Nash Equilibrium Topology of Multi-Agent Systems With Competitive Groups

Jingying Ma, Yuanshi Zheng, and Long Wang

***Abstract*—Competition is ubiquitous in nature. This paper studies competition phenomena of multi-agent systems consisting of three groups of agents. In order to achieve maximal influence, the first and the second groups send information to the third group, which leads to competition. First, we formulate this competition as a noncooperative game in which the first and the second groups are two players. Players decide agents who send and receive information. Consequently, the interaction topology of the system is generated from players' strategies. Therefore, we define the interaction topology decided by Nash equilibrium of the game as the equilibrium topology of the system. Second, the necessary condition is established for equilibrium topology. For the case that the third group's interaction graph is a tree or has a center vertex, interchangeable Nash equilibrium solutions are obtained. Moreover, due to competition, the agents of the third group might reach consensus under the equilibrium topology. Finally, when the third group's interaction graph is bidirected, the necessary and sufficient condition is given for the equilibrium topology. The equilibrium topology is also presented for the scenario where the third group's interaction graph is a bidirected circulant graph.**

***Index Terms*—Equilibrium topology, multi-agent systems, noncooperative game.**

I. Introduction

In THE last decade, distributed coordination and cooperative control of multi-agent systems (MASs) have captured tremendous attention from a wide range of academic disciplines, such as biology, engineering, social science, etc. [1]–[3]. This is mainly due to their diverse applications, such as tracking control of robotic teams [2], unmanned air vehicles (UAVs) formations [4], distributed power dispatch, and distributed energy management of smart grids [5]–[7]. Fundamental aspects of multi-agent systems pertain to consensus problem [1], [8], [9], formation control problem [4], containment control problem [10], [11], controllability analysis [12], [13], flocking [14], and so on.

In multi-agent systems, each agent is an individual who makes decision independently. When agents have the same interest, agents will cooperate with their local neighbors by sharing information. Consensus is a fundamental cooperative behavior, which means that a team of agents agree on a common goal of interest, e.g., the incremental cost in power generations, the heading of a UAV formation, or the target position of a robotic team. In [1], a simple model was introduced to study the behavior of consensus. Subsequently, some theoretical explanations were provided by using graph theory [8]. Cooperative game theory is also utilized to ensure consensus reaching [15]. Up to now, numerous results on consensus were reported, to name but a few, consensus with switching topologies [16], finite-time consensus [17], [18], optimal consensus problem [19]–[21], group consensus [22], consensus for heterogeneous multi-agent systems [23], and for switched multi-agent systems [24], [25]. In [6], a consensus protocol was proposed for smart-grid computing. By selecting the incremental cost of each generation unit as the consensus variable, Zhang and Chow [7] gave an incremental cost consensus algorithm to solve economic dispatch problem in a distributed manner. As an extension of consensus, containment control of multi-agent systems has also been widely studied recently. Some researchers considered this problem under a leader-based framework. Ji *et al.* [26] proposed a hybrid stop-go strategy to achieve containment with fixed topology. Notarstefano *et al.* [27] investigated containment control of first-order MASs with switching topologies. Other researchers investigated containment control problem by classifying the agents into boundary agents and internal agents. Liu *et al.* [10] showed that the states of internal agents converge to a convex combination of the boundary agents with weakly connected topologies.

What will happen when agents have different interests? This may produce noncooperation behaviors. For examples, price negotiation in a smart grid consisting of electric power companies and their customers and competition of two political parties, which run for election in social networks. Some methodologies, such as noncooperation game theory, signed graph theory, and optimization theory might be applied to analyze their behaviors. By virtue of game theory, Mohsenian-Rad *et al.* [28] presented a distributed demand-side energy management system for smart grids with digital communication infrastructure. Gu [29] employed differential game theory to achieve formation control. The notion of graphic game was introduced in [30]. Gharesifard

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and Cortés [31] considered a zero-sum game for two networks engaged in a strategic scenario. Clark and Poovendran [32] formulated the problem of maximizing influence in social networks with competitive ideas as a Stackelberg game. In [33], signed graph theory was employed to consider group synchronization problem for multi-agent systems with competitive interactions. Ma *et. al* investigated noncooperative behaviors of multi-agent systems with two competitive leaders in [34] and [35]. For a leader and its opponent, the authors considered the problem of minimizing the influence of the opponent. The problem was formulate as three optimization problems [34]. In [35], Ma *et al.* proposed a zero-sum game where two competitive leaders are players. Both of the two leaders want to maximize their influence on the followers. The Nash equilibrium solutions are given when the followers' interaction graph is a bidirected star graph or a circulant graph.

Inspired by the aforementioned papers, we investigate competition behavior of multi-agent systems, which consist of three groups of agents: $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$. Agents of $\mathcal{V}_1$ and $\mathcal{V}_2$ can send information to at most $m\,(\geq 1)$ agents of $\mathcal{V}_3$. Then, they can influence agents in $\mathcal{V}_3$. Agents of $\mathcal{V}_1$ and $\mathcal{V}_2$ want to exert maximum influence on $\mathcal{V}_3$, which leads to competition between them. Since agents of $\mathcal{V}_1$ ($\mathcal{V}_2$) cooperate and reach consensus, they have the same interests. Consequently, the competition between agents of $\mathcal{V}_1$ and $\mathcal{V}_2$ is actually a competition between two groups—$\mathcal{V}_1$ and $\mathcal{V}_2$. We can formulate a finite noncooperative game to analyze the competition between two groups. The main contribution of this paper is threefold. First, we develop a noncooperative game where $\mathcal{V}_1$ and $\mathcal{V}_2$ are two players and their influence power on $\mathcal{V}_3$ are payoffs. Because the players' decisions will determine the interaction topology of the system, seeking Nash equilibrium solution of the game is equivalent to choosing equilibrium topology for the system. Second, the necessary condition is established for equilibrium topology. It should be noticed that the game might have Nash equilibria, which are not interchangeable, whereas, when the graph of $\mathcal{V}_3$ is a tree or has a center vertex, exchangeable equilibrium topologies are obtained. Furthermore, agents of $\mathcal{V}_3$ might reach consensus under the equilibrium topology, which is different from the existed results of containment control. Third, the necessary and sufficient condition is obtained for the equilibrium topology when the graph of $\mathcal{V}_3$ is bidirected. Moreover, for the scenario where $\mathcal{V}_3$ is a bidirected circulant graph, we prove that all strategy pairs are interchangeable Nash equilibria. It is worth emphasizing that the current work differs from that in [35], which is extended to a general framework in this paper, mainly in the following two points. First, in this paper, competition between two groups is considered, whereas Ma *et al.* [35] focuses on the case of competition between two leaders. Second, in this paper, the interaction graph of $\mathcal{V}_3$ is a direct graph, whereas in [35], the interaction graph of followers is bidirected.

This paper is organized as follows. In Section II, we introduce some notions of graph theory and noncooperative game, and state our problem. In Sections III and IV, we present our main results and give some illustrative examples. Some conclusions are drawn in Section V.

Notation: Throughout this paper, we denote the set of real numbers by $\mathbb{R}$, the set of $n \times m$ real matrices by $\mathbb{R}^{n\times m}$. Denote by $\mathbf{1}_n$ (or $\mathbf{0}_n$) the column vector with all entries equal to one (or all zeros). I_n denotes an n-dimensional identity matrix. For a column vector $\mathbf{b} = [b_1, b_2, \ldots, b_n]^T$, $\operatorname{diag}\{\mathbf{b}\}$ is a diagonal matrix with $b_i, i = 1, \ldots, n$, on its diagonal. $\| \mathbf{b} \|_1 = \sum_{i=1}^n |b_i|$ is 1-norm of $\mathbf{b}$. Let $\mathbf{e}_i$ denote the canonical vector with a 1 in the ith entry and 0s elsewhere. A matrix is positive (resp. nonnegative) if all its entries are positive (resp. nonnegative). For a square matrix A, $\operatorname{adj} A$ and $\det A$ are the adjugate and the determinant of A, respectively. For two matrices A and B, $A \otimes B$ is Kronecker product of A and B. For two sets S_1 and S_2, denote $S_1 \times S_2$ as the Cartesian product and $S_1 \setminus S_2 = S_1 - S_2$. Let $\mathcal{I}_n = \{1, 2, \ldots, n\}$.

II. Preliminaries

A. Graph Theory

In this section, we present some basic notions of algebraic graph, which will be used in this paper. For more details, interested readers are referred to [36] for a thorough study of graph theory.

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be a directed graph consisting of a vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ and an edge set $\mathcal{E} = \{(v_j, v_i) \in \mathcal{V} \times \mathcal{V}\}$. In this paper, we assume that there are no self-loops. For an edge (v_j, v_i), v_j is called the parent vertex of v_i. A directed tree is a graph, where every vertex, except the root, has exactly one parent. A directed path in a graph $\mathcal{G}$ is a sequence $v_{i_1}, \ldots, v_{i_k}$ of vertices such that for $s = 1, \ldots, k-1$, $(v_{i_s}, v_{i_{s+1}}) \in \mathcal{E}$. A graph $\mathcal{G}$ is strongly connected if between every pair of distinct vertices v_i, v_j, there is a directed path that begins at v_i and ends at v_j. $\mathcal{A} = [a_{ij}]$ is the adjacency matrix with $a_{ij} = 1$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix is defined by $\mathcal{L} = [l_{ij}]$ with $l_{ii} = \sum_{j=1}^n a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$. It is easy to see that $\mathcal{L}\mathbf{1}_n = \mathbf{0}$.$\mathcal{G}$ is a bidirected graph, if $\mathcal{A}^T = \mathcal{A}$. For a connected bidirected graph $\mathcal{G}$, we have: 1) $\det \mathcal{L} = 0$; and 2) $\operatorname{adj} \mathcal{L} = \tau(\mathcal{G})\mathbf{1}_n\mathbf{1}_n^T$, where $\tau(\mathcal{G})$ is the number of spanning trees in graph $\mathcal{G}$ [36]. A spanning tree of $\mathcal{G}$ is a directed tree, which consists of all the vertices and some edges in $\mathcal{G}$. A vertex is called the root vertex of $\mathcal{G}$ if it is the root of a spanning tree. Suppose that graph $\mathcal{G}$ has a spanning tree. Denote by $\mathcal{V}^{(r)}$ the set of root vertices. It is easy to see that, by arranging the indices of agent, the Laplacian matrix of $\mathcal{G}$ can be written as $\mathcal{L} = \begin{pmatrix} \mathcal{L}^{(r)} & 0 \\ \mathcal{L}^{(rf)} & \mathcal{L}^{(f)} \end{pmatrix}$, where $\mathcal{L}^{(r)}$ is the Laplacian matrix of the graph induced by $\mathcal{V}^{(r)}$. A vertex v_{i^*} is called the central root vertex of $\mathcal{G}$ if it satisfies: 1) $\mathcal{V}^{(r)} = \{v_{i^*}\}$; and 2) it is the parent vertex of all other vertices.

Lemma 1: Suppose that $\mathcal{G}$ has a spanning tree. $\Delta^{(r)}$ and $\Delta^{(f)}$ are two nonnegative and nonzero diagonal matrices with same dimensions with $\mathcal{L}^{(r)}$ and $\mathcal{L}^{(f)}$, respectively. Then, the following conditions are satisfied.

1) $\mathcal{L}^{(r)} + \Delta^{(r)}$, $\mathcal{L}^{(f)}$, and $\mathcal{L}^{(f)} + \Delta^{(f)}$ are invertible.
2) $(\mathcal{L}^{(r)} + \Delta^{(r)})^{-1}$ is positive.
3) $(\mathcal{L}^{(f)})^{-1}$ and $(\mathcal{L}^{(f)} + \Delta^{(f)})^{-1}$ are nonnegative.

