$.*

Proof. Let $\bar{x}(t) = \frac{1}{n}\sum_{j=1}^{n} x_j(t)$, $\bar{v}(t) = \frac{1}{n}\sum_{j=1}^{n} v_j(t)$, $\tilde{x}_i(t) = x_i(t) - \bar{x}(t)$, $\tilde{v}_i(t) = v_i(t) - \bar{v}(t)$. System (1) with protocol (2) can be rewritten as

$$\begin{cases} \dot{\tilde{x}}_i(t) = \tilde{v}_i(t), \\ \dot{\tilde{v}}_i(t) = -\alpha \tilde{v}_i(t), & t \neq t_l, \\ \triangle\tilde{v}_i(t_l) = k \sum_{j=1}^{n} a_{ij} q(x_j(t_l) - x_i(t_l)), \end{cases} \tag{3}$$

where $\triangle\tilde{v}_i(t_l) = \triangle\tilde{v}_i(t_l^+) - \triangle\tilde{v}_i(t_l)$.

Define $x(t) = [x_1(t), \ldots, x_n(t)]^T, v(t) = [v_1(t), \ldots, v_n(t)]^T, \hat{x}(t) = B^T x(t)$, $\hat{v}(t) = B^T v(t)$, $\tilde{x}(t) = x(t) - 1_n\bar{x}(t)$, $\tilde{v}(t) = v(t) - 1_n\bar{v}(t)$ and $q(x(t)) - x(t) = \Lambda(t)x(t)$, where $\Lambda(t) = \mathrm{diag}\{\varepsilon_1(t), \ldots, \varepsilon_m(t)\}$ and $\varepsilon_i(t) \in [-\delta, +\delta], i \in \mathcal{I}_m$. Then, we have

$$\begin{aligned} \triangle\tilde{v}(t_l) &= -kBWq(\hat{x}(t_l)) \\ &= -kBW(I + \Lambda(t_l))\hat{x}(t_l) \\ &= -kBW(I + \Lambda(t_l))B^T x(t_l) \\ &= -kBW(I + \Lambda(t_l))B^T \tilde{x}(t_l), \end{aligned}$$

where the final equality is due to the fact that $B^T 1_n = 0$.

The aggregate dynamics of system (3) is represented by

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{v}(t), \\ \dot{\tilde{v}}(t) = -\alpha \tilde{v}(t), & t \neq t_l, \\ \triangle\tilde{v}(t_l) = -kL(t_l)\tilde{x}(t_l), \end{cases} \tag{4}$$

where $L(t_l) = BW(I + \Lambda(t_l))B^T$.

Let $y(t)=(\tilde{x}^T(t),\tilde{v}^T(t))^T$. According to (4), we have for $\forall t\in(t_l,t_{l+1}]$

$$\begin{aligned}y(t)&=\begin{pmatrix}I_n & \frac{1-e^{-\alpha(t-t_l)}}{\alpha}I_n\\ 0 & e^{-\alpha(t-t_l)}I_n\end{pmatrix}\begin{pmatrix}I_n & 0\\ -kL(t_l) & I_n\end{pmatrix}\\&\times\prod_{s=0}^{l-1}\left(\begin{pmatrix}I_n & \frac{1-e^{-\alpha(t_{s+1}-t_s)}}{\alpha}I_n\\ 0 & e^{-\alpha(t_{s+1}-t_s)}I_n\end{pmatrix}\begin{pmatrix}I_n & 0\\ -kL(t_s) & I_n\end{pmatrix}\right)y(t_0)\\&=\Phi(t,t_l)\prod_{s=0}^{l-1}\Phi(t_{s+1},t_s)y(t_0),\end{aligned}\tag{5}$$

where

$$\Phi(t_{s+1},t_s)=\begin{pmatrix}I_n-\frac{1-e^{-\alpha(t_{s+1}-t_s)}}{\alpha}kL(t_s) & \frac{1-e^{-\alpha(t_{s+1}-t_s)}}{\alpha}I_n\\ -e^{-\alpha(t_{s+1}-t_s)}kL(t_s) & e^{-\alpha(t_{s+1}-t_s)}I_n\end{pmatrix}.$$

Inspired by the proof of [27], we note that $y(t)=(I_2\otimes Q)(x^T(t),v^T(t))^T$, where $Q=I_n-\frac{1}{n}1_n1_n^T$. From $Q^2=Q$ and $\Phi(t_{s+1},t_s)(I_2\otimes Q)=(I_2\otimes Q)\Phi(t_{s+1},t_s)$, an equivalent solution of (5) is

$$\begin{aligned}y(t)&=\Phi(t,t_l)\prod_{s=0}^{l-1}\Phi(t_{s+1},t_s)(I_2\otimes Q)(x^T(t_0),v^T(t_0))^T\\&=\Phi(t,t_l)\prod_{s=0}^{l-1}(\Phi(t_{s+1},t_s)(I_2\otimes Q))y(t_0)\\&=\Phi(t,t_l)\prod_{s=0}^{l-1}H_sy(t_0),\end{aligned}\tag{6}$$

where

$$H_s=\begin{pmatrix}Q-\frac{1-e^{-\alpha h_s}}{\alpha}kL(t_s) & \frac{1-e^{-\alpha h_s}}{\alpha}Q\\ -e^{-\alpha h_s}kL(t_s) & e^{-\alpha h_s}Q\end{pmatrix}.$$

