

# Bipartite Consensus in Networks of Agents With Antagonistic Interactions and Quantization

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**Abstract**—This brief deals with the consensus problem in a network of agents with cooperative and antagonistic interactions subject to quantization. By employing the techniques from non-smooth analysis, we prove that all agents can be guaranteed to asymptotically reach bipartite consensus for any logarithmic quantizer accuracy under connected and structurally balanced topology and the states of all agents asymptotically converge to zero under connected and structurally unbalanced topology. In addition, finite-time bipartite consensus is considered for single-integrator agents with binary quantized information. The simulation results are given to demonstrate the effectiveness of the theoretical results.

**Index Terms**—Cooperative and antagonistic interactions, bipartite consensus, logarithmic quantizer, finite-time consensus.

## I. INTRODUCTION

**N**ETWORKED multi-agent systems consist of a group of dynamic agents, which communicate with each other through a distributed protocol to coordinate their behavior. These systems have received significant attention due to its promising applications in areas such as formation control of satellite clusters and motion coordination of robots. Many research topics related to multi-agent coordination have arisen, such as consensus, flocking, formation, controllability [1]–[3].

Consensus, as a fundamental problem in these topics, has been extensively investigated [4]–[9]. Zheng *et al.* [6] studied asymptotical consensus of multi-agent systems, which is composed of both continuous-time and discrete-time dynamic agents. In many applications, it is often required that the desired behavior can be achieved in a finite time [10]. In [9], finite-time robust consensus was solved for nonlinear multi-agent systems by designing a discontinuous observer. Due to

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finite memories capacity and limited communication channels in practical applications, the information received by each agent is not accurate and might have been truncated or quantized. Because of the quantization, undesirable system behavior, e.g., oscillations, may happen even the same system is stable without quantization. Earlier results on quantized consensus can be traced back to the work in [11], where the notion of quantized consensus was introduced. Since the dynamics of the agents is naturally described by continuous-time systems in many applications, increasing attention has been focused on the quantized consensus in a continuous-time setting [12]–[17].

In the above literatures, it is assumed that agents achieve consensus through collaboration and the edges of graph have nonnegative weights. However, it is very easy to find many contexts where the agents may also display antagonistic interactions with some of their neighboring agents. For example, in social network the relationship between two individuals may be friendly or hostile. The interaction between genes in biological systems, where cooperation and antagonism relationship exist in the form of activators/inhibitors, is also a good example [18]. To describe such networks, signed graph are often used, where an edge with positive (negative) weights correspond to cooperative (antagonistic) interactions between agents. Altafini [19] introduced the concept of bipartite consensus for multi-agent networks with antagonistic interactions. Bipartite consensus means that all the agents can reach agreement regarding a quantity which is the same for all agents in modulus but not in sign [19]. So far, some works have been done for bipartite consensus under different contexts [20]–[26].

Although much effort has been made toward solving the bipartite consensus problem, to the best of our knowledge, quantization effects have not been taken into account in the existent results. In this brief, we focus on the bipartite consensus in network of agents involving quantized information. Distributed protocols using logarithmically quantized information are proposed for single- and double-integrator agents, respectively. By using non-smooth analysis, we prove that all agents can be guaranteed to achieve bipartite consensus under connected and structurally balanced topology and the states of all agents converge to zero under connected and structurally unbalanced topology. In addition, a distributed protocol with binary quantized information is proposed for single-integrator agents to reach bipartite consensus in finite time.

This brief is organized as follows. In Section II, some mathematical preliminaries are presented. The bipartite consensus of single-integrator agents with quantization communication

is discussed in Section III-A. In Section III-B, we consider the bipartite consensus problem for double-integrator agents. In Section IV, the simulation results are given to show the effectiveness of the obtained results. Section V is a brief conclusion.