Proof: The proof is similar to that of [35, Lemma 2], and it is omitted. ■

B. Noncooperative Finite Games

In this section, we recall basic theoretic notions of noncooperative finite game following [37].

A two-person noncooperative finite game is denoted by $\mathbb{G} = (P, S, U)$, where $P = \{P_1, P_2\}$ is the set of players, $S = S_1 \times S_2$, S_k is the set of pure strategies of player $P_k \in P$, and $U = (u_1, u_2)$, $u_k : S \to \mathbb{R}$ is the payoff function of player $P_k \in P$. Each player makes decision to maximize its payoff function by considering the possible rational choice of the other player. A strategy pair $(s_1, s_2) \in S$ means that P_1 and P_2 independently choose strategies $s_1 \in S_1$ and $s_2 \in S_2$, respectively. For player P_1, $\hat{s}_1(s_2)$ is the best response strategy to a strategy $s_2 \in S_2$, if $u_1(\hat{s}_1(s_2), s_2) \geq u_1(s_1, s_2)$ holds for all $s_1 \in S_1$. Likewise, $\hat{s}_2(s_1)$ is the best response strategy to a strategy $s_1 \in S_1$, if $u_2(s_1, \hat{s}_2(s_1) \geq u_2(s_1, s_2)$ holds for all $s_2 \in S_2$.

Definition 1: [37] A strategy pair (s_1^*, s_2^*) is said to constitute a noncooperative (pure) Nash equilibrium solution for a two-person nonzero-sum game $\mathbb{G} = (P, S, U)$, if s_1^* is the best strategy to s_2^*, and vice versa.

A pair of strategies $(s_1, s_2) \in S$ is said to be better than another pair of strategies $(s_1', s_2') \in S$, if $u_1(s_1, s_2) \geq u_1(s_1', s_2')$ and $u_2(s_1, s_2) \geq u_2(s_1', s_2')$ hold and at least one of these inequalities is strict. A Nash equilibrium strategy pair is said to be admissible if there exists no better Nash equilibrium strategy pair. Let (s_1, s_2) and (s_1', s_2') be two Nash equilibrium solutions. (s_1, s_2) and (s_1', s_2') are said to be interchangeable, if (s_1', s_2) and (s_1, s_2') are also two Nash equilibrium solutions. When a game has at least two admissible and noninterchangeable Nash equilibrium solutions, there exist dilemmas, i.e., each player's decision for seeking the maximum payoff may lead to a lower payoff for both two players [37]. This kind of dilemmas cannot be completely avoided unless changing the mechanism of the game, e.g., some cooperation is allowed, or there is a hierarchy in decision making.

Let u_1, u_2 be payoff functions of $\mathbb{G}_1$, and w_1, w_2 be payoff functions of $\mathbb{G}_2$. Two finite two-person games $\mathbb{G}_1$ and $\mathbb{G}_2$ are said to be strategically equivalent, if 1) each player has the same strategy set, in both games; and 2) $u_i(s_1, s_2) = \alpha_i w_i(s_1, s_2) + \beta_i$, for all $(s_1, s_2) \in S_1 \times S_2$, where $\alpha_i > 0$ and $\beta_i \in \mathbb{R}$, $i \in \{1, 2\}$ are constants.

A game $\mathbb{G}$ is said a two-person zero-sum game, if $u_1(s_1, s_2) + u_2(s_1, s_2) = 0$ holds for all $(s_1, s_2) \in S$. For two-person zero-sum games, we refer to $\underline{U} = \max_{s_1 \in S_1} \min_{s_2 \in S_2} u_1(s_1, s_2)$ and $\overline{U} = \min_{s_2 \in S_2} \max_{s_1 \in S_1} u_1(s_1, s_2)$.

Lemma 2: [37] All strategically equivalent finite games have the same set of Nash equilibria. Moreover, if the sum of payoff functions is a constant, then the game is strategically equivalent to a two-person zero-sum game.

Lemma 3: [37] Suppose that a two-person zero-sum game $\mathbb{G}$ satisfies $\underline{U} = \overline{U}$. Then, the following statements hold.

1) $\mathbb{G}$ has a (pure) Nash equilibrium point.
2) The strategy pair (s_1^*, s_2^*) is a Nash equilibrium if and only if $s_1^* \in \{s^* \in S_1, \min_{s_2 \in S_2} u_1(s_1^*, s_2) = \underline{U}\}$ and $s_2^* \in \{s_2^* \in S_2, \max_{s_1 \in S_1} u_1(s_1, s_2^*) = \overline{U}\}$.
3) All Nash equilibrium solutions are interchangeable.

C. Problem Statement

Consider a multi-agent system consisting of $n_1 + n_2 + n_3$ agents. The agents are categorized into three groups: $\mathcal{V}_1 = \{v_1^{(1)}, v_2^{(1)}, \ldots, v_{n_1}^{(1)}\}$, $\mathcal{V}_2 = \{v_1^{(2)}, v_2^{(2)}, \ldots, v_{n_2}^{(2)}\}$, and $\mathcal{V}_3 = \{v_1^{(3)}, v_2^{(3)}, \ldots, v_{n_3}^{(3)}\}$. The interaction of $\mathcal{V}_1$ is modeled by a directed graph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$. Likewise, $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ and $\mathcal{G}_3 = (\mathcal{V}_3, \mathcal{E}_3)$ are the interaction graphs of $\mathcal{V}_2$ and $\mathcal{V}_3$, respectively. Denote $\mathcal{A}_1 = \{a_{ij}^{(1)}\}_{n_1 \times n_1}$, $\mathcal{A}_2 = \{a_{ij}^{(2)}\}_{n_2 \times n_2}$, and $\mathcal{A}_3 = \{a_{ij}^{(3)}\}_{n_3 \times n_3}$ as the adjacent matrices of $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$, respectively. The following assumptions are made throughout this paper.

A1: $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ have a spanning tree.

A2: There does not exist any information flow between $\mathcal{V}_1$ and $\mathcal{V}_2$, and from $\mathcal{V}_3$ to $\mathcal{V}_1$ or $\mathcal{V}_2$.

A3: There exist information flows from $\mathcal{V}_1$ and $\mathcal{V}_2$ to $\mathcal{V}_3$.

Remark 1: A1 describes the network structure of three groups.*A2* and*A3* indicate the information interaction among three groups.

According to *A3*, we define two matrices $\mathcal{B}$ and $\mathcal{D}$ to describe information flows from $\mathcal{V}_1$ and $\mathcal{V}_2$ to $\mathcal{V}_3$, respectively. Define $\mathcal{B} = \{b_{ij}\}_{n_3 \times n_1}$ where $b_{ij} = 1$ if agent $v_j^{(1)} \in \mathcal{V}_1$ sends information to agent $v_i^{(3)} \in \mathcal{V}_3$, while $b_{ij} = 0$ otherwise. Let $\mathcal{D} = \{d_{ik}\}_{n_3 \times n_2}$ where $d_{ik} = 1$ if agent $v_k^{(2)} \in \mathcal{V}_2$ sends information to agent $v_i^{(3)} \in \mathcal{V}_3$, while $d_{ik} = 0$ otherwise.

By *A1* , we know that $\mathcal{G}_3$ has a root vertex set. Without loss of generality, we assume that

A4: $\mathcal{V}_3^{(r)} = \{v_1^{(3)}, \ldots, v_{n_r}^{(3)}\}$ is the root vertex set of $\mathcal{G}_3$.

Denote $\mathcal{G}_3^{(r)}$ by the subgraph of $\mathcal{G}_3$ induced by $\mathcal{V}_3^{(r)}$. It follows that the Laplacian matrix of $\mathcal{G}_3$ is

$$\mathcal{L}_3 = \begin{pmatrix} \mathcal{L}_3^{(r)} & 0 \\ \mathcal{L}_3^{(rf)} & \mathcal{L}_3^{(f)} \end{pmatrix} \tag{1}$$

where $\mathcal{L}_3^{(r)}$ is the Laplacian matrix of $\mathcal{G}_3^{(r)}$. As a result, we have $\mathcal{B} = \binom{\mathcal{B}_1}{\mathcal{B}_2}$ and $\mathcal{D} = \binom{\mathcal{D}_1}{\mathcal{D}_2}$, where $\mathcal{B}_1 \in \mathbb{R}^{n_r \times n_1}$ and $\mathcal{D}_1 \in \mathbb{R}^{n_r \times n_2}$ indicate the information flows from $\mathcal{V}_1$ and $\mathcal{V}_2$ to $\mathcal{V}_3^{(r)}$, respectively.

It is easy to see that all agents of $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ consist of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$, $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \{\text{edges from } \mathcal{V}_1 \text{ and } \mathcal{V}_2 \text{ to } \mathcal{V}_3\}$. The Laplacian matrix of $\mathcal{G}$ can be written as

$$\begin{pmatrix} \mathcal{L}_1 & 0 & 0 & 0 \\ 0 & \mathcal{L}_2 & 0 & 0 \\ -\mathcal{B}_1 & -\mathcal{D}_1 & \mathcal{L}_3^{(r)} + K & 0 \\ -\mathcal{B}_2 & -\mathcal{D}_2 & \mathcal{L}_3^{(rf)} & \mathcal{L}_3^{(f)} + H \end{pmatrix} \tag{2}$$

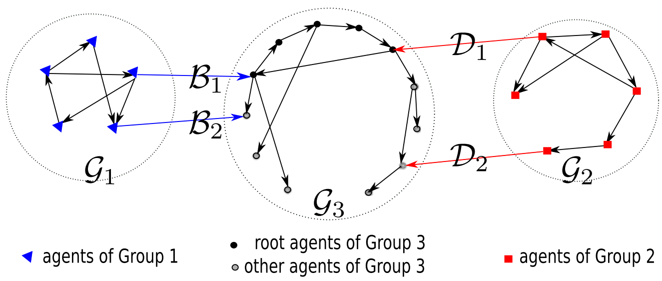


Fig. 1. Graph $\mathcal{G}$ decided by $\mathcal{B}$ and $\mathcal{D}$.

where $\mathcal{L}_i$ is the Laplacian matrix of $\mathcal{G}_i, i = 1, 2$, $K = \text{diag}\{\mathcal{B}_1 \mathbf{1}_{n_1} + \mathcal{D}_1 \mathbf{1}_{n_2}\}$, $H = \text{diag}\{\mathcal{B}_2 \mathbf{1}_{n_1} + \mathcal{D}_2 \mathbf{1}_{n_2}\}$.

Remark 2: If $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ are given, we can easily find that $\mathcal{G}$ is decided by $\mathcal{B}$ and $\mathcal{D}$ (see Fig. 1). Moreover, we have following conditions.

1) If $\mathcal{V}_1$ and $\mathcal{V}_2$ do not send information to $\mathcal{V}_3^{(r)}$, then $\mathcal{B}_1 = \mathbf{0}_{n_r \times n_1}$ and $\mathcal{D}_1 = \mathbf{0}_{n_r \times n_2}$.
2) If there exists at least one agent of $\mathcal{V}_1$ ($\mathcal{V}_2$) who sends information to $\mathcal{V}_3^{(r)}$, then $\mathcal{B}_1 \neq \mathbf{0}_{n_r \times n_1}$ ($\mathcal{D}_1 \neq \mathbf{0}_{n_r \times n_2}$).