Since the communication graph is undirected, the matrix $L(t_s)$ is a real symmetric matrix and there exists an orthogonal matrix $U(t_s)\in R^{n\times n}$ such that $U^T(t_s)L(t_s)U(t_s)=\mathrm{diag}(0,\Delta(t_s))$, where $\Delta(t_s)=\mathrm{diag}(\lambda_2(L(t_s)),\lambda_3(L(t_s)),\dots,\lambda_n(L(t_s)))$. To calculate the eigenvalues of H_s, we solve the following equation:

$$\begin{aligned}&\det(\lambda I_{2n}-H_s)=\det\big(\lambda I_{2n}-\big(I_2\otimes U^T(t_s)\big)H_s(I_2\otimes U(t_s))\big)\\&=\lambda^2\det\begin{pmatrix}(\lambda-1)I_{n-1}+\frac{1-e^{-\alpha h_s}}{\alpha}k\Delta(t_s) & -\frac{1-e^{-\alpha h_s}}{\alpha}I_{n-1}\\ e^{-\alpha h_s}k\Delta(t_s) & (\lambda-e^{-\alpha h_s})I_{n-1}\end{pmatrix}\\&=\lambda^2\prod_{i=2}^{n}\left(\lambda^2-(1+e^{-\alpha h_s}-\frac{1-e^{-\alpha h_s}}{\alpha}k\lambda_i(L(t_s))\lambda+e^{-\alpha h_s}\right)\\&=0.\end{aligned}$$

Hence, the eigenvalues of H_s are $\lambda_{\pm}^1(H_s)=0$ and $\lambda_{\pm}^i(H_s)=\frac{P_s^i}{2}\pm\frac{1}{2}\sqrt{(P_s^i)^2-4e^{-\alpha h_s}}$, where $P_s^i=1+e^{-\alpha h_s}-\frac{1-e^{-\alpha h_s}}{\alpha}k\lambda_i(L(t_s))$, $i=2,3,\dots,n$. Note $L(t_s)=BW(I+\Lambda(t_s))B^T$. From Lemma 2, one obtains that $\max\lambda_i(L(t_s))=\max\lambda_n(L(t_s))=\lambda_n((1+\delta)BWB^T)=(1+\delta)\lambda_n(L)$ and $\min\lambda_2(L(t_s))=\lambda_2((1-\delta)BWB^T)=(1-\delta)\lambda_2(L)$. Hence, from the condition $k(1+\delta)\lambda_n(L)<\alpha$, we have $k\lambda_i(L(t_s))<\alpha$, that is $\frac{k\lambda_i(L(t_s))}{\alpha}<1<\frac{1+e^{-\frac{\alpha}{2}h_s}}{1-e^{-\frac{\alpha}{2}h_s}}$. The condition $h_s<\frac{2}{\alpha}\ln\frac{\alpha+(1-\delta)k\lambda_2(L)}{\alpha-(1-\delta)k\lambda_2(L)}$ implies that $h_s<\frac{2}{\alpha}\ln\frac{\alpha+k\lambda_i(L(t_s))}{\alpha-k\lambda_i(L(t_s))}$, that is $\frac{1-e^{-\frac{\alpha}{2}h_s}}{1+e^{-\frac{\alpha}{2}h_s}}<\frac{k\lambda_i(L(t_s))}{\alpha}$. Since $\frac{1-e^{-\frac{\alpha}{2}h_s}}{1+e^{-\frac{\alpha}{2}h_s}}<\frac{k\lambda_i(L(t_s))}{\alpha}<\frac{1+e^{-\frac{\alpha}{2}h_s}}{1-e^{-\frac{\alpha}{2}h_s}}$, then $(P_s^i)^2-4e^{-\alpha h_s}=(1-e^{-\alpha h_s})^2(\frac{k\lambda_i(L(t_s))}{\alpha})^2-2(1-e^{-\alpha h_s})(1+e^{-\alpha h_s})\frac{k\lambda_i(L(t_s))}{\alpha}+(1-e^{-\alpha h_s})^2<0$. Thus, $|\lambda_{\pm}^i(H_s)|=\sqrt{e^{-\alpha h_s}}\le\sqrt{e^{-\alpha\tau}}\triangleq\rho_0<1, i=2,3,\dots,n$. Therefore, $\rho(H_s)\le\rho_0$. Since for $\forall\xi>0$, *there must exist a matrix norm* $\|\cdot\|$, *such that* $\|H_s\|\le\rho(H_s)+\xi$. *We can take* ξ *sufficient small, such that* $\|H_s\|\le\rho_0+\xi\triangleq\gamma<1$. Then, $\|y(t)\|\le\|\Phi(t,t_l)\|\prod_{s=0}^{l-1}\|H_s\|\|y(t_0)\|\le\|\Phi(t,t_l)\|\|y(t_0)\|\gamma^l$. Since $\gamma<1$, we have $\lim_{t\to+\infty}\|y(t)\|=0$, which implies that the consensus is achieved.

