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Containment control of switched multi-agent systems

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In this paper, we consider the containment control problem for the switched multi-agent system which is composed of continuous-time and discrete-time subsystems. Continuous-time protocol based on the relative state measurements of agents are designed for the switched multi-agent system with multiple stationary and dynamic leaders, respectively. By using graph theory and matrix theory, some necessary and sufficient conditions are obtained for solving the containment control problem under arbitrary switching. When the leaders are dynamic, impulsive protocol are also proposed for the switched multi-agent system. The simulation results are given to verify the effectiveness of the theoretical results.

Keywords: containment control; switched multi-agent systems; continuous-time; discrete-time; directed networks

1. Introduction

In recent years, distributed cooperative control of multi-agent systems has attracted a great deal of attention due to its broad applications in vehicle systems, sensor networks, social networks, and so on. To date, many research topics about multi-agent systems have arisen, to name but a few, consensus problem (Guan, Ji, Zhang, & Wang, 2013, 2014; Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Xiao & Wang, 2008; Zheng, Zhu, & Wang, 2011), controllability analysis (Ji, Lin, & Yu, 2012; Liu, Xie, et al., 2012, 2014), optimal control (Hengster-Movric & Lewis, 2014; Ma, Zheng, & Wang, 2014; Ma, Zheng, Wu, & Wang, 2014), rendezvous control (Xiao, Wang, & Chen, 2012).

Consensus problem is a key problem in these topics. Consensus means that a group of agents reach an agreement upon some quantities of interest using information of neighbours. By using matrix theory, graph theory, Lyapunov direct method, etc., consensus problem has been widely investigated under different contexts, such as quantised consensus (Li & Xie, 2012; Zhu, Zheng, & Wang, 2015), group consensus (Yu & Wang, 2010), sampled-data-based consensus (Gao, Wang, Xie, & Wu, 2009; Gao, Ma, Zou, Mo, & Yu, 2013). The references mentioned above often consider the consensus problems for a group of agents without any leader. However, one or multiple leaders might exist in the multi-agent systems in some practical applications. When a group of agents are led by one leader, they will follow the leader under the consensus protocol. This is the so-called leader-following consensus problem and there have been numerous studies. In Hong, Hu, and Gao (2006), tracking control problem was studied for multi-agent systems with an active leader under time-varying undirected topology. The consensus problem of heterogeneous multi-agent systems with a time-varying group reference velocity was studied in Zheng and Wang (2012). When a group of followers are led by multiple leaders, the objective is to drive the followers into the geometric space formed by the leaders, which is called containment control problem. Containment control has numerous potential applications. For example, multiple mobile robots move from one place to another. To save the cost, only a few of them are equipped with vision sensors and take on leader roles to form a safety area by detecting the dangerous obstacles, while the other robots just need stay in the moving safety area formed by the leaders. Then, all the mobile robots can arrive at the destination safely (Liu, Xie, et al., 2012). In Ji, Ferrari-Trecate, Egerstedt, and Buffa (2008), a Stop–Go control strategy was proposed to solve containment control problem for a group of first-order agents under fixed undirected network topology. Notarstefano, Egerstedt, and Haque (2011) investigated the containment control problem of first-order multi-agent system under undirected switching topologies. They proved that the containment could be achieved if the time-varying graph was jointly connected. The containment control of a group of double integrator agents was investigated in the presence of both stationary and dynamic leaders under directed fixed and switching topologies in Cao, Stuart, and Ren (2011). Some necessary and sufficient conditions were established to guarantee the achievement of containment control in multi-agent systems with multiple

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stationary and dynamic leaders under fixed directed topologies in Liu, Xie, et al. (2012). The containment control of heterogeneous multi-agent systems was discussed under directed networks in Zheng and Wang (2013). Containment control of multi-agent systems with input saturation was considered in Su, Jia, and Chen (2014), Su and Chen (2015). In Li, Guan, Liao, and Zhang (2014), impulsive protocols were designed for second-order multi-agent systems to solve containment control problem. Liu, Su, et al. (2014) considered the containment control of second-order multi-agent systems with communication delays.