Let $x_i(t) \in \mathbb{R}$, $y_j(t) \in \mathbb{R}$, and $z_k(t) \in \mathbb{R}$ be the states of $v_i^{(1)} \in \mathcal{V}_1, v_j^{(2)} \in \mathcal{V}_2$, and $v_k^{(3)} \in \mathcal{V}_3$, respectively. The dynamics of agents are represented by

$$\begin{cases} \dot{x}_i = u_i^{(x)}, \; i \in \mathcal{I}_{n_1} \\ \dot{y}_j = u_j^{(y)}, \; j \in \mathcal{I}_{n_2} \\ \dot{z}_k = u_k^{(z)}, \; k \in \mathcal{I}_{n_3} \end{cases} \tag{3}$$

where $u_i^{(x)}, u_j^{(y)}, u_k^{(z)} \in \mathbb{R}$ are control protocols of agents $v_i^{(1)}$, $v_j^{(2)}$, and $v_k^{(3)}$, respectively. In this paper, we propose the following control protocols:

$$\begin{cases} u_i^{(x)} = \sum_{h=1}^{n_1} a_{jh}^{(1)}(x_h - x_i), & i \in \mathcal{I}_{n_1} \\ u_j^{(y)} = \sum_{h=1}^{n_2} a_{jh}^{(2)}(y_h - y_j), & j \in \mathcal{I}_{n_2} \\ u_k^{(z)} = \sum_{h=1}^{n_3} a_{kh}^{(3)}(z_h - z_k) + \sum_{h=1}^{n_1} b_{kh}(x_h - z_k) & \\ \qquad + \sum_{h=1}^{n_2} d_{kh}(y_h - z_k), & k \in \mathcal{I}_{n_3}. \end{cases} \tag{4}$$

Define vector notations $X = [x_1, \ldots, x_{n_1}]^T$, $Y = [y_1, \ldots, y_{n_2}]^T$, and $Z = [Z_r^T, Z_f^T]^T$, where $Z_r = [z_1, \ldots, z_{n_r}]^T$ and $Z_f = [z_{n_r+1}, \ldots, z_{n_3}]^T$. By [10, Th. 1], the following results are given.

Lemma 4: Suppose that *A1–A4* hold. Then, agents of $\mathcal{V}_1$ and agents of $\mathcal{V}_2$ achieve consensus, respectively, i.e.

$$\lim_{t\to\infty} X(t) = \mathbf{1}_{n_1} f_1^T X(0) \text{ and } \lim_{t\to\infty} Y(t) = \mathbf{1}_{n_2} f_2^T Y(0)$$

where $f_1 \in \mathbb{R}^{n_1}$ and $f_2 \in \mathbb{R}^{n_2}$ are the left eigenvector of $\mathcal{L}_1$ and $\mathcal{L}_2$ corresponding to eigenvalue 0, respectively.

1) Assume that $\mathcal{B}_1 = \mathbf{0}_{n_r \times n_1}$ and $\mathcal{D}_1 = \mathbf{0}_{n_r \times n_2}$. Agents of $\mathcal{V}_3^{(r)}$ achieve consensus, i.e., $\lim_{t\to\infty} Z_r(t) = \mathbf{1}_{n_r} f_r^T Z_r(0)$, where $f_r \in \mathbb{R}^{n_r}$ is the left eigenvector of $\mathcal{L}_3^{(r)}$ corresponding to eigenvalue 0. Agents $v_{n_r+1}^{(3)}, \ldots, v_{n_3}^{(3)}$ will converge to convex combinations of $X(t), Y(t)$, and $Z_r(t)$, i.e.,

$$\lim_{t\to\infty} z_k(t) = \sum_{i=1}^{n_1} \alpha_{ki}^{(1)} x_i(t) + \sum_{j=1}^{n_2} \alpha_{kj}^{(2)} y_j(t) + \sum_{s=1}^{n_r} \beta_{ks} z_s(t)$$
$$k = n_r + 1, n_r + 2, \ldots, n_3$$

where

$$\begin{cases} \left[\alpha_{k1}^{(1)}, \ldots, \alpha_{k,n_1}^{(1)}\right] \\ \quad = \mathbf{e}_{k-n_r}^T \left(\mathcal{L}_3^{(f)} + \text{diag}\{\mathcal{B}_2 \mathbf{1}_{n_1} + \mathcal{D}_2 \mathbf{1}_{n_2}\}\right)^{-1} \mathcal{B}_2 \\ \left[\alpha_{k1}^{(2)}, \ldots, \alpha_{k,n_2}^{(2)}\right] \\ \quad = \mathbf{e}_{k-n_r}^T \left(\mathcal{L}_3^{(f)} + \text{diag}\{\mathcal{B}_2 \mathbf{1}_{n_1} + \mathcal{D}_2 \mathbf{1}_{n_2}\}\right)^{-1} \mathcal{D}_2 \\ [\beta_{k1}, \ldots, \beta_{k,n_r}] \\ \quad = \mathbf{e}_{k-n_r}^T \left(\mathcal{L}_3^{(f)} + \text{diag}\{\mathcal{B}_2 \mathbf{1}_{n_1} + \mathcal{D}_2 \mathbf{1}_{n_2}\}\right)^{-1} \mathcal{L}_3^{(rf)}. \end{cases} \tag{5}$$

2) Assume that $\mathcal{B}_1 \neq \mathbf{0}_{n_r \times n_1}$ or $\mathcal{D}_1 \neq \mathbf{0}_{n_r \times n_2}$. Then, agents of $\mathcal{V}_3$ will converge to convex combinations of $X(t)$ and $Y(t)$, i.e.,

$$\lim_{t\to\infty} z_k(t) = \sum_{j=1}^{n_1} \alpha_{ki}^{(1)} x_i(t) + \sum_{j=1}^{n_2} \alpha_{kj}^{(2)} y_j(t),$$
$$i = 1, 2, \ldots, n_3$$

where

$$\begin{cases} \left[\alpha_{k1}^{(1)}, \ldots, \alpha_{k,n_1}^{(1)}\right] = \mathbf{e}_k^T (\mathcal{L}_3 + \text{diag}\{\mathcal{B}\mathbf{1}_{n_1} \\ \qquad + \mathcal{D}\mathbf{1}_{n_2}\})^{-1} \mathcal{B} \\ \left[\alpha_{k1}^{(2)}, \ldots, \alpha_{k,n_2}^{(2)}\right] = \mathbf{e}_k^T (\mathcal{L}_3 + \text{diag}\{\mathcal{B}\mathbf{1}_{n_1} \\ \qquad + \mathcal{D}\mathbf{1}_{n_2}\})^{-1} \mathcal{D}. \end{cases} \tag{6}$$

Remark 3: From (5) and (6), we find that the convergence states of $\mathcal{V}_3$ can be written as

$$\lim_{t\to\infty} z_k(t) = \sum_{i=1}^{n_1} \alpha_{ki}^{(1)} x_i(t) + \sum_{j=1}^{n_2} \alpha_{kj}^{(2)} y_j(t) + \sum_{s=1}^{n_r} \beta_{ks} z_s(t),$$
$$i = 1, 2, \ldots, n_3. \tag{7}$$

When $\mathcal{B}_1 = \mathbf{0}_{n_r \times n_1}$ and $\mathcal{D}_1 = \mathbf{0}_{n_r \times n_2}$, we have

$$\begin{cases} \alpha_{ki}^{(1)} = 0, \alpha_{kj}^{(2)} = 0, \sum_{s=1}^{n_r} \beta_{ks} = 1, k \in \mathcal{I}_{n_r}, i \in \mathcal{I}_{n_1}, j \in \mathcal{I}_{n_2} \\ \sum_{i=1}^{n_1} \alpha_{ki}^{(1)} + \sum_{j=1}^{n_2} \alpha_{kj}^{(2)} + \sum_{s=1}^{n_r} \beta_{ks} = 1, k \in \mathcal{I}_{n_3} \backslash \mathcal{I}_{n_r} \end{cases} \tag{8}$$

whereas if $\mathcal{B}_1 \neq \mathbf{0}_{n_r \times n_1}$ or $\mathcal{D}_1 \neq \mathbf{0}_{n_r \times n_2}$, then

$$\sum_{i=1}^{n_1} \alpha_{ki}^{(1)} + \sum_{j=1}^{n_2} \alpha_{kj}^{(2)} = 1, \beta_{ks} = 0, \; k \in \mathcal{I}_{n_3}, s \in \mathcal{I}_{n_r}. \tag{9}$$

By (7), we have the convergence state of agent $v_k^{(3)}$ decided by $x_i(t)$, $y_j(t)$, and $z_k(t)$, $i \in \mathcal{I}_{n_1}, j \in \mathcal{I}_{n_2}, k \in \mathcal{I}_{n_r}$. Thus, it is easy to know that coefficients $\alpha_{ki}^{(1)}$ and $\alpha_{kj}^{(2)}$ measure how agents of $v_i^{(1)}$ and $v_j^{(2)}$ exert influence on the agent $v_k^{(3)}$, respectively. Therefore, $\sum_{k=1}^{n_3}\alpha_{ki}^{(1)}$ ($\sum_{k=1}^{n_3}\alpha_{kj}^{(2)}$) represents agent $v_i^{(1)}$'s ($v_j^{(2)}$'s) influence on $\mathcal{V}_3$. According to Lemma 4, agents of $\mathcal{V}_1$ ($\mathcal{V}_2$) will reach consensus, which implies that they have the same interest of maximizing the influence. $\sum_{i=1}^{n_1}\sum_{k=1}^{n_3}\alpha_{ki}^{(1)}$ ($\sum_{j=1}^{n_2}\sum_{k=1}^{n_3}\alpha_{kj}^{(2)}$) can be employed to measure the influence power of $\mathcal{V}_1$ ($\mathcal{V}_2$) on $\mathcal{V}_3$. Note that the greater $\sum_{i=1}^{n_1}\sum_{k=1}^{n_3}\alpha_{ki}^{(1)}$ ($\sum_{j=1}^{n_2}\sum_{k=1}^{n_3}\alpha_{kj}^{(2)}$) is, the more powerful influence of $\mathcal{V}_1$ ($\mathcal{V}_2$) is. Since an agent of $\mathcal{V}_1$ and an agent of $\mathcal{V}_2$ have different convergence states, they influence agents of $\mathcal{V}_3$ by different manners. This produces conflict between $\mathcal{V}_1$ and $\mathcal{V}_2$. Inspired by this fact, we propose the following noncooperative finite game where $\mathcal{V}_1$ and $\mathcal{V}_2$ are two competitive player.

Definition 2: For multi-agent system (3) and (4), we define the following.

1) *Players:* Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two players, i.e., $P = \{\mathcal{V}_1, \mathcal{V}_2\}$. Players make their decisions independently and simultaneously. Meanwhile, each one unilaterally seeks the maximum payoff, by also taking into account the possible rational choice of the other player.
2) *Strategies:* Each player choose at most m pairs of agents $(v_j^{(r)}, v_k^{(3)})(r = 1, 2)$ where $v_j^{(r)}$ will send information to $v_k^{(3)}$. In other words, the strategy sets of $\mathcal{V}_1$ and $\mathcal{V}_2$ are $\mathcal{S}_1 = \{\mathcal{B}\,|\,\mathbf{1}_{n_3}^T\mathcal{B}\mathbf{1}_{n_1} \le m\}$ and $\mathcal{S}_2 = \{\mathcal{D}\,|\,\mathbf{1}_{n_3}^T\mathcal{D}\mathbf{1}_{n_2} \le m\}$, respectively.
3) *Payoffs:* Payoff functions of $\mathcal{V}_1$ and $\mathcal{V}_2$ are $u_1(\mathcal{B},\mathcal{D}) = \sum_{i=1}^{n_1}\sum_{k=1}^{n_3}\alpha_{ki}^{(1)}$ and $u_2(\mathcal{B},\mathcal{D}) = \sum_{j=1}^{n_2}\sum_{k=1}^{n_3}\alpha_{kj}^{(2)}$, respectively.