We can further calculate the agents' asymptotic states. Since the communication graph is connected, one can check that $\dot{\bar{v}}(t)=-\alpha\bar{v}(t)$. Hence, $\bar{v}(t)=e^{-\alpha(t-t_0)}\bar{v}(t_0)$. Combined with the fact that $\lim_{t\to+\infty}|v_i(t)-v_j(t)|=0$, we have $\lim_{t\to\infty}v_i(t)=0$, $i\in\mathcal{I}_n$. Since $\dot{\bar{x}}(t)=\bar{v}(t)$, then $\bar{x}(t)=\bar{x}(t_0)+\frac{1-e^{-\alpha(t-t_0)}}{\alpha}\bar{v}(t_0)$, it has $\lim_{t\to\infty}x_i(t)=\bar{x}(t_0)+\frac{1}{\alpha}\bar{v}(t_0)$. □

3.2. Dynamic consensus

In this section, we adopt the following impulsive protocol:

$$\begin{aligned}u_i(t)=&\sum_{l=0}^{\infty}\left[k_1\sum_{j=1}^{n}a_{ij}q(x_j(t)-x_i(t))+k_2\sum_{j=1}^{n}a_{ij}q(v_j(t)-v_i(t))\right]\\&\times\delta(t-t_l).\end{aligned}\tag{7}$$

Under protocol (7), system (1) can be rewritten as:

$$\begin{cases}\dot{x}_i(t)=v_i(t_l),\\ \dot{v}_i(t)=0, & t\neq t_l,\\ \triangle v_i(t_l)=k_1\sum_{j=1}^{n}a_{ij}q(x_j(t_l)-x_i(t_l))\\ \quad+k_2\sum_{j=1}^{n}a_{ij}q(v_j(t_l)-v_i(t_l)).\end{cases}\tag{8}$$

Take $t_{l+1}-t_l\equiv h>0$, $l\in N$. System (8) can be represented compactly by

$$\begin{pmatrix}x(t_{l+1})\\ v(t_{l+1})\end{pmatrix}=\begin{pmatrix}I_n & hI_n\\ 0 & I_n\end{pmatrix}\begin{pmatrix}x(t_l)\\ v(t_l)\end{pmatrix}-\begin{pmatrix}hk_1BW & hk_2BW\\ k_1BW & k_2BW\end{pmatrix}\begin{pmatrix}q(\hat{x}(t_l))\\ q(\hat{v}(t_l))\end{pmatrix}.\tag{9}$$

Lemma 3. *System (1) achieves consensus under protocol (7) if and only if discrete-time system (9) achieves consensus.*

Proof. Sufficiency. Since system (9) achieves consensus, we have $\lim_{l\to+\infty}|x_i(t_l)-x_j(t_l)|=0$, $\lim_{l\to+\infty}|v_i(t_l)-v_j(t_l)|=0$. For $\forall t>t_0$, $\exists l\in N$, such that $t\in(t_l,t_{l+1}]$. When $t\to\infty$, it means that $l\to\infty$ and $t_l\to\infty$. Thus, $\lim_{t\to+\infty}|v_i(t)-v_j(t)|=0, i,j\in\mathcal{I}_n$ and $\lim_{t\to+\infty}|\triangle v_i(t_l)|=0$, $\forall i\in\mathcal{I}_n$. For $\forall i,j\in\mathcal{I}_n$, it has

$$\begin{aligned}\lim_{t\to+\infty}|x_i(t)-x_j(t)|=&\lim_{t\to+\infty}|x_i(t_l)-x_j(t_l)+(t-t_l)(v_i(t_l)-v_j(t_l))\\&+(t-t_l)(\triangle v_i(t_l)-\triangle v_j(t_l))|\\=&\,0.\end{aligned}$$

Necessity. The necessity is obvious and thus is omitted for brevity. □

Theorem 2. *Suppose that the communication graph* $\mathcal{G}$ *is connected. System (1) achieves consensus under protocol (7) if* $h<\frac{4}{\lambda_n(L)k_1}-\frac{2k_2}{k_1}$ *and* $\delta<\frac{2}{\lambda_n(L)(d_1+\sqrt{d_1^2+4d_2})}$, *where* d_1 *and* d_2 *are defined in (13).*

Proof. Define the quantization error by

$$q(\hat{x}(t_l))-\hat{x}(t_l)=\Lambda_x\hat{x}(t_l),\ q(\hat{v}(t_l))-\hat{v}(t_l)=\Lambda_v\hat{v}(t_l),$$

where $\Lambda_x=\mathrm{diag}\{\varepsilon_1,\dots,\varepsilon_m\}$, $\Lambda_v=\mathrm{diag}\{\varepsilon_{m+1},\dots,\varepsilon_{2m}\}$ and $\varepsilon_i\in[-\delta,+\delta]$, $i\in\mathcal{I}_{2m}$. Then, system (9) is rewritten as

$$\begin{aligned}\begin{pmatrix}x(t_{l+1})\\ v(t_{l+1})\end{pmatrix}=&\begin{pmatrix}I_n-hk_1L & hI_n-hk_2L\\ -k_1L & I_n-k_2L\end{pmatrix}\begin{pmatrix}x(t_l)\\ v(t_l)\end{pmatrix}-\begin{pmatrix}hk_1BW & hk_2BW\\ k_1BW & k_2BW\end{pmatrix}\\&\times\begin{pmatrix}\Lambda_x & 0_{m\times m}\\ 0_{m\times m} & \Lambda_v\end{pmatrix}\begin{pmatrix}\hat{x}(t_l)\\ \hat{v}(t_l)\end{pmatrix}.\end{aligned}\tag{10}$$