*Notation:* Let  $R^n$  be the set of  $n$ -dimensional real vectors and  $R^{n \times m}$  be the set of  $n \times m$  real matrix.  $A^T$  denotes the transpose of matrix  $A$ .  $|\cdot|$  stands for the absolute value of a real number and  $\|\cdot\|$  denotes the 2-norm both for vectors and.  $diag\{a_1, \dots, a_n\}$  is a diagonal matrix.  $\mathcal{I}_n = \{1, 2, \dots, n\}$ .  $sign(\cdot)$  is the sign function. Let  $B(x, \delta)$  be the open ball of radius  $\delta$  centered at  $x$ ,  $\mathfrak{B}(R^n)$  be the collection of all subsets of  $R^n$ ,  $\mu(S)$  be the Lebesgue measure of  $S$ ,  $co$  be the convex hull,  $\bar{co}$  be the closure of set  $co$ . For a set  $S$ ,  $S \leq 0$  ( $S < 0$ ) means that  $v \leq 0$  ( $v < 0$ ) for all  $v \in S$ .

## II. PRELIMINARIES

Let a signed graph  $\mathcal{G}$  be a triple  $\mathcal{G}(\mathcal{A}) = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  with a set of vertices  $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ , a set of  $m$  edges  $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ , and a weighted adjacency matrix  $\mathcal{A} = (a_{ij}) \in R^{n \times n}$ . If  $(v_i, v_j) \in \mathcal{E}$ , then vertices  $v_i$  and  $v_j$  can exchange information, namely, they are adjacent. The adjacency matrix  $\mathcal{A}$  is a symmetric matrix with adjacency element  $a_{ij} \neq 0$  if  $v_i, v_j$  are adjacent, and  $a_{ij} = 0$  otherwise. A path from  $v_i$  to  $v_j$  is a sequence of distinct vertices starting with  $v_i$  and ending with  $v_j$  such that any two consecutive vertices are adjacent. For  $v_i = v_j$ , the path is a cycle. A cycle is positive if it contains an even number of negative edge weights. It is negative otherwise. We say a signed graph  $\mathcal{G}$  is connected if between any two distinct vertices, there is a path between them. A signed graph  $\mathcal{G}$  is said structurally balanced if there exists a bipartition  $\mathcal{V}_1, \mathcal{V}_2$  of vertices, where  $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$  and  $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ , such that  $a_{ij} \geq 0$  for  $\forall v_i, v_j \in \mathcal{V}_l$ ,  $(l \in \{1, 2\})$  and  $a_{ij} \leq 0$  for  $\forall v_i \in \mathcal{V}_l, v_j \in \mathcal{V}_q, l \neq q$ ,  $(l, q \in \{1, 2\})$ . It is said structurally unbalanced otherwise. The Laplacian matrix of signed graph  $\mathcal{G}$  is defined as  $L = (l_{ij}) \in R^{n \times n}$  with the elements  $l_{ii} = \sum_{j=1, j \neq i}^N |a_{ij}|$  and  $l_{ij} = -a_{ij}$  if  $i \neq j$ .

*Lemma 1* [19]: A connected signed graph  $\mathcal{G}(\mathcal{A})$  is structurally balanced if and only if  $\exists D = diag\{\sigma_1, \sigma_2, \dots, \sigma_n\}$  such that  $DAD$  has all nonnegative entries, where  $\sigma_i \in \{1, -1\}$ ,  $i \in \mathcal{I}_n$ . A connected signed graph  $\mathcal{G}(\mathcal{A})$  is structurally unbalanced if and only if one or more circles of  $\mathcal{G}(\mathcal{A})$  are negative.

In this brief, we consider the scenario where the data is acquired through digital communication. Logarithmic quantizer is a type of quantizers used frequently. Define the set of quantization levels as [27]

$$U = \{\pm u_i, u_i = \left(\frac{1-\delta}{1+\delta}\right)^i u_0, i = \pm 1, \pm 2, \dots\} \cup \{\pm u_0\} \cup \{0\},$$

where  $u_0 > 0$  and accuracy parameter  $0 < \delta < 1$ .

A logarithmic quantizer  $q : R \rightarrow R$  is a map defined as:

$$q(a) = \begin{cases} u_i & \text{if } \frac{1}{1+\delta} u_i < a \leq \frac{1}{1-\delta} u_i, a > 0, \\ 0 & \text{if } a = 0, \\ -q(-a) & \text{if } a < 0, \end{cases}$$

and for a vector  $x = [x_1, \dots, x_n]^T \in R^n$ ,  $q(x) = [q(x_1), \dots, q(x_n)]^T$ . From the definition, it is known that the logarithmic quantizer has a countable number of levels. It is

capable of adjusting the size of the quantization step according to the input value.