In many practical systems, switching is a common phenomenon. For example, those systems with abrupt parameter variations can be modelled as switched systems. For multi-agent systems, lots of work have been done for distributed coordination of multi-agent systems with switching topologies (Guan et al., 2013, 2014; Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Zheng & Wang, 2012). However, switching behaviours can occur not only in network topology, but also can exist on the dynamical behaviours of agents. In this paper, we proposed a novel type of switched multi-agent system, in which the dynamics of agents switches between continuous-time dynamics and discrete-time dynamics. In the real world, many applications contain such switched systems. For example, a continuous-time plant controlled either by a physically implemented regulator or by a computer-implemented one together with a switching rule between them. Because in any computer-aided systems, the controller only can be implemented in a discrete-time model. When the sampling period is not necessary small, we just need dealing with the value changed on sampling points, and correspondingly consider the discretisation model of the continuous dynamics. Therefore, the entire system can be considered as a switched system composed of both continuous-time and discrete-time subsystems (Zhai, Lin, Michel, & Yasuda, 2004). For a continuous-time multi-agent system, if we sometimes use computer to activate all the agents in a discrete-time manner, then the entire multi-agent system can be seen as a switched multi-agent system, which is composed of a continuous-time subsystem and a discrete-time subsystem. Since a switched system may be instable even though all the subsystems are stable, convergence analysis for such switched multi-agent system become more difficult than the multi-agent system only having continuous-time dynamics or discrete-time dynamics. In Zhai et al. (2004), the authors studied the stability of switched systems which are composed of a continuous-time subsystem and a discrete-time subsystem. Some algebraic conditions were given for solving the stability problem under arbitrary switching. The controllability and observability of such switched system have been investigated in Zhu, Xing, and Guan (2008). Zheng and Wang (2014) investigated the consensus problem of switched multi-agent system composed of continuous-time and discrete-time subsystems. Inspired by the work above, we try to study the containment control problem of the switched multi-agent system with stationary and dynamic leaders, respectively. First, continuous-time protocols based on the relative state measurements of agents are proposed to solve the containment control problem under arbitrary switching by using the matrix theory and the graph theory. Second, impulsive protocols are designed for the switched multi-agent system with dynamic leaders. Based on an impulsive control theory, necessary and sufficient condition is given to guarantee the achievement of containment control. The simulation results are given to verify the effectiveness of the theoretical results.

This paper is organised as follows. In Section 2, some mathematical preliminaries are presented. The containment control problem of the switched multi-agent system is discussed in Section 3. In Section 4, the simulation results are given to show the effectiveness of the obtained results. Section 5 is a brief conclusion.

Notation: Throughout this paper, we let C be the set of all complex numbers, N be the set of non-negative integers, N_+ be the set of positive integers, R^n be the n-dimensional Euclidean space, $R^{n\times m}$ be the set of $n \times m$ real matrix, I_n be the $n \times n$ identity matrix. The superscript 'T' represents the transpose. For $\lambda \in C$, the notation $Re(\lambda)$, $Im(\lambda)$ and $|\lambda|$ are the real part, the imaginary part and the modulus of λ. For a matrix A, $\Lambda(A)$ denotes the eigenvalue set of A. $triag\{a_1, \ldots, a_n\}$ represent the upper triangular matrices. Given two matrices P and Q, we denote their Kronecker product with $P \otimes Q$.

2. Preliminaries

In this section, we first review some basic concepts and properties from the graph theory, and then give some definitions which will be useful in the sequel (Godsil & Royal, 2001; Liu, Xie, et al., 2012; Rockafellar, 1970; Zheng & Wang, 2013).

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted directed graph of order N, with the set of nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, a set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ and a non-negative weighted adjacency matrix $\mathcal{A} = (a_{ij}) \in R^{N\times N}$ with adjacency element $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. For every $(v_j, v_i) \in \mathcal{E}$, v_j is called the parent of v_i, while v_i is called the child of v_j, and v_j is a neighbour of v_i. A graph with the property that $(v_i, v_j) \in \mathcal{E}$ implies $(v_j, v_i) \in \mathcal{E}$ is said to be undirected. A directed path from v_{i_1} to v_{i_k} is a sequence $v_{i_1}, v_{i_2}, v_{i_3}, \ldots, v_{i_k}$ of vertices, such that any two consecutive vertices with property that $(v_{i_s}, v_{i_{s+1}}) \in \mathcal{E}$, $s = 1, \ldots, k-1$. A directed tree is a directed graph, where every vertex, except one special vertex without any parent, which is called the root, has exactly one parent. A directed forest is a directed graph consisting of one or more directed trees, no two of which have a vertex in common. A directed spanning tree (directed spanning forest) is a directed tree (directed forest), which consists of all the nodes and some edges in $\mathcal{G}$. The matrix $\Delta = (\Delta_{ij})$ of the graph $\mathcal{G}$ is a diagonal matrix with $\Delta_{ii} = \sum_{j=1}^{N} a_{ij}$. The Laplacian matrix $\mathcal{L} = (l_{ij}) \in R^{N\times N}$ of the graph is defined as $\mathcal{L} := \Delta - \mathcal{A}$.