There exists an orthogonal matrix $U=[\frac{1_n}{\sqrt{n}},U_1]\in R^{n\times n}$, such that $U^TLU=\mathrm{diag}(0,\Delta)$, where $\Delta=\mathrm{diag}(\lambda_2(L),\lambda_3(L),\dots,\lambda_n(L))$. Then,

by applying the state transformation $r(t)=U^Tx(t)$, $p(t)=U^Tv(t)$ and $y(t)=(r^T(t),\ p^T(t))^T$, the Eq. (10) becomes

$$y(t_{l+1})=(M_1+M_2)y(t_l), \tag{11}$$

where

$$M_1=\begin{pmatrix} 1 & 0_{n-1}^T & h & 0_{n-1}^T \\ 0_{n-1} & I_{n-1}-hk_1\Delta & 0_{n-1} & hI_{n-1}-hk_2\Delta \\ 0 & 0_{n-1}^T & 1 & 0_{n-1}^T \\ 0_{n-1} & -k_1\Delta & 0_{n-1} & I_{n-1}-k_2\Delta \end{pmatrix},$$

$$M_2=\begin{pmatrix} 0 & 0_{n-1}^T & 0 & 0_{n-1}^T \\ 0_{n-1} & hk_1U_1^TBW\Lambda_xB^TU_1 & 0_{n-1} & hk_2U_1^TBW\Lambda_vB^TU_1 \\ 0 & 0_{n-1}^T & 0 & 0_{n-1}^T \\ 0_{n-1} & k_1U_1^TBW\Lambda_xB^TU_1 & 0_{n-1} & k_2U_1^TBW\Lambda_vB^TU_1 \end{pmatrix}.$$

Since the communication graph among agents is connected, we know that $\lim_{l\to\infty}x_i(t_l)=\lim_{l\to\infty}x_j(t_l)$, $\forall i,j\in\mathcal{I}_n$ if and only if $\lim_{l\to\infty}Lx(t_l)=0$. From $x(t)=Ur(t)$, the latter holds if and only if $\lim_{l\to\infty}LUr(t_l)=0$. Since $LU=U\mathrm{diag}(0,\Delta)$, $\lim_{l\to\infty}LUr(t_l)=0$ holds if and only if $\lim_{l\to\infty}\mathrm{diag}(0,\Delta)r(t_l)=0$, equivalently $\lim_{l\to\infty}r_i(t_l)=0$ for $i=2,3,\dots,n$. Similarly, we have $\lim_{l\to\infty}v_i(t_l)=\lim_{l\to\infty}v_j(t_l)$, $\forall i,j\in\mathcal{I}_n$ if and only if $\lim_{l\to\infty}p_i(t_l)=0$ for $i=2,3,\dots,n$. Thus, we obtain that system (10) can achieve consensus if the following system

$$y(t_{l+1})=(M_3+E\Lambda F)y(t_l)y(t_l), \tag{12}$$

is stable, where

$$M_3=\begin{pmatrix} I_{n-1}-hk_1\Delta & hI_{n-1}-hk_2\Delta \\ -k_1\Delta & I_{n-1}-k_2\Delta \end{pmatrix},$$

$$E=\begin{pmatrix} hk_1 & hk_2 \\ k_1 & k_2 \end{pmatrix}\otimes(U_1^TBW^{\frac{1}{2}}),$$

$$\Lambda=\mathrm{diag}(\Lambda_x,\Lambda_y),\quad F=I_2\otimes(W^{\frac{1}{2}}B^TU_1).$$

It is easy to get that $\lambda_{\max}(E^TE)=(k_1^2+k_2^2)(1+h^2)\lambda_n(L)$, $\lambda_{\max}(F^TF)=\lambda_n(L)$. To calculate the eigenvalues of M_3, we solve the following equation:

$$\begin{aligned}\det(\lambda I_{2(n-1)}-M_3)&=\det\begin{pmatrix}(\lambda-1)I_{n-1}+hk_1\Delta & -hI_{n-1}+hk_2\Delta \\ k_1\Delta & (\lambda-1)I_{n-1}+k_2\Delta\end{pmatrix}\\ &=\prod_{i=2}^{n}\left(\lambda^2+(hk_1\lambda_i+k_2\lambda_i-2)\lambda+1-k_2\lambda_i\right)\\ &=0.\end{aligned}$$

Let $a_i(\lambda)=\lambda^2+(hk_1\lambda_i+k_2\lambda_i-2)\lambda+1-k_2\lambda_i$, $i=2,3,\dots,n$. Define $r_i(\sigma)=(\sigma-1)^2a_i\left(\frac{1+\sigma}{1-\sigma}\right)=hk_1\lambda_i\sigma^2+2\lambda_ik_2\sigma-hk_1\lambda_i-2k_2\lambda_i+4$. Then the Schur stable of $a_i(\mu)$ is equivalent to the Hurwitz stable of $r_i(\sigma)$. The roots of $r_i(\sigma)=0$ are $\frac{-k_2}{hk_1}\pm\sqrt{(\frac{k_2}{hk_1}+1)^2-\frac{4}{hk_1\lambda_i}}$. Thus, $r_i(\sigma)$ is Hurwitz stable if and only if $(\frac{k_2}{hk_1}+1)^2-\frac{4}{hk_1\lambda_i}<(\frac{-k_2}{hk_1})^2$, that is $h<\frac{4}{\lambda_ik_1}-\frac{2k_2}{k_1}$. Therefore, M_3 is Schur stable.