We denote this game as $\mathbb{G} = (P, S, U)$, where $S = \mathcal{S}_1 \times \mathcal{S}_2$ and $U = (u_1, u_2)$.

Remark 4: In game $\mathbb{G}$, players are two groups $\mathcal{V}_1$ and $\mathcal{V}_2$. A strategy pair $(\mathcal{B},\mathcal{D})$ corresponds with an interaction graph of multi-agent system (3)–(4) (see Fig. 1). Let this graph be $\mathcal{G}_{(\mathcal{B},\mathcal{D})}$. The Laplacian matrix of $\mathcal{G}_{(\mathcal{B},\mathcal{D})}$ is written in (2).

Remark 5: This game is common in real world. Consider two political parties run for election. The decision of a voter might be influenced by his/her neighbors in social networks. Aiming to win the election, each party reach consensus among their members. Meanwhile, they choose some members to broadcast their political opinion to some voters who will exercise a great influence on others.

III. Properties of Game $\mathbb{G}$

In the following context, we will develop some properties of game $\mathbb{G}$.

Property 1: The sum of two players' payoffs is not greater than n_3.

Proof: It suffices to prove

$$u_1(\mathcal{B},\mathcal{D}) + u_2(\mathcal{B},\mathcal{D}) \le n_3 \tag{10}$$

holds for all $(\mathcal{B},\mathcal{D}) \in \mathcal{S}_1 \times \mathcal{S}_2$. Recalling Lemma 4, there are two cases that should be considered.

Case 1: If $\mathcal{B}_1 = \mathbf{0}_{n_r\times n_1}$ and $\mathcal{D}_1 = \mathbf{0}_{n_r\times n_2}$, by Lemma 4 and (8), we have

$$\sum_{k=1}^{n_3}\left(\sum_{i=1}^{n_1}\alpha_{ki}^{(1)} + \sum_{j=1}^{n_2}\alpha_{kj}^{(2)} + \sum_{s=1}^{n_r}\beta_{ks}\right) = n_3.$$

Consequently,

$$u_1(\mathcal{B},\mathcal{D}) + u_2(\mathcal{B},\mathcal{D}) = n_3 - \sum_{k=1}^{n_3}\sum_{s=1}^{n_r}\beta_{ks} \le n_3 - n_r < n_3. \tag{11}$$

Case 2: If $\mathcal{B}_1 \ne \mathbf{0}_{n_r\times n_1}$ or $\mathcal{D}_1 \ne \mathbf{0}_{n_r\times n_2}$, by Lemma 4 and (9), we have

$$u_1(\mathcal{B},\mathcal{D}) + u_2(\mathcal{B},\mathcal{D}) = n_3. \tag{12}$$

Thus, it follows from (11) and (12) that (10) holds. ■

Property 2: Suppose that $\mathcal{G}_3$ is strongly connected. Then, the following statements hold.

1) Game $\mathbb{G}$ is equivalent to a zero-sum game.
2) All Nash equilibrium solutions are interchangeable.

Proof: Since the interaction graph of $\mathcal{V}_3$ is strongly connected, we have $\mathcal{V}_3^{(r)} = \mathcal{V}_3$. Hence, we have $\mathcal{B}_1 \ne \mathbf{0}_{n_r\times n_1}$ and $\mathcal{D}_1 \ne \mathbf{0}_{n_r\times n_2}$. As a result,

$$u_1(\mathcal{B},\mathcal{D}) + u_2(\mathcal{B},\mathcal{D}) = n_3 \tag{13}$$

holds for each strategy pair $(\mathcal{B},\mathcal{D}) \in \mathcal{S}_1 \times \mathcal{S}_2$, which means that the sum of payoff functions is a constant. Therefore, it follows from Lemma 2 that game $\mathbb{G}$ is equivalent to a zero-sum game. By Lemma 3, we can obtain straightforward that all Nash equilibrium solutions are interchangeable. ■

Remark 6: By Property 2, $u_1(B,D) + u_2(B,D) < n_3$ when $\mathcal{B}_1 = \mathbf{0}_{n_r\times n_1}$ and $\mathcal{D}_1 = \mathbf{0}_{n_r\times n_2}$. It means that the sum of payoffs varies with strategy pair. Consequently, the game may not be a zero-sum game. By Property 3, we obtain that the game is equivalent to a zero-sum game when $\mathcal{G}_3$ is strongly connected. Recalling [35], the graph of followers is undirected connected and the game is equivalent to a zero-sum game. Hence, we extend the work of [35] to a general framework in this paper.

Property 3: Suppose that strategies $\mathcal{B}, \mathcal{B}' \in \mathcal{S}_1$ and $\mathcal{D}, \mathcal{D}' \in \mathcal{S}_2$.

1) If $\mathcal{B}\mathbf{1}_{n_1} = \mathcal{B}'\mathbf{1}_{n_1}$ and $\mathcal{D}\mathbf{1}_{n_2} = \mathcal{D}'\mathbf{1}_{n_2}$, then

$$u_1(\mathcal{B},\mathcal{D}) = u_1(\mathcal{B}',\mathcal{D}) \text{ and } u_2(\mathcal{B},\mathcal{D}) = u_2(\mathcal{B},\mathcal{D}'). \tag{14}$$

2) If $\mathcal{B}\mathbf{1}_{n_1} = \mathcal{D}'\mathbf{1}_{n_2}$ and $\mathcal{D}\mathbf{1}_{n_2} = \mathcal{B}'\mathbf{1}_{n_1}$, then

$$u_1(\mathcal{B},\mathcal{D}) = u_2(\mathcal{B}',\mathcal{D}'). \tag{15}$$

3) If $\mathcal{B}\mathbf{1}_{n_1} = \mathcal{D}\mathbf{1}_{n_2}$, then

$$u_1(\mathcal{B},\mathcal{D}) = u_2(\mathcal{B},\mathcal{D}). \tag{16}$$

Moreover, if $\mathcal{B}_1\mathbf{1}_{n_1} = \mathcal{D}_1\mathbf{1}_{n_2} \ne \mathbf{0}_{n_r}$, then the agents of group $\mathcal{V}_3$ will reach consensus asymptotically to

$$\frac{1}{2}\left[\lim_{t\to\infty}(x_1(t) + y_1(t))\right]. \tag{17}$$

Proof: It follows from (5) and (6) that

$$u_1(\mathcal{B},\mathcal{D}) = \begin{cases} \mathbf{1}_{n_3-n_r}^T \left(\mathcal{L}_3^{(f)} + \mathrm{diag}\{\mathcal{B}_2\mathbf{1}_{n_1} + \mathcal{D}_2\mathbf{1}_{n_2}\}\right)^{-1} \mathcal{B}_2\mathbf{1}_{n_1}, \\ \quad \text{for } \mathcal{B}_1 = \mathbf{0}_{n_r\times n_1} \text{ and } \mathcal{D}_1 = \mathbf{0}_{n_r\times n_2} \\ \mathbf{1}_{n_3}^T \left(\mathcal{L}_3 + \mathrm{diag}\{\mathcal{B}\mathbf{1}_{n_1} + \mathcal{D}\mathbf{1}_{n_2}\}\right)^{-1} \mathcal{B}\mathbf{1}_{n_1}, \\ \quad \text{for } \mathcal{B}_1 \neq \mathbf{0}_{n_r\times n_1} \text{ or } \mathcal{D}_1 \neq \mathbf{0}_{n_r\times n_2} \end{cases} \tag{18}$$

and

$$u_2(\mathcal{B},\mathcal{D}) = \begin{cases} \mathbf{1}_{n_3-n_r}^T \left(\mathcal{L}_3^{(f)} + \mathrm{diag}\{\mathcal{B}_2\mathbf{1}_{n_1} + \mathcal{D}_2\mathbf{1}_{n_2}\}\right)^{-1} \mathcal{D}_2\mathbf{1}_{n_2}, \\ \quad \text{for } \mathcal{B}_1 = \mathbf{0}_{n_r\times n_1} \text{ and } \mathcal{D}_1 = \mathbf{0}_{n_r\times n_2} \\ \mathbf{1}_{n_3}^T \left(\mathcal{L}_3 + \mathrm{diag}\{\mathcal{B}\mathbf{1}_{n_1} + \mathcal{D}\mathbf{1}_{n_2}\}\right)^{-1} \mathcal{D}\mathbf{1}_{n_1}, \\ \quad \text{for } \mathcal{B}_1 \neq \mathbf{0}_{n_r\times n_1} \text{ or } \mathcal{D}_1 \neq \mathbf{0}_{n_r\times n_2}. \end{cases} \tag{19}$$

Thus, we can straightforward obtain that (14)–(16) hold. By Lemma 4, we have $\sum_{i=1}^{n_1}\alpha_{ki}^{(1)} = \mathbf{e}_k^T(\mathcal{L}_3 + \mathrm{diag}\{\mathcal{B}\mathbf{1}_{n_1} + \mathcal{D}\mathbf{1}_{n_2}\})^{-1}\mathcal{B}\mathbf{1}_{n_1}$, $\sum_{j=1}^{n_2}\alpha_{kj}^{(2)} = \mathbf{e}_k^T(\mathcal{L}_3 + \mathrm{diag}\{\mathcal{B}\mathbf{1}_{n_1} + \mathcal{D}\mathbf{1}_{n_2}\})^{-1}\mathcal{D}\mathbf{1}_{n_2}$, $\sum_{i=1}^{n_1}\alpha_{ki}^{(1)} + \sum_{j=1}^{n_2}\alpha_{kj}^{(2)} = 1$, and

$$\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = \cdots = \lim_{t\to\infty} x_{n_1}(t)$$
$$\lim_{t\to\infty} y_1(t) = \lim_{t\to\infty} y_2(t) = \cdots = \lim_{t\to\infty} y_{n_2}(t).$$

Therefore, we know that $\sum_{i=1}^{n_1}\alpha_{ki}^{(1)} = \sum_{j=1}^{n_2}\alpha_{kj}^{(2)} = \frac{1}{2}$ if $\mathcal{B}\mathbf{1}_{n_1} = \mathcal{D}\mathbf{1}_{n_2} \neq \mathbf{0}$, which implies that (17) holds. ■

Property 4:

1) Suppose that player $\mathcal{V}_2$ do not change its strategy. Then, player $\mathcal{V}_1$ can increase its payoff by adding new edges from $\mathcal{V}_1$ to $\mathcal{V}_3$.
2) Suppose that player $\mathcal{V}_1$ do not change its strategy. Then, player $\mathcal{V}_2$ can increase its payoff by adding new edges from $\mathcal{V}_2$ to $\mathcal{V}_3$.

Proof: For a strategy $\mathcal{B} \in \mathcal{S}_1$, let $\mathcal{B}'$ be the new strategy, which is obtained by adding new edges from $\mathcal{V}_1$ to $\mathcal{V}_3$. Likewise, for a strategy $\mathcal{D} \in \mathcal{S}_2$, let $\mathcal{D}'$ be the new strategy gotten by adding new edges from $\mathcal{V}_2$ to $\mathcal{V}_3$. It follows that $\mathcal{B}' - \mathcal{B}$ and $\mathcal{D}' - \mathcal{D}$ are nonzero nonnegative matrices. It suffices to prove $u_1(\mathcal{B}',\mathcal{D}) > u_1(\mathcal{B},\mathcal{D})$ and $u_2(\mathcal{B},\mathcal{D}') > u_2(\mathcal{B},\mathcal{D})$ hold for all $\mathcal{B} \in \mathcal{S}_1, \mathcal{D} \in \mathcal{S}_2$.