Definition 2.1 (Liu, Xie, et al., 2012)**:** For an n agent system, an agent is called a leader if the agent has no neighbour, and an agent is called a follower if the agent has at least one neighbour.

Suppose that the multi-agent systems has m followers and $n - m$, $n > m$, leaders. Without loss of generality, we assume that the agents indexed by $1, \dots, m$ are followers, and the agents indexed by $m + 1, \dots, n$ are leaders. Denote the set of leaders as $\mathcal{R}$ and the set of followers as $\mathcal{F}$, respectively. The communication topology among the n agents is presented by a directed graph $\mathcal{G}$ and $\mathcal{L}$ is the Laplacian matrix of the graph. Since the leaders have no neighbours, $\mathcal{L}$ can be partitioned as $\mathcal{L} = \begin{pmatrix} \mathcal{L}_{\mathcal{F}\mathcal{F}} & \mathcal{L}_{\mathcal{F}\mathcal{R}} \\ 0_{(n-m)\times m} & 0_{(n-m)\times(n-m)} \end{pmatrix}$, where $\mathcal{L}_{\mathcal{F}\mathcal{F}} \in R^{m\times m}$ and $\mathcal{L}_{\mathcal{F}\mathcal{R}} \in R^{m\times(n-m)}$.

Lemma 2.2 (Liu, Xie, et al., 2012)**:** *Suppose that the communication graph $\mathcal{G}$ has a directed spanning forest. Then, all the eigenvalues of $\mathcal{L}_{\mathcal{F}\mathcal{F}}$ have positive real parts, each entry of $-\mathcal{L}_{\mathcal{F}\mathcal{F}}^{-1}\mathcal{L}_{\mathcal{F}\mathcal{R}}$ is non-negative, and each row of $-\mathcal{L}_{\mathcal{F}\mathcal{F}}^{-1}\mathcal{L}_{\mathcal{F}\mathcal{R}}$ has a sum equal to* 1.

By Lemma 2.2, if the graph $\mathcal{G}$ contains a directed spanning forest, the m eigenvalues of $\mathcal{L}_{\mathcal{F}\mathcal{F}}$ can be ordered as $0 < Re(\lambda_1) \le Re(\lambda_2) \le \cdots \le Re(\lambda_m)$.

Definition 2.3 (Rockafellar, 1970)**:** A subset K of R^m is said to be convex if $(1 - \lambda)x + \lambda y \in \boldsymbol{K}$ whenever $x \in K$, $y \in K$ and $0 < \lambda < 1$. The convex hull of a finite set of points $X = \{x_1, ..., x_n\}$ in R^m is the minimal convex set containing all points in X, denoted by $Co\{X\}$. Particularly, $Co\{X\} = \{\sum_{i=1}^{n} a_i x_i \mid x_i \in X, a_i \ge 0, \sum_{i=1}^{n} a_i = 1\}$

Definition 2.4 (Zheng & Wang, 2013)**:** The multi-agent system is said to solve the containment control problem if for any initial conditions, the states of the followers converge to the convex hull spanned by those of the leaders under a certain control input.