Since M_3 is Schur stable, there exists a matrix $P>0$ satisfies $M_3^TPM_3-P=-I$. Take $V(t)=y^T(t)Py(t)$. We have

$$\begin{aligned}V(t_{l+1})-V(t_l)&=y^T(t_l)(M_3+E\Lambda F)^TP(M_3+E\Lambda F)y(t_l)-y^T(t_l)Py(t_l)\\ &=-y^T(t_l)\left(M_3^TPM_3-P\right)y(t_l)+2y^T(t_l)F^T\Lambda E^TPM_3y(t_l)\\ &\quad+y^T(t_l)F^T\Lambda E^TPE\Lambda Fy(t_l)\\ &=-y^T(t_l)y(t_l)+2y^T(t_l)F^T\Lambda E^TPM_3y(t_l)\\ &\quad+y^T(t_l)F^T\Lambda E^TPE\Lambda Fy(t_l).\end{aligned}$$

Note that

$$\begin{aligned}2y^T(t_l)F^T\Lambda E^TPM_3y(t_l)&\le y^T(t_l)F^T\Lambda Fy(t_l)\\ &\quad+y^T(t_l)M_3^TPE\Lambda E^TPM_3y(t_l)\\ &\le\left[\delta\lambda_n(L)+\delta\lambda_{\max}(E^TE)\lambda_{\max}(P)(\lambda_{\max}(P)-1)\right]y^T(t_l)y(t_l)\\ &=d_1\lambda_n(L)\delta y^T(t_l)y(t_l),\end{aligned}$$

and

$$\begin{aligned}y^T(t_l)F^T\Lambda E^TPE\Lambda Fy(t_l)&\le\left[\delta^2\lambda_{\max}(P)\lambda_{\max}(E^TE)\lambda_{\max}(F^TF)\right]\\ &\quad\times y^T(t_l)y(t_l)\\ &=d_2\lambda_n^2(L)\delta^2y^T(t_l)y(t_l),\end{aligned}$$

where

$$\begin{aligned}d_1&=1+(k_1^2+k_2^2)(1+h^2)\lambda_{\max}(P)(\lambda_{\max}(P)-1),\\ d_2&=\lambda_{\max}(P)(k_1^2+k_2^2)(1+h^2).\end{aligned} \tag{13}$$

Thus

$$V(t_{l+1})-V(t_l)\le(d_2\lambda_n^2(L)\delta^2+d_1\lambda_n(L)\delta-1)y^T(t_l)y(t_l).$$

It follows from $\delta<\frac{2}{\lambda_n(L)(d_1+\sqrt{d_1^2+4d_2})}$ that $V(t_{l+1})-V(t_l)<0$. Therefore, system (12) is stable, that is system (1) achieves consensus under protocol (7). Following the proof of Theorem 1, we can obtain that $\lim_{t\to\infty}v_i(t)=\bar v(t_0)$ and $\lim_{t\to\infty}x_i(t)=\bar x(t_0)+(t-t_0)\bar v(t_0)$, $i\in\mathcal{I}_n$. □

Remark 1. If we want to force the agents to reach a desired state, we can consider the case that there is a leader in MASs. If the communication graph among the followers is undirected and connected, similar to the proof above, it is easy to get some sufficient conditions for the states of all agents asymptotically approach the state of the leader.

The impulsive consensus protocol needs the abrupt change of states at sampling instants. However, in practice, some agents may not bear the sudden changes of states. Thus, we propose the following consensus protocol:

$$u_i(t)=\begin{cases}\frac{k_1}{s}\sum_{j=1}^n a_{ij}q(x_j(t_l)-x_i(t_l)) & t_l<t\le t_l+s,\\ \quad+\frac{k_2}{s}\sum_{j=1}^n a_{ij}q(v_j(t_l)-v_i(t_l)), & \\ 0, & t_l+s<t\le t_{l+1},\end{cases} \tag{14}$$

where $0<s<h$.

Since the control input of each agent during $(t_l,t_l+s]$ is time-invariant, we have

$$\begin{cases}x_i(t_l+s)=x_i(t_l)+sv_i(t_l)+\frac{s^2}{2}u_i(t_l),\\ v_i(t_l+s)=v_i(t_l)+su_i(t_l).\end{cases} \tag{15}$$

Thus, the aggregate dynamics of the system (1) under consensus protocol (14) is represented by

$$\begin{aligned}\begin{pmatrix}x(t_l+s)\\ v(t_l+s)\end{pmatrix}&=\begin{pmatrix}I_n & sI_n\\ 0_{n\times n} & I_n\end{pmatrix}\begin{pmatrix}x(t_l)\\ v(t_l)\end{pmatrix}\\ &\quad-\begin{pmatrix}\frac{s}{2}k_1BW & \frac{s}{2}k_2BW\\ k_1BW & k_2BW\end{pmatrix}\begin{pmatrix}q(\hat x(t_l))\\ q(\hat v(t_l))\end{pmatrix}.\end{aligned} \tag{16}$$

and

$$\begin{pmatrix}x(t_{l+1})\\ v(t_{l+1})\end{pmatrix}=\begin{pmatrix}I_n & (h-s)I_n\\ 0_{n\times n} & I_n\end{pmatrix}\begin{pmatrix}x(t_l+s)\\ v(t_l+s)\end{pmatrix}. \tag{17}$$

Thus, one obtains

$$\begin{aligned}\begin{pmatrix}x(t_{l+1})\\ v(t_{l+1})\end{pmatrix}&=\begin{pmatrix}I_n & hI_n\\ 0_{n\times n} & I_n\end{pmatrix}\begin{pmatrix}x(t_l)\\ v(t_l)\end{pmatrix}\\ &\quad-\begin{pmatrix}\left(h-\frac{s}{2}\right)k_1BW & \left(h-\frac{s}{2}\right)k_2BW\\ k_1BW & k_2BW\end{pmatrix}\begin{pmatrix}q(\hat x(t_l))\\ q(\hat v(t_l))\end{pmatrix}.\end{aligned} \tag{18}$$

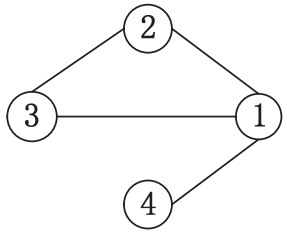


Fig. 1. The communication graph among agents.