Because of the discontinuity of the quantized signals, the solutions in this brief are understood in Filippov sense. Given the vector differential equation

$$\dot{x}(t) = f(x(t)), \quad (1)$$

where  $x \in R^n, f : R^n \rightarrow R^n$  is measurable and locally essentially bounded. A Filippov solution of the equation (1) on  $[t_0, t_1]$  is defined to be an absolutely continuous function  $x : [t_0, t_1] \rightarrow R^n$  such that

$$\dot{x} \in F[f](x), \quad (2)$$

where the Filippov set-valued map  $F[f](x) = \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \bar{co}\{f(B(x, \delta)) \setminus S\}$ .

Let  $V : R^n \rightarrow R$  be a locally Lipschitz continuous and regular function and  $\Omega_V$  be the set of measure zero where the gradient of  $V$  with respect to  $x$  is not defined. The generalized gradient of  $V$  at  $x$  is defined by  $\partial V(x) = co\{\lim_{i \rightarrow \infty} \nabla V(x_i) | x_i \rightarrow x, x_i \notin \Omega_V\}$ . The set-valued Lie derivative  $\dot{V} : R^n \rightarrow \mathfrak{B}(R^n)$  of  $V$  with respect to  $f$  at  $x$  is defined as  $\dot{V}(x) = \{a \in R : \text{there exists } w \in F[f](x) \text{ such that } p^T w = a \text{ for all } p \in \partial V(x)\}$ .

*Lemma 2* [7]: Let  $V : R^n \rightarrow R$  be a locally Lipschitz continuous and regular function. Let  $x_0 \in S \subset R^d$ , with  $S$  compact and strongly invariant for (2). Assume that for all  $x \in S$  either  $\max \dot{V} \leq 0$  or  $\dot{V} = \emptyset$ . Let  $Z = \{x \in R^n : 0 \in \dot{V}(x)\}$ . Then, any Filippov solution of (1) starting at  $x_0 \in S$  converges to the largest weakly invariant set  $M$  contained in  $\bar{Z} \cap S$ .

*Lemma 3* [7]: Under the same assumptions of Lemma 2, further assume that there exists a neighborhood  $U$  of  $Z \cap S$  in  $S$  such that  $\max \dot{V} \leq -\varepsilon < 0$  almost everywhere on  $U \setminus (Z \cap S)$ . Then, any Filippov solution of (1) starting at  $x_0 \in S$  reaches  $Z \cap S$  in finite time.

## III. MAIN RESULTS

### A. Bipartite Consensus of Single-Integrator Agents

In this section, we consider bipartite consensus of single-integrator agents:

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{I}_n, \quad (3)$$

where  $x_i, u_i \in R$  are the position state and control input of agent  $i$ , respectively. We present a distributed protocol as follows:

$$u_i = -k \sum_{j=1}^n |a_{ij}| q(x_i - sign(a_{ij})x_j), \quad (4)$$

where  $k > 0$  is the control gain.

From the definition of the Filippov set-valued map, we have for  $a \geq 0 \in R$ ,  $F[q](a) = q(a)$  if  $q(\cdot)$  is continuous on  $a$ , and  $F[q](a) = [(1-\delta)a, (1+\delta)a]$  otherwise. Moreover,  $F[q](-a) = -F[q](a)$  for  $a \geq 0$ . Thus, we have that  $aF[q](a) \geq 0$  and the equality holds only when  $a = 0$ .

*Definition 1:* System (3) is said to reach bipartite consensus if for any initial conditions, we have  $\lim_{t \rightarrow \infty} |x_i(t)| = \lim_{t \rightarrow \infty} |x_j(t)|$ ,  $\forall i, j \in \mathcal{I}_n$ .