Since $\mathcal{B}' - \mathcal{B}$ is a nonzero nonnegative matrix, we have $(\mathcal{B}' - \mathcal{B})\mathbf{1}_{n_1} = \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_k}$. Without loss of generality, we assume that $(\mathcal{B}' - \mathcal{B})\mathbf{1}_{n_1} = \mathbf{e}_j$. Recalling Lemma 4, we will consider three situations: 1) $\mathcal{B}_1 \neq \mathbf{0}_{n_r\times n_1}$ or $\mathcal{D}_1 \neq \mathbf{0}_{n_r\times n_2}$; 2) $\mathcal{B}_1 = \mathcal{B}_1' = \mathbf{0}_{n_r\times n_1}$ and $\mathcal{D}_1 = \mathbf{0}_{n_r\times n_2}$; and 3) $\mathcal{B}_1 = \mathbf{0}_{n_r\times n_1}$, $\mathcal{D}_1 = \mathbf{0}_{n_r\times n_2}$ and $\mathcal{B}_1' \neq \mathbf{0}_{n_r\times n_1}$.

Case 1: We assume that $\mathcal{B}_1 \neq \mathbf{0}_{n_r\times n_1}$ or $\mathcal{D}_1 \neq \mathbf{0}_{n_r\times n_2}$. It follows from (18) that $u_1(\mathcal{B}',\mathcal{D}) - u_1(\mathcal{B},\mathcal{D}) = \mathbf{1}_{n_3}^T[(Q + \mathrm{diag}\{\mathbf{e}_j\})^{-1} - Q^{-1}]\mathcal{B}\mathbf{1}_{n_1} + \mathbf{1}_{n_3}^T(Q + \mathrm{diag}\{\mathbf{e}_j\})^{-1}\mathbf{e}_j$, where $Q = \mathcal{L}_3 + \mathrm{diag}\{\mathcal{B}\mathbf{1}_{n_1} + \mathcal{D}\mathbf{1}_{n_2}\}$. By the matrix inversion lemma in [38], we obtain

$$(Q + \mathrm{diag}\{\mathbf{e}_j\})^{-1} - Q^{-1} = \frac{Q^{-1}\mathbf{e}_j\mathbf{e}_j^T Q^{-1}}{1 + \mathbf{e}_j^T Q^{-1}\mathbf{e}_j}. \tag{20}$$

It follows from Lemma 1 that Q^{-1} and $(Q + \mathrm{diag}\{\mathbf{e}_j\})^{-1}$ are nonnegative matrices. Together with (20), we can obtain that $\mathbf{1}_{n_3}^T[(Q + \mathrm{diag}\{\mathbf{e}_j\})^{-1} - Q^{-1}]\mathcal{B}\mathbf{1}_{n_1} \geq 0$ and $\mathbf{1}_{n_3}^T(Q + \mathrm{diag}\{\mathbf{e}_j\})^{-1}\mathbf{e}_j > 0$. Hence, $u_1(\mathcal{B}',\mathcal{D}) > u_1(\mathcal{B},\mathcal{D})$.

Case 2: We assume that $\mathcal{B}_1 = \mathcal{B}_1' = \mathbf{0}_{n_r\times n_1}$ and $\mathcal{D}_1 = \mathbf{0}_{n_r\times n_2}$. Similar to the proof of case 1, we can prove that $u_1(\mathcal{B}',\mathcal{D}) > u_1(\mathcal{B},\mathcal{D})$.

Case 3: Suppose that $\mathcal{B}_1 = \mathbf{0}_{n_r\times n_1}$, $\mathcal{D}_1 = \mathbf{0}_{n_r\times n_2}$, and $\mathcal{B}_1' \neq \mathbf{0}_{n_r\times n_1}$. It follows that $\mathcal{B}' = \left(\begin{smallmatrix}\mathcal{B}_1' \\ \mathcal{B}_2\end{smallmatrix}\right)$ and $\mathcal{B}_1'\mathbf{1}_{n_1} = \mathbf{p}_j$, where $\mathbf{p}_j$ is an n_r-dimensional canonical vector with 1 in the jth entry and 0s elsewhere. Therefore, we get $\mathcal{L}_3 + \mathrm{diag}\{\mathcal{B}'\mathbf{1}_{n_1} + \mathcal{D}\mathbf{1}_{n_2}\} = \left(\begin{smallmatrix} R & \mathbf{0}_{n_r\times(n_3-n_r)} \\ \mathcal{L}_3^{(rf)} & Q \end{smallmatrix}\right)$ and $(\mathcal{L}_3 + \mathrm{diag}\{\mathcal{B}'\mathbf{1}_{n_1} + \mathcal{D}\mathbf{1}_{n_2}\})^{-1} = \left(\begin{smallmatrix} R^{-1} & \mathbf{0}_{n_r\times(n_3-n_r)} \\ -Q^{-1}\mathcal{L}_3^{(rf)}R^{-1} & Q^{-1} \end{smallmatrix}\right)$, where $R = \mathcal{L}_3^{(r)} + \mathrm{diag}\{\mathbf{p}_j\}$ and $Q = \mathcal{L}_3^{(f)} + \mathrm{diag}\{\mathcal{B}_2\mathbf{1}_{n_1} + \mathcal{D}_2\mathbf{1}_{n_2}\}$. Together with (18), we have

$$u_1(\mathcal{B}',\mathcal{D}) - u_1(\mathcal{B},\mathcal{D}) = \mathbf{1}_{n_r}^T R^{-1}\mathbf{p}_j - \mathbf{1}_{n_3-n_r}^T Q^{-1}\mathcal{L}_3^{(rf)}R^{-1}\mathbf{p}_j.$$

According to Lemma 1, we know that R^{-1} is a positive matrix and Q^{-1} is a nonnegative matrix. As a result, we have $\mathbf{1}_{n_r}^T R^{-1}\mathbf{p}_j > 0$ and $-\mathbf{1}_{n_3-n_r}^T Q^{-1}\mathcal{L}_3^{(rf)}R^{-1}\mathbf{p}_j \geq 0$, which implies that $u_1(\mathcal{B}',\mathcal{D}) > u_1(\mathcal{B},\mathcal{D})$.

Similar to the above proof, we can prove that $u_2(\mathcal{B},\mathcal{D}') > u_2(\mathcal{B},\mathcal{D})$ holds for all $\mathcal{B} \in \mathcal{S}_1$. ■

By Property 4, we can straightforward obtain the following result.

Property 5: A best response strategy of game $\mathbb{G}$ always contains m agents of $\mathcal{V}_3$.

IV. Equilibrium Topologies

According to Remark 4, a strategy pair $(\mathcal{B},\mathcal{D})$ corresponds to an interaction graph $\mathcal{G}_{(\mathcal{B},\mathcal{D})}$. Therefore, the following definition is given.

Definition 3: If a strategy pair $(\mathcal{B}^*,\mathcal{D}^*)$ is a Nash equilibrium solution of game $\mathbb{G}$, then the corresponding interaction graph $\mathcal{G}_{(\mathcal{B}^*,\mathcal{D}^*)}$ is called the equilibrium topology of system (3)–(4).

A. Necessary Condition for Being an Equilibrium Topology

Theorem 1: Suppose that $\mathcal{G}_{(\mathcal{B}^*,\mathcal{D}^*)}$ is the equilibrium topology. Then, the following statements hold.

1) $\mathcal{G}_{(\mathcal{B}^*,\mathcal{D}^*)}$ contains m edges that begin at $\mathcal{V}_1$ and end at $\mathcal{V}_3$, and m edges that begin at $\mathcal{V}_2$ and end at $\mathcal{V}_3$.
2) $\mathcal{G}_{(\mathcal{B}^*,\mathcal{D}^*)}$ contains at least one edge that begins at $\mathcal{V}_1$ or $\mathcal{V}_2$ and ends at the root vertex set of $\mathcal{G}_3$.

Proof: First, we know that $\mathcal{B}^*$ is the best response of $\mathcal{D}^*$, and vice versa. By Property 5, it follows that $\mathcal{G}_{(\mathcal{B}^*,\mathcal{D}^*)}$ contains m edges that begin at $\mathcal{V}_1$ and end at $\mathcal{V}_3$, and m edges that begin at $\mathcal{V}_2$ and end at $\mathcal{V}_3$.

Second, we assume that neither $\mathcal{B}^*$ nor $\mathcal{D}^*$ contains root vertices of $\mathcal{G}_3$. According to the definition of Nash equilibrium

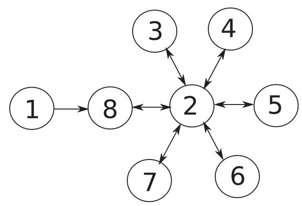


Fig. 2. Directed graph $\mathcal{G}_3$ for Example 1.

solution, we have

$$\begin{cases} u_1(\mathcal{B}^*, \mathcal{D}^*) \ge u_1(\mathcal{B}, \mathcal{D}^*), \mathcal{B} \in \mathcal{S}_1 \\ u_2(\mathcal{B}^*, \mathcal{D}^*) \ge u_2(\mathcal{B}^*, \mathcal{D}), \mathcal{D} \in \mathcal{S}_2. \end{cases} \tag{21}$$

Assume that $(\mathcal{B}', \mathcal{D}') \in S$ satisfies $\mathcal{B}'\mathbf{1}_{n_1} = \mathcal{D}^*\mathbf{1}_{n_2}$ and $\mathcal{D}'\mathbf{1}_{n_2} = \mathcal{B}^*\mathbf{1}_{n_2}$. By Property 3, it is easy to know that $(\mathcal{B}', \mathcal{D}')$ is also a Nash equilibrium solution. Therefore, we have

$$\begin{cases} u_1(\mathcal{B}', \mathcal{D}') \ge u_1(\mathcal{B}, \mathcal{D}'), \mathcal{B} \in \mathcal{S}_1 \\ u_2(\mathcal{B}', \mathcal{D}') \ge u_2(\mathcal{B}', \mathcal{D}), \mathcal{D} \in \mathcal{S}_2. \end{cases} \tag{22}$$

Assume that strategy $\hat{\mathcal{B}}$ contains a root agent of $\mathcal{G}_3$. It follows that $\hat{\mathcal{B}}_1 \ne \mathbf{0}_{n_r \times n_1}$. By Property 1, we have

$$u_1(\mathcal{B}^*, \mathcal{D}^*) + u_2(\mathcal{B}^*, \mathcal{D}^*) = n_3 - n_r < n_3 = u_1(\hat{\mathcal{B}}, \mathcal{D}^*) + u_2(\hat{\mathcal{B}}, \mathcal{D}^*).$$

Consequently, we obtain that

$$u_1(\mathcal{B}^*, \mathcal{D}^*) < u_1(\hat{\mathcal{B}}, \mathcal{D}^*) \tag{23}$$

or

$$u_2(\mathcal{B}^*, \mathcal{D}^*) < u_2(\hat{\mathcal{B}}, \mathcal{D}^*). \tag{24}$$

When inequality (23) holds, it conflicts with (21). Let $\hat{\mathcal{D}} \in \mathcal{S}_2$ satisfy $\hat{\mathcal{D}}\mathbf{1}_{n_2} = \hat{\mathcal{B}}\mathbf{1}_{n_1}$. When (24) holds, it follows that $u_1(\mathcal{B}', \mathcal{D}') < u_1(\mathcal{B}', \hat{\mathcal{D}})$. Since $u_1(\mathcal{B}', \mathcal{D}') + u_2(\mathcal{B}', \mathcal{D}') = n_3 - n_r$ and $u_1(\mathcal{B}', \hat{\mathcal{D}}) + u_2(\mathcal{B}', \hat{\mathcal{D}}) = n_3$, we have $u_2(\mathcal{B}', \mathcal{D}') < u_2(\mathcal{B}', \mathcal{D}') + n_r < u_2(\mathcal{B}', \hat{\mathcal{D}})$, which conflicts with (22). Consequently, $\mathcal{G}_{(\mathcal{B}^*, \mathcal{D}^*)}$ contains at least one edge that begins at $\mathcal{V}_1$ or $\mathcal{V}_2$ and ends at the root vertex set of $\mathcal{G}_3$. ■

Theorem 1 proposes a necessary condition for a strategy pair $(\mathcal{B}^*, \mathcal{D}^*)$ being Nash equilibrium solution.