Similarly, the system (18) achieves consensus if and only if the following system

$$y(t_{l+1}) = (M_4 + E_1\Lambda F)y(t_l), \tag{19}$$

is stable, where

$$M_4 = \begin{pmatrix} I_{n-1} - (h - \frac{s}{2})k_1\Delta & hI_{n-1} - (h - \frac{s}{2})k_2\Delta \\ -k_1\Delta & I_{n-1} - k_2\Delta \end{pmatrix},$$

$$E_1 = \begin{pmatrix} (h - \frac{s}{2})k_1 & (h - \frac{s}{2})k_2 \\ k_1 & k_2 \end{pmatrix} \otimes (U_1^T B W^{\frac{1}{2}}).$$

Let $P > 0$ be the unique positive definite solution to the Lyapunov equation $M_4^T P M_4 - P = -I$. Define

$$d_3 = 1 + (k_1^2 + k_2^2)(1 + (h - s)^2)\lambda_{\max}(P)(\lambda_{\max}(P) - 1),$$
$$d_4 = \lambda_{\max}(P)(k_1^2 + k_2^2)(1 + (h - s)^2). \tag{20}$$

Similar to the proof of Theorem 2, the following result can be obtained directly.

Theorem 3. *Suppose that the communication graph $\mathcal{G}$ is connected. System (1) achieves consensus under intermittent protocol (14) if* $s < \frac{2k_2}{k_1}, h - s < \frac{4}{\lambda_n(L)k_1} - \frac{2k_2}{k_1}$ *and* $\delta < \frac{2}{\lambda_n(L)(d_3 + \sqrt{d_3^2 + 4d_4})}$.

4. Numerical simulations

Example 1 (Static consensus)**.** Consider a connected graph $\mathcal{G}$ given by Fig. 1, where the weight of each edge is 1. We can easily get $\lambda_2(L) = 1$ and $\lambda_4(L) = 4$. Take control gain $\alpha = 1$ and accuracy parameter $\delta = 0.1$. According to Theorem 1, we can take $k = 0.2 < \frac{\alpha}{(1+\delta)\lambda_4(L)} = 0.23$. For simplicity, take the uniform impulsive interval $h_k = 0.5 < \frac{2}{\alpha} \ln \frac{\alpha + (1-\delta)k\lambda_2(L)}{\alpha - (1-\delta)k\lambda_2(L)} = 0.73$. The simulation results are given in Fig. 2.

Example 2 (Dynamic consensus)**.** We choose $h = 2$ and $k_1 = k_2 = 0.2$ to satisfy the conditions in Theorem 2. Solve the Lyapunov equation $M_3^T P M_3 - P = -I$, we get the matrix $P > 0$ and $\lambda_{\max}(P) = 21.0139$. According to definition in (13), we get $d_1 = 169.2280$ and $d_2 = 8.4056$. Thus, we take $\delta = 0.001$. We can see from Fig. 3 that all agents reach consensus.

5. Conclusion

In this paper, the quantized consensus problem was considered for MASs with undirected communication graph. Impulsive consensus protocols using the quantized relative state measurements were proposed for MASs to achieve static and dynamic consensus, respectively. We proved that all agents can achieve consensus by selecting appropriate control gains, quantizer accuracy and impulsive intervals. In addition, an intermittent consensus protocol was also proposed to avoid the abrupt change of states. The future