**Theorem 1:** Consider system (3) with protocol (4) over a signed graph  $\mathcal{G}$ . If  $\mathcal{G}$  is connected and structurally balanced, the states of all agents can reach bipartite consensus asymptotically. If  $\mathcal{G}$  is connected and structurally unbalanced, the states of all agents converge to zero asymptotically.

**Proof:** Take  $V(x) = \frac{1}{2} \sum_{i=1}^n x_i^2$  and it has

$$\dot{\tilde{V}} = \sum_{i=1}^n x_i F[\dot{x}_i] \subseteq -k \sum_{i=1}^n \sum_{j=1}^n x_i |a_{ij}| F[q](x_i - \text{sign}(a_{ij})x_j).$$

Note that

$$\begin{aligned} & \sum_{i=1}^n \sum_{j=1}^n x_i |a_{ij}| F[q](x_i - \text{sign}(a_{ij})x_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n x_i a_{ij} \text{sign}(a_{ij}) F[q](x_i - \text{sign}(a_{ij})x_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n x_i a_{ij} F[q](\text{sign}(a_{ij})x_i - x_j) \\ &= \sum_{j=1}^n \sum_{i=1}^n x_j a_{ji} F[q](\text{sign}(a_{ji})x_j - x_i) \\ &= - \sum_{i=1}^n \sum_{j=1}^n x_j a_{ij} F[q](x_i - \text{sign}(a_{ij})x_j). \end{aligned}$$

Thus, we obtain that

$$\begin{aligned} \dot{\tilde{V}}(x) &\leq -\frac{k}{2} \sum_{i=1}^n \sum_{j=1}^n a_{ij} (\text{sign}(a_{ij})x_i - x_j) F[q](x_i - \text{sign}(a_{ij})x_j) \\ &= -\frac{k}{2} \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| (x_i - \text{sign}(a_{ij})x_j) F[q](x_i - \text{sign}(a_{ij})x_j). \end{aligned}$$

Note that  $aF[q](a) \geq 0$  for  $\forall a \in R$  and the equality holds when  $a = 0$ . Thus,  $\dot{\tilde{V}}(x) \leq 0$ , where the equality holds only when  $x_i = \text{sign}(a_{ij})x_j$  for  $\forall a_{ij} \neq 0$ . If  $\mathcal{G}$  is connected and structurally balanced, according to Lemma 1, there exists a diagonal matrix  $D$  such that  $DAD$  is nonnegative. Hence, it follows that  $\sigma_i \sigma_j a_{ij} = |a_{ij}|$  which yields that  $\sigma_i \sigma_j = \text{sign}(a_{ij})$  for  $\forall a_{ij} \neq 0$ . Since  $\sigma_i^2 = 1$ , then  $x_i = \text{sign}(a_{ij})x_j$  for  $\forall a_{ij} \neq 0$  implies that  $\sigma_i x_i = \sigma_j x_j$  for  $\forall i, j \in \mathcal{I}_n$ . Thus, the bipartite consensus is achieved according to Lemma 2. We can further calculate the asymptotic states for all the agents. Let  $c(t) = \frac{1}{n} \sum_{i=1}^n \sigma_i x_i(t)$  and it is easy to verify that  $\sum_{i=1}^n \sigma_i \dot{x}_i(t) = 0$ . Thus,  $\dot{c}(t) \in \frac{1}{n} F[\sum_{i=1}^n \sigma_i \dot{x}_i] = \{0\}$ . Therefore,  $c(t)$  is time invariant, i.e.,  $c(t) = c(0)$ . Thus, we have that  $\lim_{t \rightarrow +\infty} |x_i(t)| = |c(0)|$ ,  $\forall i \in \mathcal{I}_n$ .