Example 1: Suppose $\mathcal{V}_1 = \{v_1^{(1)}\}$, $\mathcal{V}_2 = \{v_1^{(2)}\}$, and $\mathcal{V}_3 = \{1, 2, 3, 4, 5, 6, 7, 8\}$. $\mathcal{G}_3$ is shown in Fig. 2. Therefore, strategy sets of $\mathcal{V}_1$ and $\mathcal{V}_2$ are $\mathcal{S}_1 = \mathcal{S}_2 = \{\mathbf{e}_i, i = 1, 2, \ldots, 8\}$, where $\mathbf{e}_i$ is the canonical vector of $\mathbb{R}^8$. By using (18) and (19), we have

$$U_1 = [u_1(\mathbf{e}_i, \mathbf{e}_j)]_{8\times 8} = \begin{pmatrix} 4 & 3.67 & 4.5 & 4.5 & 4.5 & 4.5 & 4.5 & 4.5 \\ 4.33 & 2.6 & 3 & 3 & 3 & 3 & 3 & 3.8 \\ 3.5 & 2 & 2 & 2.33 & 2.33 & 2.33 & 2.33 & 3 \\ 3.5 & 2 & 2.33 & 2 & 2.33 & 2.33 & 2.33 & 3 \\ 3.5 & 2 & 2.33 & 2.33 & 2 & 2.33 & 2.33 & 3 \\ 3.5 & 2 & 2.33 & 2.33 & 2.33 & 2 & 2.33 & 3 \\ 3.5 & 2 & 2.33 & 2.33 & 2.33 & 2.33 & 2 & 3 \\ 3.5 & 1.6 & 2 & 2 & 2 & 2 & 2 & 2.33 \end{pmatrix}$$

and $U_2 = [u_2(\mathbf{e}_i, \mathbf{e}_j)]_{8\times 8} = U_1^T$. It is easy to see that this game has two Nash equilibria $(\mathbf{e}_1, \mathbf{e}_2)$ and $(\mathbf{e}_2, \mathbf{e}_1)$. It is shown that two Nash equilibria contain vertex 1, which is the root vertex of $\mathcal{G}_3$. This result illustrates the effectiveness of theoretical results in Theorem 1. Since $u_1(\mathbf{e}_1, \mathbf{e}_2) < u_1(\mathbf{e}_2, \mathbf{e}_1)$ and $u_2(\mathbf{e}_1, \mathbf{e}_2) > u_2(\mathbf{e}_2, \mathbf{e}_1)$, we have $\mathbb{G}$ admits two admissible Nash equilibrium solutions, which are obviously not interchangeable.

Remark 7: In Example 1, it is shown that $\mathbb{G}$ has multiple admissible Nash equilibrium solutions, which are not interchangeable. Considering that there is not cooperation between players by the nature of the problem, two players make decisions independently and simultaneously. Hence, $\mathcal{V}_1$ might stick to $\mathbf{e}_2$, and $\mathcal{V}_2$ might also adopt $\mathbf{e}_2$, thus yielding an outcome of $(\mathbf{e}_2, \mathbf{e}_2)$, which is not a Nash equilibrium solution. This is indeed one of dilemmas of noncooperative nonzero-sum decision making [37]. In this example, the reason of being a dilemma is that Nash equilibrium solutions of game $\mathbb{G}$ are not interchangeable. There is really no remedy for it unless changing the mechanism of the game, for instance, players are admitted to communicate and negotiate before making decisions when facing such dilemmas.

Although Example 1 exists multiple admissible Nash equilibrium solutions that are not interchangeable, Nash equilibrium solutions might be interchangeable under some situations. In the next content, we will give some special cases where the game has interchangeable Nash equilibria.

B. Equilibrium Topologies Under Some Special Cases

1) $\mathcal{G}_3$ is a Directed Tree:

Theorem 2: Suppose that $\mathcal{G}_3$ is a directed tree and $m = 1$. Then, the following statements hold.

1) Graph $\mathcal{G}_{(\mathcal{B}, \mathcal{D})}$ is the equilibrium topology, if and only if both of two strategies contain the root vertex of $\mathcal{G}_3$.
2) All Nash equilibrium solutions are interchangeable.

Proof: Without loss of generality, we assume that $v_1^{(3)}$ is the root vertex of $\mathcal{G}_3$. Suppose that $\mathcal{B}^*$ and $\mathcal{D}^*$ contain $v_1^{(3)}$. It follows that $\mathcal{B}^*\mathbf{1}_{n_1} = \mathcal{D}^*\mathbf{1}_{n_2} = [1\ 0\ 0\ \ldots\ 0]^T$. By Property 3, we have $u_1(\mathcal{B}^*, \mathcal{D}^*) = u_2(\mathcal{B}^*, \mathcal{D}^*) = \frac{n_3}{2}$. Thus, from the definition of Nash equilibrium solution, it is suffices to prove $u_1(\mathcal{B}, \mathcal{D}^*) \le \frac{n_3}{2}$ and $u_2(\mathcal{B}^*, \mathcal{D}) \le \frac{n_3}{2}$ hold for all $\mathcal{B} \in \mathcal{S}_1$, $\mathcal{D} \in \mathcal{S}_2$. Suppose that strategy $\mathcal{B}$ contains $v_i^{(3)}$. Then, we have $\mathcal{B}\mathbf{1}_{n_1} = \mathbf{e}_i$. Let $T_i = \{v_j^{(3)} \in \mathcal{V}_3 \,|\, \text{there exists a path begins at } v_i^{(3)} \text{ and ends in } v_j^{(3)}\}$ and $|T_i| = k_i$, where $|\cdot|$ is the cardinality of a set. Since $\mathcal{G}_3$ is a tree, we have $\{\begin{smallmatrix} k_i = n_3 - 1, i = 1, \\ k_i < n_3 - 1, i \ne 1 \end{smallmatrix}$. If $i = 1$, it is obviously to see that $u_1(\mathcal{B}, \mathcal{D}^*) = \frac{k_i + 1}{2} = \frac{n_3}{2}$. For the case of $i \ne 1$, by Theorem 1 in [10], we know $\lim_{t\to\infty} z_j(t) = \{\begin{smallmatrix} \frac{1}{2}x_1(t) + \frac{1}{2}y_1(t), & j \in T_i \cup \{i\} \\ x_1(t), & \text{else} \end{smallmatrix}$ for strategy pair $(\mathcal{B}, \mathcal{D}^*)$. It follows that $u_1(\mathcal{B}, \mathcal{D}^*) = \frac{k_i + 1}{2} < \frac{n_3}{2}$. Therefore, we have $u_1(\mathcal{B}, \mathcal{D}^*) \le \frac{n_3}{2}$ holds for all $\mathcal{B} \in \mathcal{S}_1$. Similarity, it is easy to prove that $u_2(\mathcal{B}^*, \mathcal{D}) \le \frac{n_3}{2}$ holds for all $\mathcal{D} \in \mathcal{S}_2$. Thus, together with Theorem 1, we have: 1) $(\mathcal{B}, \mathcal{D})$ is a Nash equilibrium solution if both of two strategies contain $v_1^{(3)}$; and 2) $(\mathcal{B}, \mathcal{D})$ is not a Nash equilibrium solution if at least one strategy does not contain $v_1^{(3)}$. Therefore, a strategy pair is a Nash equilibrium if and only if both of the two strategies contain the root agent of $\mathcal{G}_3$.

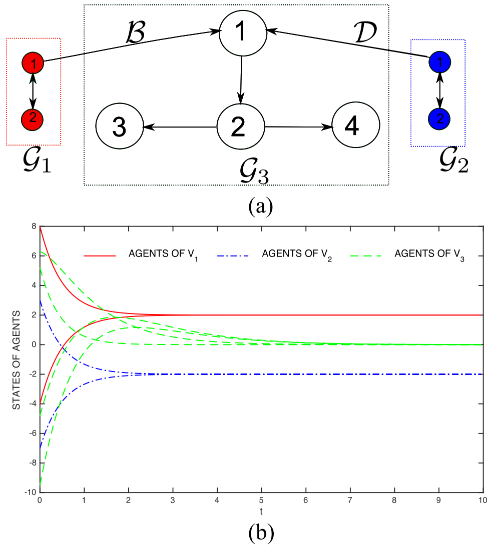


Fig. 3. (a) Interaction graph $\mathcal{G}_{(\mathcal{B}^*,\mathcal{D}^*)}$ and (b) state trajectories of all the agents for Example 2.

Let $(\mathcal{B}_1^*, \mathcal{D}_1^*)$ and $(\mathcal{B}_2^*, \mathcal{D}_2^*)$ be two Nash equilibrium solutions. We have $\mathcal{B}_1^*, \mathcal{B}_2^*, \mathcal{D}_1^*$, and $\mathcal{D}_2^*$ that contain the root agent of $\mathcal{G}_3$. Consequently, we observe that $(\mathcal{B}_1^*, \mathcal{D}_2^*)$ and $(\mathcal{B}_1^*, \mathcal{D}_2^*)$ are also two Nash equilibrium solutions, which means that $(\mathcal{B}_1^*, \mathcal{D}_1^*)$ and $(\mathcal{B}_2^*, \mathcal{D}_2^*)$ are interchangeable. ■

Corollary 1: Suppose that $\mathcal{G}_3$ is a tree and $m = 1$. Then, agents of $\mathcal{V}_3$ will achieve consensus under the equilibrium topology.

Proof: By Property 3 and Theorem 2, we can obtain this result straightforward. ■

Remark 8: A tree structural graph means that there exist a hierarchy among agents. In a hierarchical system, the most powerful agent is the root agent. As a result, each player will choose this agent to propagate its information. Therefore, the results of Theorem 2 seems intuitional.

Example 2: $\mathcal{G}_1, \mathcal{G}_2$, and $\mathcal{G}_3$ are shown in Fig. 3(a). It is obvious that $\mathcal{G}_3$ is a directed tree. Assume that $\mathcal{V}_1$ and $\mathcal{V}_2$ can only select one agent in $\mathcal{V}_3$ to connect, respectively, i.e., $m = 1$. Therefore, strategy sets of $\mathcal{V}_1$ and $\mathcal{V}_2$ are $\mathcal{S}_1 = \mathcal{S}_2 = \{\mathcal{B} = [b_{ij}]_{4\times 2}, b_{ij} \in \{0,1\}, \mathbf{1}_4^T \mathcal{B}\mathbf{1}_2 = 1\}$. We know that each player has eight strategies. Therefore, we have

$$U_1 = [u_1(\mathcal{B},\mathcal{D})]_{8\times 8} = \begin{pmatrix} 2 & 1.5 & 0.5 & 0.5 \\ 2.5 & 1 & 0.5 & 0.5 \\ 3.5 & 1.25 & 0.33 & 0.5 \\ 3.5 & 1.25 & 0.5 & 0.33 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and $U_2 = [u_2(\mathcal{B},\mathcal{D})]_{8\times 8} = U_1^T$. It is easy to see that this game has four Nash equilibrium solutions. A strategy pair $(\mathcal{B},\mathcal{D})$ is a Nash equilibrium solution, if and only if $\mathcal{B} \in \mathcal{S}^*$ and $\mathcal{D} \in \mathcal{S}^*$, where

$$\mathcal{S}^* = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

It is obvious that all Nash equilibrium solutions are interchangeable. Consider a Nash equilibrium solution

$$(\mathcal{B}^*, \mathcal{D}^*) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right).$$

The corresponding interaction graph is shown in Fig. 3(a). The state trajectories of agents are presented in Fig. 3(b), where agents of $\mathcal{V}_3$ achieve consensus. Those results illustrate the effectiveness of theoretical results in Theorem 2 and Corollary 1.