Lemma 2.5 (Ogata, 1995 (Hermite–Biehler theorem))**:** *The polynomial $\gamma(\sigma)$ is Hurwitz stable if and only if the related pair $m(\omega)$, $n(\omega)$ is interlaced, and $m(0)n'(0) - m'(0)n(0) > 0$, where $m(\omega)$, $n(\omega)$ are the real and imaginary parts of $\gamma(i\omega)$, respectively.*

3. Main results

3.1 Containment control with stationary leaders

Consider a multi-agent system which consists of n identical agents with first-order dynamics. Suppose that agent i takes the switched dynamics, which is composed of continuous-time dynamics as

$$\dot{x}_i(t) = u_i(t), \quad i \in \{1, \dots, n\} \tag{1}$$

and discrete-time dynamics as

$$x_i(t+1) = x_i(t) + u_i(t), \quad i \in \{1, \dots, n\}, \tag{2}$$

where x_i, $u_i \in R$ are the position and control input of agent i, respectively. All results in this paper still hold for x_i, $u_i \in R^m$ by using the Kronecker product operations.

The linear protocol has been widely applied to multi-agent systems. We present a linear protocol for the switched multi- agent system as follows:

$$u_i(t) = \begin{cases} k \sum\limits_{j\in\mathcal{F}\bigcup\mathcal{R}} a_{ij}(x_j(t) - x_i(t)), & i \in \mathcal{F}, \\ 0, & i \in \mathcal{R}, \end{cases} \tag{3}$$

where $k > 0$ is the control gain to be designed.

Let $x_{\mathcal{F}} = [x_1, x_2, \dots, x_m]^T$ and $x_{\mathcal{R}} = [x_{m+1}, x_{m+2}, \dots, x_n]^T$. Suppose that the dynamics of each agent switches simultaneously from one to another. Then, the entire system can be considered as a switched multi-agent system which is composed of continuous-time subsystem,

$$\begin{cases} \dot{x}_{\mathcal{F}}(t) = -k\mathcal{L}_{\mathcal{F}\mathcal{F}}x_{\mathcal{F}}(t) - k\mathcal{L}_{\mathcal{F}\mathcal{R}}x_{\mathcal{R}}(t), & i \in \mathcal{F}, \\ \dot{x}_{\mathcal{R}}(t) = 0, & i \in \mathcal{R}, \end{cases} \tag{4}$$

and discrete-time subsystem

$$\begin{cases} x_{\mathcal{F}}(t+1) = (I_m - k\mathcal{L}_{\mathcal{F}\mathcal{F}})x_{\mathcal{F}}(t) - k\mathcal{L}_{\mathcal{F}\mathcal{R}}x_{\mathcal{R}}(t), & i \in \mathcal{F}, \\ x_{\mathcal{R}}(t+1) = x_{\mathcal{R}}(t), & i \in \mathcal{R}. \end{cases} \tag{5}$$

Theorem 3.1: *The switched multi-agent system* (4)–(5) *can solve the containment control problem under arbitrary switching if and only if the communication graph $\mathcal{G}$ contains a directed spanning forest and $0 < k < \min_{\lambda\in\Lambda(\mathcal{L}_{\mathcal{F}\mathcal{F}})} \frac{2Re(\lambda)}{|\lambda|^2}$.*

Proof: Sufficiency is proved as follows.

Let $\delta(t) = x_{\mathcal{F}}(t) + \mathcal{L}_{\mathcal{F}\mathcal{F}}^{-1}\mathcal{L}_{\mathcal{F}\mathcal{R}}x_{\mathcal{R}}(t)$. We can obtain from (4) that

$$\dot{\delta}(t) = -k\mathcal{L}_{\mathcal{F}\mathcal{F}}\delta(t) \tag{6}$$

and from (5) that

$$\delta(t+1) = (I - k\mathcal{L}_{\mathcal{F}\mathcal{F}})\delta(t). \tag{7}$$

For any time $t > 0$, we can always divide the time interval $[0, t]$ as $t = t_c + t_d$, where $t_c \in R$ is the total duration time on subsystem (4) and $t_d \in N$ is the total duration time on subsystem (5). Then, from (6) and (7) one has

$$\delta(t) = e^{-k\mathcal{L}_{\mathcal{F}\mathcal{F}}t_c}(I - k\mathcal{L}_{\mathcal{F}\mathcal{F}})^{t_d}\delta(0). \tag{8}$$