If  $\mathcal{G}$  is connected and structurally unbalanced, according to Lemma 1, one or more circles of  $\mathcal{G}(\mathcal{A})$  are negative. Thus,  $x_i = \text{sign}(a_{ij})x_j$ ,  $\forall a_{ij} \neq 0$  implies that  $x_i = 0$ ,  $\forall i \in \mathcal{I}_n$ . From Lemma 2, we have that  $\lim_{t \rightarrow \infty} x_i(t) = 0$ ,  $\forall i \in \mathcal{I}_n$ . This completes the proof. ■

In practice, it is often required that the consensus can be reached in a finite time. Thus, we consider a distributed protocol using binary quantized information as follows:

$$u_i = -k \sum_{j=1}^n |a_{ij}| \text{sign}(x_i - \text{sign}(a_{ij})x_j), \quad (5)$$

where  $k > 0$  is the control gain. Because of the discontinuity of the sign function, we consider the solution of differential

inclusion  $\dot{x}(t) \in -kF(g(x))$ , where  $g(x)$  denote the right-hand side of (5).

**Theorem 2:** Consider system (3) with protocol (5) over a signed graph  $\mathcal{G}$ . If  $\mathcal{G}$  is connected and structurally balanced, the states of all agents can reach bipartite consensus in finite time. If  $\mathcal{G}$  is connected and structurally unbalanced, the states of all agents converge to zero in finite time.

**Proof:** Consider the Lyapunov function

$$V(x) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| |x_i - \text{sign}(a_{ij})x_j|.$$

It is easy to verify that  $V(x)$  is single-valued, locally Lipschitz and regular. Hence, one has  $\partial V(x) \subseteq [(\partial V)_1, (\partial V)_2, \dots, (\partial V)_n]^T$  with  $(\partial V)_i = \sum_{j=1}^n |a_{ij}| SGN(x_i - \text{sign}(a_{ij})x_j)$ , where  $SGN(s)$  is defined as  $SGN(s) = [-1, 1]$  if  $s = 0$ , otherwise  $SGN(s) = \text{sign}(s)$ .

Now we consider the following two cases. The first case is that  $x_i \neq \text{sign}(a_{ij})x_j$ , for  $\forall a_{ij} \neq 0$ . In this case,  $\dot{\tilde{V}}(x) = -k \|\nabla V(x)\|^2 < 0$ , where  $\nabla V(x) = \{\sum_{j=1}^n |a_{1j}| \text{sign}(x_1 - \text{sign}(a_{1j})x_j), \dots, \sum_{j=1}^n |a_{nj}| \text{sign}(x_n - \text{sign}(a_{nj})x_j)\}^T$ . The second case is that  $\exists a_{ij} \neq 0$  such that  $x_i = \text{sign}(a_{ij})x_j$ . Suppose  $\dot{\tilde{V}}(x) \neq \emptyset$  and take  $a \in \dot{\tilde{V}}(x)$ . Then, by the definition of set-valued Lie derivative, there exists  $\omega \in -kF(g(x))$  such that  $a = p^T \omega$  for all  $p \in \partial V(x)$ . Since  $F(g(x)) = \partial V(x)$ , choose  $p = -\frac{1}{k} \omega$ . Thus, we have  $a = -\frac{1}{k} \|\omega\|^2 \leq 0$  and  $a = 0$  only if  $\omega = 0$ , that is  $0 \in \partial V(x)$ . Hence, if  $\dot{\tilde{V}}(x) \neq \emptyset$ , then  $\dot{\tilde{V}}(x) \leq 0$  and the equality holds only when  $0 \in \partial V(x)$ .

Note that  $0 \in \partial V(x)$  implies that  $0 \in (\partial V)_i$ ,  $\forall i \in \mathcal{I}_n$ . It follows that  $0 \in \sum_{i=1}^n x_i (\partial V)_i = \sum_{i=1}^n x_i \sum_{j=1}^n |a_{ij}| SGN(x_i - \text{sign}(a_{ij})x_j) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| (x_i - \text{sign}(a_{ij})x_j) SGN(x_i - \text{sign}(a_{ij})x_j) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| |x_i - \text{sign}(a_{ij})x_j|$ . Hence, one has  $x_i = \text{sign}(a_{ij})x_j$  for  $\forall a_{ij} \neq 0$ . Similar to the deduce in Theorem 1, it implies that  $\sigma_i x_i = \sigma_j x_j$  for  $\forall i, j \in \mathcal{I}_n$  under a connected and structurally balanced graph and  $x_i = 0$ ,  $\forall i \in \mathcal{I}_n$  under a connected and structurally unbalanced graph.