2) $\mathcal{G}_3$ has a Central Root Vertex:

Theorem 3: Suppose that $\mathcal{G}_3$ has a central root vertex and $m = 1$. Then, the following statements hold.

1) Graph $\mathcal{G}_{(\mathcal{B},\mathcal{D})}$ is the equilibrium topology if and only if both of two strategies contain the central root vertex.
2) All Nash equilibrium solutions are interchangeable.

Proof: Without loss of generality, we assume that the central root vertex is $v_1^{(3)}$. We need to prove that $(\mathcal{B}^*, \mathcal{D}^*)$ is a Nash equilibrium if and only if $\mathcal{B}^*\mathbf{1}_{n_1} = \mathcal{D}^*\mathbf{1}_{n_2} = \mathbf{e}_1$. According to the definition of Nash equilibrium, it suffices to prove $u_1(\mathcal{B}^*, \mathcal{D}^*) \geq u_1(\mathcal{B}, \mathcal{D}^*)$ and $u_2(\mathcal{B}^*, \mathcal{D}^*) \geq u_2(\mathcal{B}^*, \mathcal{D})$ hold for all $\mathcal{B} \in \mathcal{S}_1$ and $\mathcal{D} \in \mathcal{S}_2$.

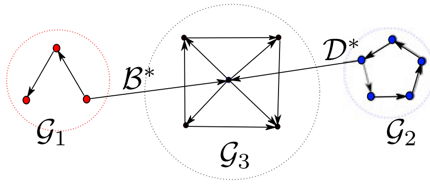
Let strategies $\mathcal{B}$ and $\mathcal{D}$ contain $v_j^{(3)}$ $(j \neq 1)$. This implies $\mathcal{B}\mathbf{1}_{n_1} = \mathcal{D}\mathbf{1}_{n_2} = \mathbf{e}_j$. Since $v_1^{(3)}$ is a central root vertex, we have $\mathcal{L}_3 = \left(\begin{smallmatrix} 0 & \mathbf{0}_{n_3-1}^T \\ -\mathbf{1}_{n_3-1} & \mathcal{L}_3^{(f)} \end{smallmatrix} \right)$. As a result, $u_1(\mathcal{B}^*, \mathcal{D}^*) = \frac{1}{2}[1 + \mathbf{1}_{n_3-1}^T (\mathcal{L}_3^{(f)})^{-1}\mathbf{1}_{n_3-1}]$ and $u_1(\mathcal{B}^*, \mathcal{D}) = 1 + \mathbf{1}_{n_3-1}^T(\mathcal{L}_3^{(f)} + \mathrm{diag}\{\mathbf{p}_{j-1}\})^{-1}\mathbf{1}_{n_3-1}$, where $\mathbf{p}_{j-1}$ is a (n_3-1)-dimensional canonical vector with 1 in the $j-1$th entry and 0s elsewhere. It follows that

$$\begin{aligned} &u_1(\mathcal{B}^*, \mathcal{D}) - u_1(\mathcal{B}^*, \mathcal{D}^*) \\ &= \frac{1}{2}\left\{ 1 + \mathbf{1}_{n_3-1}^T \left(\mathcal{L}_3^{(f)} + \mathrm{diag}\{\mathbf{p}_{j-1}\}\right)^{-1} \mathbf{1}_{n_3-1} \right. \\ &\quad \left. + \mathbf{1}_{n_3-1}^T \left[\left(\mathcal{L}_3^{(f)} + \mathrm{diag}\{\mathbf{p}_{j-1}\}\right)^{-1} - \left(\mathcal{L}_3^{(f)}\right)^{-1} \right] \mathbf{1}_{n_3-1} \right\}. \end{aligned}$$

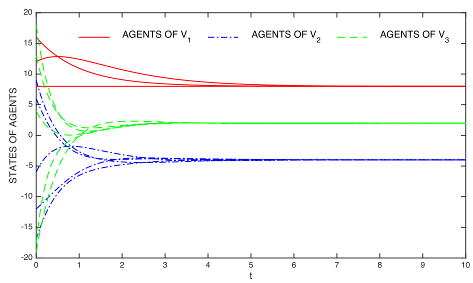
From the matrix inversion lemma [38], we obtain

$$\begin{aligned} &\left(\mathcal{L}_3^{(f)} + \mathrm{diag}\{\mathbf{p}_{j-1}\}\right)^{-1} - \left(\mathcal{L}_3^{(f)}\right)^{-1} \\ &= \frac{\left(\mathcal{L}_3^{(f)}\right)^{-1} \mathbf{p}_{j-1}\mathbf{p}_{j-1}^T \left(\mathcal{L}_3^{(f)}\right)^{-1}}{1 + \mathbf{p}_{j-1}^T \left(\mathcal{L}_3^{(f)}\right)^{-1} \mathbf{p}_{j-1}}. \end{aligned}$$

By Lemma 1, we know that $(\mathcal{L}_3^{(f)})^{-1}$ and $(\mathcal{L}_3^{(f)} + \mathrm{diag}\{\mathbf{p}_{j-1}\})^{-1}$ are nonnegative matrices. Consequently, we have $u_1(\mathcal{B}^*, \mathcal{D}) > u_1(\mathcal{B}^*, \mathcal{D}^*)$. Since $\mathcal{B}^*\mathbf{1}_{n_1} = \mathcal{D}^*\mathbf{1}_{n_2} = \mathbf{e}_1$ and $\mathcal{B}\mathbf{1}_{n_1} = \mathcal{D}\mathbf{1}_{n_2} = \mathbf{e}_j$, it follows from Properties 1 and 3 that $u_1(\mathcal{B}^*, \mathcal{D}^*) = \frac{n_3}{2}$ and $u_1(\mathcal{B}, \mathcal{D}^*) = u_2(\mathcal{B}^*, \mathcal{D}) = n_3 - u_1(\mathcal{B}^*, \mathcal{D}) < u_1(\mathcal{B}^*, \mathcal{D}^*)$. Likewise, we can prove that $u_2(\mathcal{B}^*, \mathcal{D}) < u_2(\mathcal{B}^*, \mathcal{D}^*)$. The above proof yields the following observation: 1) $(\mathcal{B}, \mathcal{D})$ is a Nash equilibrium solution if both of two strategies contain $v_1^{(3)}$; and 2) $(\mathcal{B}, \mathcal{D})$ is not a Nash equilibrium solution if at least one strategy does not contain $v_1^{(3)}$.



(a)



(b)

Fig. 4. (a) Interaction graph $\mathcal{G}_{(\mathcal{B}^*,\mathcal{D}^*)}$ and (b) state trajectories of all the agents for Example 3.

Therefore, $(\mathcal{B}^*, \mathcal{D}^*)$ is a Nash equilibrium if and only if both of two strategies contain the central root vertex.

Similar to the proof of Theorem 2, we can prove that all Nash equilibrium solutions are interchangeable. ■

Remark 9: The results in Theorem 3 is reasonable since the center root agent can send information to all other agents, which leads to its biggest influence power in $\mathcal{G}_3$. Thus, both of the two players choose this agent to spread their information.

Remark 10: It should be mentioned that there is an intersection of case 1 and case 2. For example, a directed star graph is a directed tree with center root vertex.

Corollary 2: Suppose that $\mathcal{G}_3$ has a central root vertex and $m = 1$. Then, agents of $\mathcal{V}_3$ will achieve consensus under the equilibrium topology.

Proof: By Property 3 and Theorem 3, we can obtain this result straightforward. ■

Remark 11: According to previous theoretical results of multi-agent systems, agents of $\mathcal{V}_3$ cannot reach consensus. However, by Corollaries 1 and 2, we know that agents of $\mathcal{V}_3$ might reach consensus under equilibrium topologies. The main reason might be the competition between $\mathcal{V}_1$ and $\mathcal{V}_2$.

Example 3: The interaction graphs $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ are depicted in Fig. 4(a). It is not difficult to observe that $\mathcal{G}_3$ has a central root vertex. Suppose that $\mathcal{V}_1$ and $\mathcal{V}_2$ can select one agent in $\mathcal{V}_3$ to connect, respectively, i.e., $m = 1$. It follows that $\mathcal{S}_1 = \{\mathcal{B} = [b_{ij}]_{5\times 3}, b_{ij} \in \{0,1\}, \mathbf{1}_5^T\mathcal{B}\mathbf{1}_3 = 1\}$ and $\mathcal{S}_2 = \{\mathcal{D} = [d_{ij}]_{5\times 5}, d_{ij} \in \{0,1\}, \mathbf{1}_5^T\mathcal{D}\mathbf{1}_5 = 1\}$. By computing the payoff functions of two players, we have $U_1 = [u_1(\mathcal{B},\mathcal{D})]_{15\times 25} = G \otimes (\mathbf{1}_3\mathbf{1}_5^T)$ and $U_2 = [u_2(\mathcal{B},\mathcal{D})]_{15\times 25} = G^T \otimes (\mathbf{1}_3\mathbf{1}_5^T)$, where

$$G = \begin{pmatrix} 2.5 & 4.5556 & 4.4444 & 3.9167 & 4.75 \\ 0.4444 & 0.3333 & 0.4444 & 0.4444 & 0.4167 \\ 0.5556 & 0.4815 & 0.4167 & 0.5556 & 0.5417 \\ 1.0833 & 1.0278 & 0.9444 & 0.7222 & 1.0312 \\ 0.25 & 0.25 & 0.25 & 0.25 & 0.2 \end{pmatrix}.$$

We find that all strategy pairs satisfying$\mathcal{B}\mathbf{1}_3 = \mathcal{D}\mathbf{1}_5 = \mathbf{e}_1$ are the interchangeable Nash equilibrium solutions. This is consistent with theoretical results in Theorem 3. Let us consider the Nash equivalent solution $(\mathcal{B}^*, \mathcal{D}^*)$. The corresponding interaction graph $\mathcal{G}_{(\mathcal{B}^*,\mathcal{D}^*)}$ is described in Fig. 4(a). The states of agents are presented in Fig. 4(b). We see that consensus can be achieved among agents of $\mathcal{V}_3$. This illustrates the effectiveness of the conclusion of Corollary 2.

3) $\mathcal{G}_3$ is a Bidirected Graph:

In what follows, we assume that

A5. $\mathcal{G}_3$ is a connected bidirected graph and $m = 1$.

Let $E = \{\mathbf{e}_k,\, k \in \mathcal{I}_{n_3} \,|\, \|(\mathcal{L}_3 + \mathrm{diag}\{\mathbf{e}_l\})^{-1}\mathbf{e}_k\|_1 \geq \|(\mathcal{L}_3 + \mathrm{diag}\{\mathbf{e}_k\})^{-1}\mathbf{e}_l\|_1,\, l \in \mathcal{I}_{n_3}\}$.