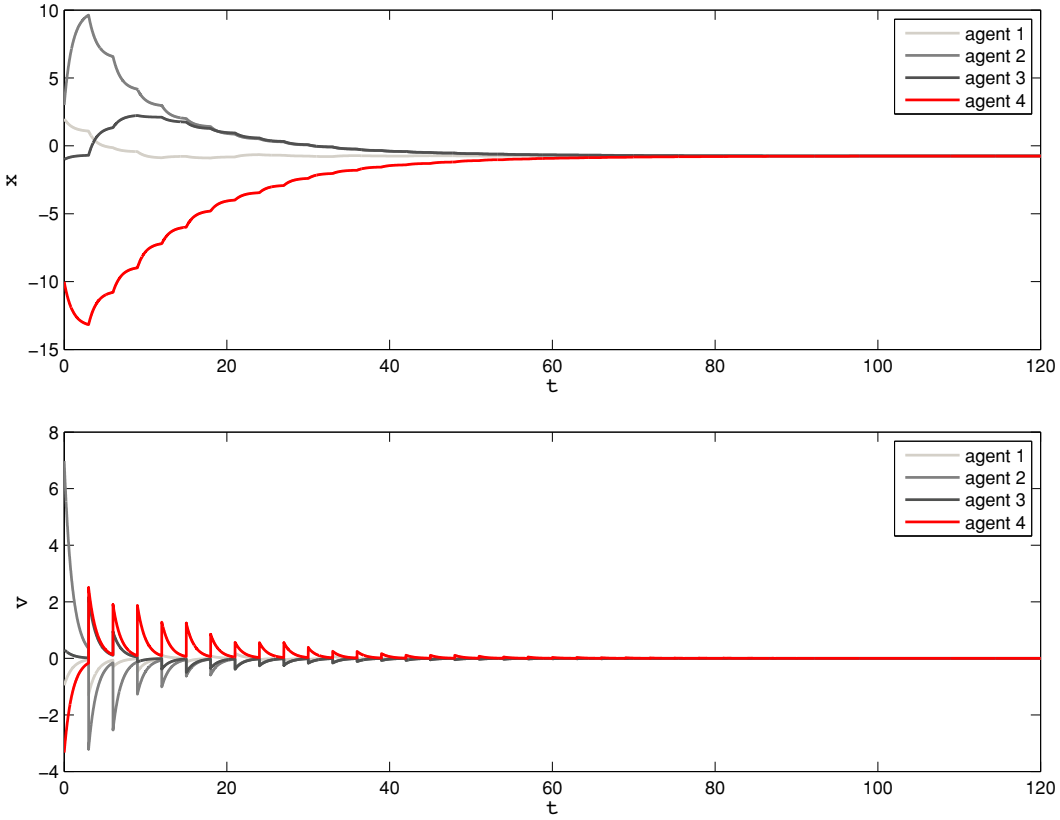


Fig. 2. Position and velocity trajectories of all the agents under protocol (2).

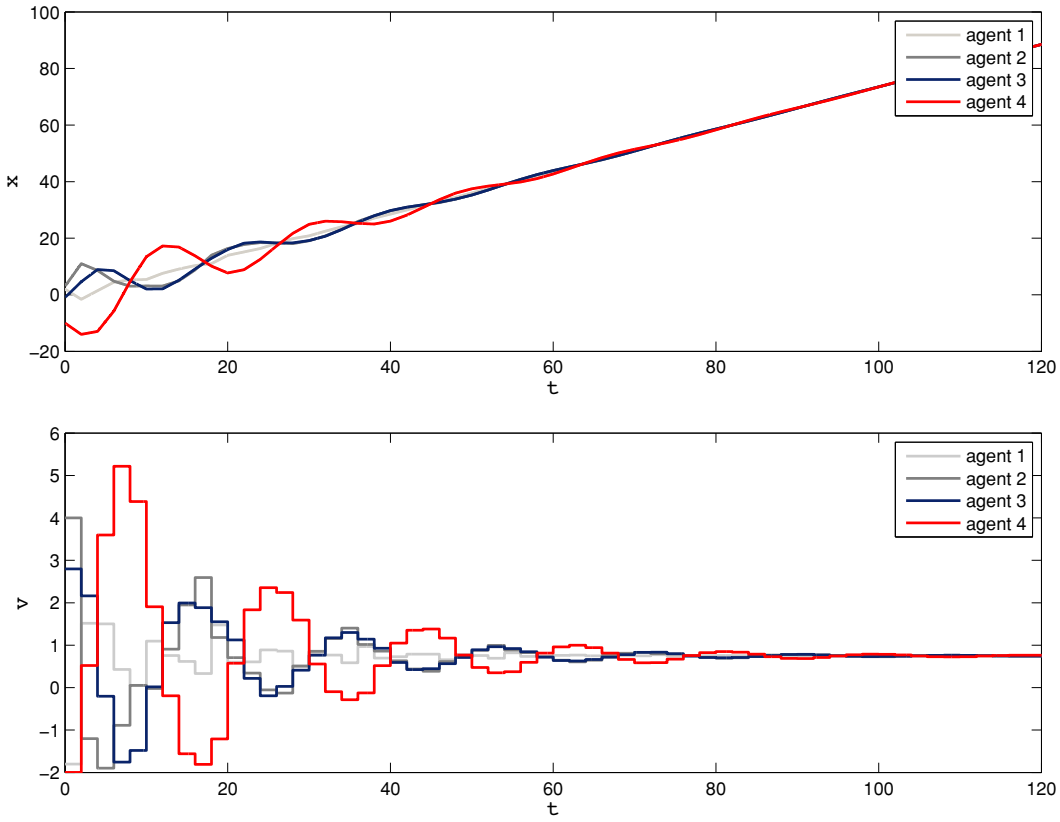


Fig. 3. Position and velocity trajectories of all the agents under protocol (7).

work will focus on impulsive consensus problem for MASs with more complex dynamics in presence of quantizers.

References

[1] R. Olfati-Saber, R.M. Murray, Consensus problems in networks of agents with switching topology and time-delays, IEEE Trans. Autom. Control 49 (9) (2004) 1520–1533.

[2] Y. Zheng, J. Ma, L. Wang, Consensus of hybrid multi-agent systems, IEEE Trans. Neural Networks Learn. Syst. (2017), doi:10.1109/TNNLS.2017.2651402.

[3] Y. Zheng, Y. Zhu, L. Wang, Consensus of heterogeneous multi-agent systems, IET Control Theory Appl. 5 (16) (2011) 1881–1888.

[4] Y. Zheng, L. Wang, Consensus of switched multi-agent systems, IEEE Trans. Circuits Syst. II Express Briefs 63 (6) (2016) 314–318.

[5] X. Lin, Y. Zheng, Finite-time consensus of switched multiagent systems, IEEE Trans. Syst., Man, and Cybern.: Systems (2016), doi:10.1109/TSMC.2016.2631659.

[6] J. He, P. Cheng, L. Shi, J. Chen, Y. Sun, Time synchronization in WSNs: a maximum-value-based consensus approach, IEEE Trans. Autom. Control 59 (3) (2014) 660–675.

[7] J. He, L. Duan, F. Hou, P. Cheng, J. Chen, Multiperiod scheduling for wireless sensor networks: a distributed consensus approach, IEEE Trans. Signal Process. 63 (7) (2015) 1651–1663.