Since the communication network $\mathcal{G}$ contains a directed spanning forest, from Lemma 2.2, one has $\lim_{t_c\to\infty} e^{-k\mathcal{L}_{\mathcal{F}\mathcal{F}}t_c} = 0$. Note that $1 - k\lambda_i$, $i = 1, 2, \ldots, m$, are the eigenvalues of matrix $I - k\mathcal{L}_{\mathcal{F}\mathcal{F}}$ and $|1 - k\lambda_i| < 1$ if and only if $0 < k < \frac{2Re(\lambda_i)}{|\lambda_i|^2}$. Thus, if $0 < k < \min_{\lambda\in\Lambda(\mathcal{L}_{\mathcal{F}\mathcal{F}})} \frac{2Re(\lambda)}{|\lambda|^2}$, $\lim_{t_d\to\infty}(I - k\mathcal{L}_{\mathcal{F}\mathcal{F}})^{t_d} = 0$. When $t \to \infty$, it means that at least $t_c \to \infty$ or $t_d \to \infty$. Thus, we obtain $\lim_{t\to\infty} \delta(t) = 0$, that is $\lim_{t\to\infty} x_{\mathcal{F}}(t) = -\mathcal{L}_{\mathcal{F}\mathcal{F}}^{-1}\mathcal{L}_{\mathcal{F}\mathcal{R}}x_{\mathcal{R}}(0)$. Therefore, the containment control problem of switched multi-agent system (4)–(5) is solved.

Necessity is proved as follows.

Similar to the proof of theorem 1 in Liu, Xie, et al. (2012), we can get that the directed graph $\mathcal{G}$ has a directed spanning forest is necessary for the switched system (4)–(5) to achieve containment control. When the switched multi-agent system can solve the containment control problem under arbitrary switching, the discrete-time multi-agent system (5) can achieve containment control. Note that the system (5) can solve the containment control problem if and only if the positions of all the followers are determined by the leaders's positions. If we rewrite system (5) as $x(t+1) = M_1x(t)$, where $x(t) = [x_{\mathcal{F}}^T(t), x_{\mathcal{R}}^T(t)]^T$, $M_1 = \begin{pmatrix} I - k\mathcal{L}_{\mathcal{F}\mathcal{F}} & -k\mathcal{L}_{\mathcal{F}\mathcal{R}} \\ 0 & I_{n-m} \end{pmatrix}$, then the rank of $\lim_{t\to\infty} M_1^t$ cannot exceed $n - m$. If the condition that $0 < k < \min_{\lambda\in\Lambda(\mathcal{L}_{\mathcal{F}\mathcal{F}})} \frac{2Re(\lambda)}{|\lambda|^2}$ does not hold, then at least one of the eigenvalues of $I - k\mathcal{L}_{\mathcal{F}\mathcal{F}}$ is on or outside the unit circle from the above proof. As a result , $\lim_{t\to\infty} M_1^t$ has a rank greater that $n - m$, which results in a contradiction. □

The undirected graph can be treated as a special directed graph. If the communication topology among followers is undirected, $\mathcal{L}_{\mathcal{F}\mathcal{F}}$ is symmetric and the eigenvalues of $\mathcal{L}_{\mathcal{F}\mathcal{F}}$ are all real numbers. Therefore, we can easily get the following corollary.

Corollary 3.2: *Suppose the communication topology among followers is undirected. Then, the switched multi-agent system* (4)–(5) *can exponentially solve the containment control problem under arbitrary switching if the communication graph $\mathcal{G}$ contains a directed spanning forest and $0 < k < \frac{2}{\lambda_m}$.*

Proof: If $0 < k < \frac{2}{\lambda_m}$, we obtain from the eigenvalues of $\mathcal{L}_{\mathcal{F}\mathcal{F}}$ that $\|e^{-k\mathcal{L}_{\mathcal{F}\mathcal{F}}t_c}\| \leq e^{-k\lambda_1 t_c}$ and $\|(I - k\mathcal{L}_{\mathcal{F}\mathcal{F}})^{t_d}\| \leq (1 - k\lambda_1)^{t_d}$. Combining these two inequalities, we have

$$\begin{aligned}\|\delta(t)\| &= \|e^{-k\mathcal{L}_{\mathcal{F}\mathcal{F}}t_c}(I - k\mathcal{L}_{\mathcal{F}\mathcal{F}})^{t_d}\delta(0)\| \\ &\leq e^{-k\lambda_1 t_c}(1 - k\lambda_1)^{t_d}\|\delta(0)\| \leq e^{-\alpha t}\|\delta(0)\|,\end{aligned}$$

where $\alpha = \min\{k\lambda_1, \ln \frac{1}{1-k\lambda_1}\}$. Hence, the switched multi-agent system can exponentially solve the containment control problem under arbitrary switching. □