Theorem 4: Suppose that A5 holds. Then, graph $\mathcal{G}_{(\mathcal{B},\mathcal{D})}$ is the equilibrium topology, if and only if $\mathcal{B}\mathbf{1}_{n_1} \in E$ and $\mathcal{D}\mathbf{1}_{n_2} \in E$.

Proof: For a strategy pair $(\mathcal{B},\mathcal{D})$, denote $\mathcal{B}\mathbf{1}_{n_1} = \mathbf{e}_i$ and $\mathcal{D}\mathbf{1}_{n_2} = \mathbf{e}_j$. From (18), we obtain $u_1(\mathcal{B},\mathcal{D}) = \mathbf{1}_{n_3}^T[\mathcal{L}_3 + \mathrm{diag}\{\mathbf{e}_i + \mathbf{e}_j\}]^{-1}\mathbf{e}_i$. Let $\det[\mathcal{L}_3 + \mathrm{diag}\{\mathbf{e}_j\}] = \tau$. It follows from $\mathrm{adj}[\mathcal{L}_3 + \mathrm{diag}\{\mathbf{e}_i + \mathbf{e}_j\}]\mathbf{e}_i = \mathrm{adj}[\mathcal{L}_3 + \mathrm{diag}\{\mathbf{e}_j\}]\mathbf{e}_i$ that $u_1(\mathcal{B},\mathcal{D}) = \frac{\tau\mathbf{1}_n^T[\mathcal{L}_3+\mathrm{diag}\{\mathbf{e}_j\}]^{-1}\mathbf{e}_i}{\det[\mathcal{L}_3+\mathrm{diag}\{\mathbf{e}_i+\mathbf{e}_j\}]}$. Since $[\mathcal{L}_3 + \mathrm{diag}\{\mathbf{e}_j\}]^{-1}$ is a nonnegative matrix, we have

$$u_1(\mathcal{B},\mathcal{D}) = \frac{\tau\left\|[\mathcal{L}_3 + \mathrm{diag}\{\mathbf{e}_j\}]^{-1}\mathbf{e}_i\right\|_1}{\det[\mathcal{L}_3 + \mathrm{diag}\{\mathbf{e}_i + \mathbf{e}_j\}]}. \tag{25}$$

Likewise, we have

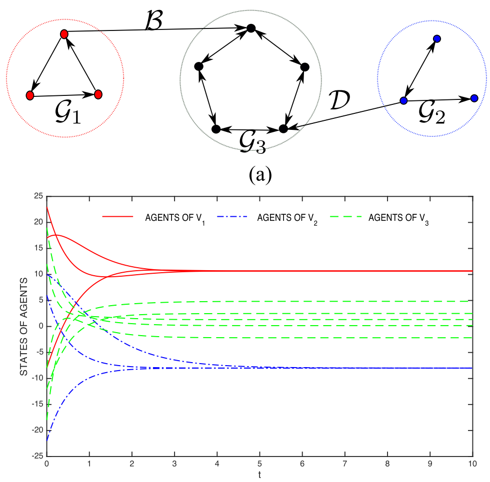
$$u_2(\mathcal{B},\mathcal{D}) = \frac{\tau\left\|[\mathcal{L}_3 + \mathrm{diag}\{\mathbf{e}_i\}]^{-1}\mathbf{e}_j\right\|_1}{\det[\mathcal{L}_3 + \mathrm{diag}\{\mathbf{e}_i + \mathbf{e}_j\}]}. \tag{26}$$

Recalling (13), it is easy to get that $\mathcal{B}\mathbf{1}_{n_1} \in E$ if and only if $\mathcal{B} \in \mathcal{S}_1^* \triangleq \{\mathcal{B} \in \mathcal{S}_1 \,|\, \min_{\mathcal{D}\in\mathcal{S}_2} u_1(\mathcal{B},\mathcal{D}) = \frac{n_3}{2}\}$,$\mathcal{D}\mathbf{1}_{n_1} \in E$ if and only if $\mathcal{D} \in \mathcal{S}_2^* \triangleq \{\mathcal{D} \in \mathcal{S}_2 \,|\, \min_{\mathcal{B}\in\mathcal{S}_1} u_2(\mathcal{B},\mathcal{D}) = \frac{n_3}{2}\}$.

By Lemma 2, game $\mathbb{G}$ is equivalent to a zero-sum game $\mathbb{G}' = (P, S, W)$ where $W = (w_1, w_2)$, $w_1(\mathcal{B},\mathcal{D}) = \frac{1}{n_3}u_1(\mathcal{B},\mathcal{D})$, and $w_2(\mathcal{B},\mathcal{D}) = -\frac{1}{n_3}w_1(\mathcal{B},\mathcal{D}) = \frac{1}{n_3}u_2(\mathcal{B},\mathcal{D}) - 1$.Therefore, it suffices to prove that $(\mathcal{B},\mathcal{D})$ is a Nash equilibrium for $\mathbb{G}'$ if and only if $\mathcal{B} \in \mathcal{S}_1^*$ and $\mathcal{D} \in \mathcal{S}_2^*$. From the definition of $\mathcal{S}_1^*$, we have $w_1(\mathcal{B},\mathcal{D}) \geq \frac{1}{2}$ for all $\mathcal{B} \in S_1^*$, $\mathcal{D} \in \mathcal{S}_2$, which implies that $\underline{U} \geq \frac{1}{2}$. Similarly, $w_1(\mathcal{B},\mathcal{D}) \leq \frac{1}{2}$ for all $\mathcal{D} \in \mathcal{S}_2^*$ and $\mathcal{B} \in \mathcal{S}_1$, thereby resulting in $\overline{U} \leq \frac{1}{2}$. On the other hand, from the definitions of $\overline{U}$ and $\underline{U}$, we know that $\overline{U} \geq \underline{U}$. Consequently, it follows that $\overline{U} = \underline{U} = \frac{1}{2}$, $\mathcal{S}_1^* = \{\mathcal{B} \in \mathcal{S}_1, \min_{\mathcal{D}\in\mathcal{S}_2} w_1(\mathcal{B}^*,\mathcal{D}) = \underline{U}\}$ and $\mathcal{S}_2^* = \{\mathcal{D} \in \mathcal{S}_2, \max_{\mathcal{B}\in\mathcal{S}_1} w_1(\mathcal{B},\mathcal{D}) = \overline{U}\}$. By Lemma 3, we know that $(\mathcal{B},\mathcal{D})$ is a Nash equilibrium for $\mathbb{G}'$ if and only if $\mathcal{B} \in \mathcal{S}_1^*$ and $\mathcal{D} \in \mathcal{S}_2^*$. Equivalently, graph $\mathcal{G}_{(\mathcal{B},\mathcal{D})}$ is a Nash equilibrium topology if and only if $\mathcal{B}\mathbf{1}_{n_1} \in E$ and $\mathcal{D}\mathbf{1}_{n_2} \in E$. ■

Theorem 5: Suppose that A5 holds and $\mathcal{G}_3$ is a circulant graph. Then, all topologies of the system are equilibrium topologies.

Proof: Let $\mathcal{B}\mathbf{1}_{n_1} = \mathbf{e}_i$ and $\mathcal{D}\mathbf{1}_{n_2} = \mathbf{e}_j$. For the case of $\mathbf{e}_i = \mathbf{e}_j$, it is obvious from Properties 1 and 2 that $u_1(\mathcal{B},\mathcal{D}) = u_2(\mathcal{B},\mathcal{D}) = \frac{n_3}{2}$. For the case of $\mathbf{e}_i \neq \mathbf{e}_j$, without loss of generality, we assume that $i < j$. Since $\mathcal{G}_3$ is a circulant graph, it follows that $\mathcal{L}_3$ is a circulant matrix. Denote $Q_i = \mathcal{L}_3 + \mathrm{diag}\{\mathbf{e}_i\}$ and



(a)

(b)

Fig. 5. (a) Interaction graph $\mathcal{G}_{(\mathcal{B},\mathcal{D})}$ and (b) state trajectories of all the agents for Example 4.

$Q_j = \mathcal{L}_3 + \mathrm{diag}\{\mathbf{e}_j\}$. For a permutation matrix

$$P = \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n_3+1-j} \\ \mathbf{0} & I_{j-i} & \mathbf{0} \end{pmatrix},$$

we have $Q_i = PQ_jP^T$.

Noticing that P is orthogonal, we obtain that $\mathbf{1}_n^T(Q_i)^{-1}\mathbf{e}_j = \mathbf{1}_n^T P(Q_j)^{-1}P^T\mathbf{e}_j$.Because $P^T\mathbf{e}_j = \mathbf{e}_i$ and $\mathbf{1}_n^T P = \mathbf{1}_n^T$, we know$\mathbf{1}_n^T(Q_i)^{-1}\mathbf{e}_j = \mathbf{1}_n^T(Q_j)^{-1}\mathbf{e}_i$. Hence, from (25) and (26), $u_1(\mathcal{B},\mathcal{D}) = u_2(\mathcal{B},\mathcal{D}) = \frac{n_3}{2}$. Consequently, we have $\mathcal{S}_1^* = \mathcal{S}_1$ and $\mathcal{S}_2^* = \mathcal{S}_2$. In other words, graph $\mathcal{G}_{(\mathcal{B},\mathcal{D})}$ is the equilibrium topology for every $(\mathcal{B},\mathcal{D}) \in \mathcal{S}_1 \times \mathcal{S}_2$. ■

Example 4: The interaction graphs $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ are depicted in Fig. 5(a) where $\mathcal{G}_3$ is a circulant graph. It follows that $\mathcal{S}_1 = \{\mathcal{B} = [b_{ij}]_{5\times 3}, b_{ij} \in \{0,1\}, \mathbf{1}_5^T\mathcal{B}\mathbf{1}_3 = 1\}$ and $\mathcal{S}_2 = \{\mathcal{D} = [d_{ij}]_{5\times 2}, d_{ij} \in \{0,1\}, \mathbf{1}_5^T\mathcal{D}\mathbf{1}_2 = 1\}$. By computing the payoff functions of two players, we have $U_1 = [u_1(\mathcal{B},\mathcal{D})]_{15\times 10} = 2.5\mathbf{1}_{15}\mathbf{1}_{10}^T$ and $U_2 = [u_2(\mathcal{B},\mathcal{D})]_{15\times 10} = U_1$. Therefore, all strategy pairs are Nash equilibria, which illustrate the effectiveness of theoretical results in Theorem 5. In particular, consider a strategy pair $(\mathcal{B},\mathcal{D})$ whose interaction graph $\mathcal{G}_{(\mathcal{B},\mathcal{D})}$ is shown in Fig. 5(a). The state trajectories of agents are shown in Fig. 5(b). By Property 3, we know that if two players choose same agents of $\mathcal{G}_3$ as their strategies, agents of $\mathcal{G}_3$ will reach consensus. In this example, two players choose different agents, which gives rise to the result of not reaching consensus among agents of $\mathcal{G}_3$.

V. Conclusion

In this paper, the agents of the multi-agent system were classified into three groups. Since the first and the second groups could influence the third group, there exists competition between them. We studied this competitive behavior in a noncooperative game theoretical framework. A two-player noncooperative game was proposed, where a strategy pair corresponds an interaction topology of the system. The necessary condition was obtained for Nash equilibrium. When the third group is a tree or has a center vertex, interchangeable equilibrium solutions were given. It was shown that the agents of third group might reach consensus under the equilibrium topology, which is different from previous theoretical results of containment control. In the future, we may consider this game for multi-agent systems with more than two competitive groups.

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