Note that if  $x_i \neq \text{sign}(a_{ij})x_j$  for  $\forall a_{ij} \neq 0$ ,  $\nabla V(x)$  takes a finite number of values over  $R^n$ . Hence, if  $\exists a_{ij} \neq 0$  such that  $x_i = \text{sign}(a_{ij})x_j$ , the set-valued map  $\partial V(x)$  admits a finite number of set-values, denoted by  $\mathcal{V}_1, \dots, \mathcal{V}_m$ . Since  $\mathcal{V}_i$ ,  $i \in \mathcal{I}_m$  is compact, there exists  $\alpha = k \min\{\min\{\|v\|^2, v \in \mathcal{V}_i\}, i \in \mathcal{I}_m\} > 0$  such that  $\dot{\tilde{V}}(x) \leq -\alpha$ . Thus, we can complete the proof by using Lemma 3. ■

### B. Bipartite Consensus of Double-Integrator Agents

In this section, we consider bipartite consensus of double-integrator agents:

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = u_i(t), \end{cases} \quad i \in \mathcal{I}_n, \quad (6)$$

where  $x_i, v_i \in R$  are the position and velocity states of agent  $i$ , respectively, and  $u_i \in R$  is the input of agent  $i$ . We adopt the following protocol:

$$u_i(t) = -k_1 \sum_{j=1}^n |a_{ij}| q(x_i - \text{sign}(a_{ij})x_j) - k_2 q(v_i), \quad i \in \mathcal{I}_n, \quad (7)$$

where  $k_1, k_2 > 0$  are the control gains.

**Definition 2:** System (6) is said to reach bipartite consensus if for any initial conditions, we have  $\lim_{t \rightarrow \infty} |x_i(t)| = \lim_{t \rightarrow \infty} |x_j(t)|$  and  $\lim_{t \rightarrow \infty} v_i(t) = 0, \forall i, j \in \mathcal{I}_n$ .

**Theorem 3:** Consider system (6) with protocol (7) over a signed graph  $\mathcal{G}$ . If  $\mathcal{G}$  is connected and structurally balanced, the states of all agents can reach bipartite consensus asymptotically. If  $\mathcal{G}$  is connected and structurally unbalanced, the states of all agents converge to zero asymptotically.

**Proof:** Let  $\hat{x}$  and  $v$  respectively denote the column vector formed by all  $\hat{x}_{ij} = x_i - \text{sign}(a_{ij})x_j$  and all  $v_i, i, j \in \mathcal{I}_n$ . Take a Lyapunov function as

$$V(\hat{x}, v) = \frac{1}{2}k_1 \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| \int_0^{\hat{x}_{ij}} q(s)ds + \frac{1}{2} \sum_{i=1}^n v_i^2.$$

The function  $V(\hat{x}, v)$  is regular since  $\int_0^x q(s)ds$  is regular everywhere. Let  $\partial V(\hat{x}, v) = [\frac{\partial V(\hat{x}, v)}{\partial \hat{x}_{12}}, \frac{\partial V(\hat{x}, v)}{\partial \hat{x}_{13}}, \dots, \frac{\partial V(\hat{x}, v)}{\partial \hat{x}_{n,n-1}}, v_1, v_2, \dots, v_n]^T$ . Thus,

$$\begin{aligned} \dot{V}(\hat{x}, v) &\leq \bigcap_{\zeta_{ij} \in \partial V(\hat{x}, v)} \zeta_{ij}^T \left[ \dot{\hat{x}}_{12}, \dot{\hat{x}}_{13}, \dots, \dot{\hat{x}}_{n,n-1}, F[\dot{v}_1], F[\dot{v}_2], \dots, F[\dot{v}_n] \right]^T \\ &\subseteq \bigcap_{\zeta_{ij} \in [q(\hat{x}_{ij}^-), q(\hat{x}_{ij}^+)]} \left( \frac{1}{2}k_1 \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| \zeta_{ij} (v_i - \text{sign}(a_{ij})v_j) \right. \\ &\quad \left. + \sum_{i=1}^n v_i F[\dot{v}_i] \right) \\ &\subseteq \bigcap_{\zeta_{ij} \in [q(\hat{x}_{ij}^-), q(\hat{x}_{ij}^+)]} \left( \frac{1}{2}k_1 \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| \zeta_{ij} (v_i - \text{sign}(a_{ij})v_j) \right. \\ &\quad \left. - k_1 \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| v_i F[q](\hat{x}_{ij}) - k_2 \sum_{i=1}^n v_i F[q](v_i) \right), \end{aligned}$$