Remark 1: To avoid to calculate the Laplacian spectrum, we have $\lambda_m < 2\max_{i\in\mathcal{F}}\{\Delta_{ii}\}$ from the Geršgorin disc theorem. Thus, the switched multi-agent system (4)–(5) can solve the containment control problem under arbitrary switching if $0 < k < \frac{1}{\max\limits_{i\in\mathcal{F}}\{\Delta_{ii}\}}$.

Remark 2: In this paper, we consider the network of the leaders without communication. In the case where the leaders interact with each other, inspired by the work in remark 1 of Zheng and Wang (2013), the distributed control inputs of leaders can be designed as $u_i = k\sum_{j\in\mathcal{R}} a_{ij}((x_j(t) - h_j) - (x_i(t) - h_i))$, $i \in \mathcal{R}$. If the graph $\mathcal{G}$ contains a directed spanning tree, positions of the leaders in switched multi-agent system (4)–(5) will converge to a desired formation and the followers will converge to the convex hull of the leaders' final positions.

3.2 *Containment control with dynamic leaders*

Consider a multi-agent system which consists of n identical agents with second-order dynamics. Suppose that agent i takes the switched dynamics, which is composed of continuous-time dynamics as

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = u_i(t), \quad i \in \{1, \ldots, n\} \end{cases} \tag{9}$$

and discrete-time dynamics as

$$\begin{cases} x_i(t+1) = x_i(t) + v_i(t), \\ v_i(t+1) = v_i(t) + u_i(t), \quad i \in \{1, \ldots, n\}, \end{cases} \tag{10}$$

where x_i, v_i, $u_i \in R$ are the position, velocity and control input of agent i, respectively.

First, we give a linear protocol as

$$u_i(t) = \begin{cases} k_1 \sum\limits_{j\in\mathcal{F}\bigcup\mathcal{R}} a_{ij}(x_j(t) - x_i(t)) \\ \qquad + k_2 \sum\limits_{j\in\mathcal{F}\bigcup\mathcal{R}} a_{ij}(v_j(t) - v_i(t)), & i \in \mathcal{F}, \\ 0, & i \in \mathcal{R}, \end{cases} \tag{11}$$

where $k_1, k_2 > 0$ are the control gains to be designed.

Let $z_i = [x_i, v_i]^T$, $z_{\mathcal{F}} = [z_1^T, \ldots, z_m^T]^T$ and $z_{\mathcal{R}} = [z_{m+1}^T, \ldots, z_n^T]^T$. Suppose that the dynamics of each agent switches simultaneously from one to another. Then, we get a switched multi-agent system which is composed of continuous-time subsystem

$$\begin{cases} \dot{z}_{\mathcal{F}}(t) = (I_m \otimes E - \mathcal{L}_{\mathcal{F}\mathcal{F}} \otimes F)z_{\mathcal{F}}(t) - (\mathcal{L}_{\mathcal{F}\mathcal{R}} \otimes F)z_{\mathcal{R}}(t), \\ \dot{z}_{\mathcal{R}} = (I_{n-m} \otimes E)z_{\mathcal{R}}(t) \end{cases} \tag{12}$$

and discrete-time subsystem

$$\begin{cases} z_{\mathcal{F}}(t+1) = (I_m \otimes G - \mathcal{L}_{\mathcal{F}\mathcal{F}} \otimes F)z_{\mathcal{F}}(t) \\ \qquad - (\mathcal{L}_{\mathcal{F}\mathcal{R}} \otimes F)z_{\mathcal{R}}(t), \\ z_{\mathcal{R}}(t+1) = (I_{n-m} \otimes G)z_{\mathcal{R}}(t), \end{cases} \tag{13}$$

where $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ k_1 & k_2 \end{pmatrix}$ and $G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Theorem 3.3: *The switched multi-agent system* (12)–(13) *can solve the containment control problem under arbitrary switching if and only if the communication graph $\mathcal{G}$ contains a directed spanning forest and*

$$\begin{cases} k_2 > k_1, \\ \left((k_1 - 2k_2)|\lambda_i|^2 + 4Re(\lambda_i)\right)(k_2 - k_1)^2 > \frac{4k_1 Im^2(\lambda_i)}{|\lambda_i|^2}, \end{cases} \tag{14}$$

where λ_i, $i = 1, 2, \ldots, m$ are the eigenvalues of $\mathcal{L}_{\mathcal{FF}}$.