[8] J. Hu, D. Chen, J. Du, State estimation for a class of discrete nonlinear systems with randomly occurring uncertainties and distributed sensor delays, Int. J. Gen. Syst. 43 (3–4) (2014) 387–401.

[9] Y. Zhang, Y.P. Tian, Consensus of data-sampled multi-agent systems with random communication delay and packet loss, IEEE Trans. Autom. Control 55 (4) (2010) 929–943.

[10] T. Li, M. Fu, L. Xie, J.F. Zhang, Distributed consensus with limited communication data rate, IEEE Trans. Autom. Control 56 (2) (2011) 279–291.

[11] J. Hu, Z. Wang, B. Shen, H. Gao, Quantized recursive filtering for a class of nonlinear systems with multiplicative noises and missing measurements, Int. J. Control 86 (4) (2013) 650–663.

[12] A. Kashyap, T. Basar, R. Srikant, Quantized consensus, Automatica 43 (7) (2007) 1192–1203.

[13] T. Li, L. Xie, Distributed coordination of multi-agent systems with quantized-observer based encoding-decoding, IEEE Trans. Autom. Control 57 (12) (2012) 3023–3037.

[14] H. Li, G. Chen, T. Huang, Z. Dong, W. Zhu, L. Gao, Event-triggered distributed average consensus over directed digital networks with limited communication bandwidth, IEEE Trans. Cybern. 46 (12) (2016) 3098–3110.

[15] H. Li, G. Chen, X. Liao, T. Huang, Leader-following consensus of discrete-time multiagent systems with encoding-decoding, IEEE Trans. Circuits Syst. II Express Briefs 63 (4) (2016) 401–405.

[16] F. Ceragioli, C. De Persis, P. Frasca, Discontinuities and hysteresis in quantized average consensus, Automatica 47 (9) (2011) 1916–1928.

[17] P. Frasca, Continuous-time quantized consensus: convergence of Krasovskii solutions, Syst. Control Lett. 61 (2) (2012) 273–278.

[18] D.V. Dimarogonas, K.H. Johansson, Stability analysis for multi-agent systems using the incidence matrix: quantized communication and formation control, Automatica 46 (4) (2010) 695–700.

[19] M. Guo, D.V. Dimarogonas, Consensus with quantized relative state measurements, Automatica 49 (8) (2013) 2531–2537.

[20] H. Liu, M. Cao, C. De Persis, Quantization effects on synchronized motion of teams of mobile agents with second-order dynamics, Syst. Control Lett. 61 (12) (2012) 1157–1167.

[21] C.D. Persis, B. Jayawardhana, Coordination of passive systems under quantized measurements, SIAM J. Control Optim. 50 (6) (2012) 3155–3177.

[22] Y. Xu, J. Wang, The synchronization of linear systems under quantized measurements, Syst. Control Lett. 62 (10) (2013) 972–980.

[23] Y. Zhu, Y. Zheng, L. Wang, Quantised consensus of heterogeneous multi-agent systems, IET Control Theory Appl. 9 (17) (2015) 2553–2560.

[24] Q. Hui, Hybrid consensus protocols: an impulsive dynamical system approach, Int. J. Control 83 (6) (2010) 1107–1116.

[25] Z.H. Guan, Z.W. Liu, G. Feng, M. Jian, Impulsive consensus algorithms for second-order multi-agent networks with sampled information, Automatica 48 (2012) 1397–1404.

[26] Z.W. Liu, Z.H. Guan, X.M. Shen, G. Feng, Consensus of multi-agent networks with aperiodic sampled communication via impulsive algorithms using position only measurements, IEEE Trans. Autom. Control 57 (10) (2012) 2639–2643.

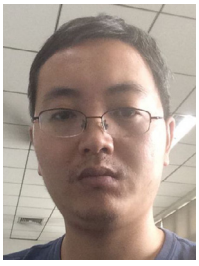
[27] H. Zhang, J. Zhou, Distributed impulsive consensus for second-order multi-agent systems with input delays, IET Control Theory Appl. 7 (16) (2012) 1978–1983.

[28] Z.H. Guan, B. Hu, M. Chi, D.X. He, X.M. Cheng, Guaranteed performance consensus in second-order multi-agent systems with hybrid impulsive control, Automatica 50 (9) (2014) 2415–2418.

[29] M. Fu, L. Xie, The sector bound approach to quantized feedback control, IEEE Trans. Autom. Control 50 (11) (2004) 1689–1711.

[30] I. Olkin, A.W. Marshall, Inequalities: Theory of Majorization and Its Applications, Academic Press, New York, 2016.

Yunru Zhu was born in Xian, Shaanxi Province. She received the BS degree in automatic from Xidian University in 2003, the MS degree from Huazhong University of Science and Technology in 2006, and PhD degree from Xidian University in 2015, respectively. Since 2006, she has been working at the School of Mechano-Electronic Engineering, Xidian University. Her current research interests are in the fields of coordination of multi-agent systems, quantized control and impulsive control.



Yuanshi Zheng was born in Jiangshan, Zhejiang Province, China on February 19, 1985. He received his bachelor, master, and doctorate degrees from Ningxia University and Xidian University in 2006, 2009, and 2012, respectively. He is currently a associate professor of Xidian University. His research interests are in the fields of coordination of multi-agent systems, consensus problems, containment control and coverage control.

Yongqiang Guan received his PhD degree in complex systems at Beihang University. He is currently a lecturer at School of Mechano-Electronic Engineering, Xidian University. His current research interests are in the fields of coordination of multiagent systems and complex networks.