where  $q(\hat{x}_{ij}^+) = \lim_{x \rightarrow \hat{x}_{ij}^+} q(x)$  and  $q(\hat{x}_{ij}^-) = \lim_{x \rightarrow \hat{x}_{ij}^-} q(x)$ .

Similar to the deduce in Theorem 2, we have

$$\sum_{i=1}^n \sum_{j=1}^n |a_{ij}| v_i F[q](\hat{x}_{ij}) = - \sum_{i=1}^n \sum_{j=1}^n a_{ij} v_j F[q](\hat{x}_{ij}).$$

Thus, it is obtained that

$$\begin{aligned} \dot{V}(\hat{x}, v) &\subseteq \bigcap_{\zeta_{ij} \in [q(\hat{x}_{ij}^-), q(\hat{x}_{ij}^+)]} \left( \frac{1}{2}k_1 \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| \zeta_{ij} (v_i - \text{sign}(a_{ij})v_j) \right. \\ &\quad \left. - \frac{1}{2}k_1 \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| (v_i - \text{sign}(a_{ij})v_j) [q(\hat{x}_{ij}^-), q(\hat{x}_{ij}^+)] \right) \\ &\quad - k_2 \sum_{i=1}^n v_i F[q](v_i) \\ &= \bigcap_{\zeta_{ij} \in [q(\hat{x}_{ij}^-), q(\hat{x}_{ij}^+)]} \left( \frac{1}{2}k_1 \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| (v_i - \text{sign}(a_{ij})v_j) \right. \\ &\quad \left. [\zeta_{ij} - q(\hat{x}_{ij}^+), \zeta_{ij} - q(\hat{x}_{ij}^-)] \right) - k_2 \sum_{i=1}^n v_i F[q](v_i) \\ &= -k_2 \sum_{i=1}^n v_i F[q](v_i), \end{aligned}$$

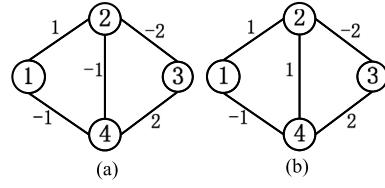


Fig. 1. Connected signed graphs. (a) Structurally balanced graph. (b) Structurally unbalanced graph.

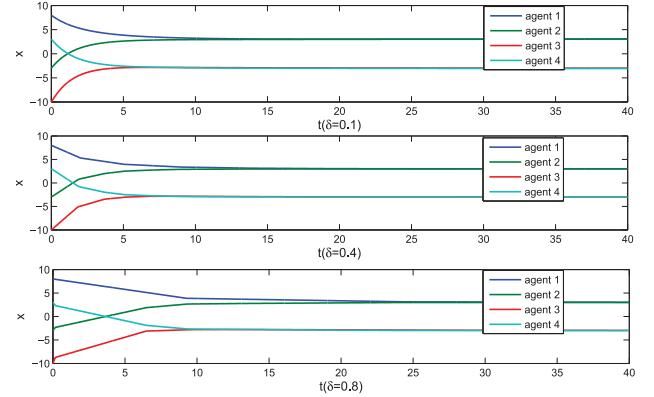


Fig. 2. State trajectories of single-integrator agents with protocol (4) under Fig. 1(a).

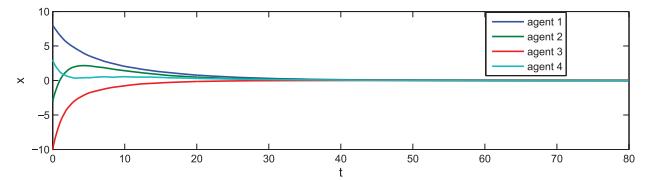


Fig. 3. State trajectories of single-integrator agents with protocol (4) under Fig. 1(b).

where the last equation follows from the fact that  $\bigcap_{\zeta_{ij} \in [q(\hat{x}_{ij}^-), q(\hat{x}_{ij}^+)]} [\zeta_{ij} - q(\hat{x}_{ij}^+), \zeta_{ij} - q(\hat{x}_{ij}^-)] = \{0\}$ .

Since  $v_i F[q](v_i) \geq 0$ , we have  $\dot{V}(\hat{x}, v) \leq 0$  and the equality holds only when  $v_i = 0, i \in \mathcal{I}_n$ . From (7), it follows that  $0 \in \sum_{j=1}^n |a_{ij}| F[q](\hat{x}_{ij}), i \in \mathcal{I}_n$ , which implies that  $x_i = \text{sign}(a_{ij})x_j$  for  $\forall a_{ij} \neq 0$ . Thus, we can follow the argument of the last part of the proof of Theorem 1 to complete the proof. ■

#### IV. NUMERICAL SIMULATION

In this section, we present computer simulations to demonstrate the effectiveness of the theoretical results.

**Example 1:** Consider a network of agents with connected communication topology shown in Fig. 1. For single-integrator agents, the initial condition is  $x(0) = (8, -3, -10, 3)^T$ . Then it follows from Theorem 1 that the agents can achieve the bipartite consensus under connected and structurally balanced topology and the consensus value of agents  $|c(0)| = 3$ . Fig. 2 shows the simulation results of protocol (4) with different values of accuracy parameter  $\delta$ . We can see that the bipartite consensus is solved asymptotically and the bigger  $\delta$  is, the more time it takes for the system to converge. Fig. 3 shows the state trajectories of agents under structurally unbalanced

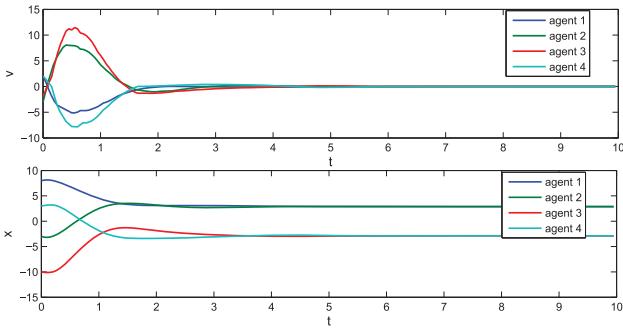


Fig. 4. State trajectories of double-integrator agents with protocol (7) under Fig. 1(a).

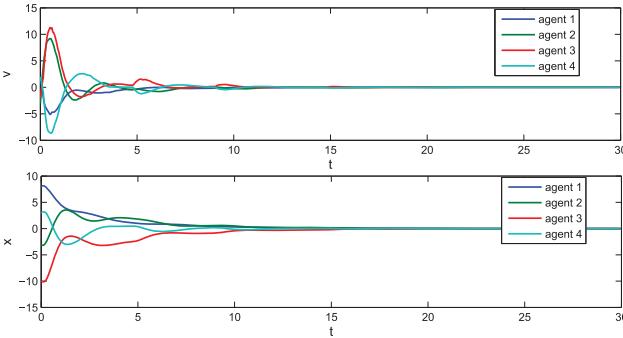


Fig. 5. State trajectories of double-integrator agents with protocol (7) under Fig. 1(b).

graph. Clearly, this figure illustrates that the states of all agents converge to zero. For double-integrator systems, the initial condition is  $x(0) = (8, -3, -10, 3)^T$ ,  $v(0) = (2, -3, -2, 1)^T$ . The simulation results shown in Fig. 4 and Fig. 5 verify the effectiveness of Theorem 3.

## V. CONCLUSION

In this brief, we consider the quantized bipartite consensus problem in networks of agents with antagonistic interactions. It has been shown that the bipartite consensus can be achieved for any quantizer accuracy under connected and structurally balanced topology. It has also been pointed out the states of all agents converge to zero under connected and structurally unbalanced topology. The future work will focus on quantized bipartite consensus via event-triggered control.

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