Proof: Sufficiency is proved as follows.

Let $\delta(t) = z_{\mathcal{F}}(t) + (\mathcal{L}_{\mathcal{FF}}^{-1}\mathcal{L}_{\mathcal{FR}} \otimes I_2)z_{\mathcal{R}}(t)$, we can obtain from (12) that $\delta(t)$ satisfies the following dynamics:

$$\begin{aligned} \dot{\delta}(t) &= \dot{z}_{\mathcal{F}}(t) + (\mathcal{L}_{\mathcal{FF}}^{-1}\mathcal{L}_{\mathcal{FR}} \otimes I_2)\dot{z}_{\mathcal{R}}(t) \\ &= (I_m \otimes E - \mathcal{L}_{\mathcal{FF}} \otimes F)z_{\mathcal{F}} - (\mathcal{L}_{\mathcal{FR}} \otimes F)z_{\mathcal{R}} \\ &\quad + (\mathcal{L}_{\mathcal{FF}}^{-1}\mathcal{L}_{\mathcal{FR}} \otimes I_2)(I_{n-m} \otimes E)z_{\mathcal{R}} \\ &= (I_m \otimes E - \mathcal{L}_{\mathcal{FF}} \otimes F)\left(\delta(t) - (\mathcal{L}_{\mathcal{FF}}^{-1}\mathcal{L}_{\mathcal{FR}} \otimes I_2)z_{\mathcal{R}}(t)\right) \\ &\quad - (\mathcal{L}_{\mathcal{FR}} \otimes F)z_{\mathcal{R}} + (\mathcal{L}_{\mathcal{FF}}^{-1}\mathcal{L}_{\mathcal{FR}} \otimes E)z_{\mathcal{R}} \\ &= (I_m \otimes E - \mathcal{L}_{\mathcal{FF}} \otimes F)\delta(t) \\ &= H_1\delta(t), \end{aligned} \tag{15}$$

where $H_1 = \boldsymbol{I_m} \otimes E - \mathcal{L}_{\mathcal{FF}} \otimes F$.

Similarly, from (13) we get $\delta(t)$ satisfies

$$\delta(t+1) = (I_m \otimes G - \mathcal{L}_{\mathcal{FF}} \otimes F)\delta(t) = H_2\delta(t), \tag{16}$$

where $H_2 = \boldsymbol{I_m} \otimes G - \mathcal{L}_{\mathcal{FF}} \otimes F$.

For any time $t = t_c + t_d$, where $t_c \in R$ is the total duration time on subsystem (12) and $t_d \in N$ is the total duration time on subsystem (13). Note that $H_1H_2 = H_2H_1$, then from (15) and (16) we obtain

$$\delta(t) = e^{H_1 t_c} H_2^{t_d}\delta(0). \tag{17}$$

Since graph $\mathcal{G}$ is directed, there exists an invertible matrix W, such that $W^{-1}\mathcal{L}_{\mathcal{FF}}W = triag\{\lambda_1, \ldots, \lambda_m\}$. Then, we have

$$\begin{aligned} &(W^{-1} \otimes I_2)H_2(W \otimes I_2) \\ &\quad = I_m \otimes G - triag\{\lambda_1, \ldots, \lambda_m\} \otimes F \\ &\quad = triag\left\{\begin{pmatrix} 1 & 1 \\ -k_1\lambda_1 & 1 - k_2\lambda_1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 1 \\ -k_1\lambda_m & 1 - k_2\lambda_m \end{pmatrix}\right\}. \end{aligned}$$

Thus, the eigenvalues of H_2 can be obtained by solving the equation

$$\begin{aligned} &det(\mu I_{2m} - H_2) \\ &\quad = \prod_{i=1}^{m} det\begin{pmatrix} \mu - 1 & -1 \\ k_1\lambda_i & \mu - 1 + k_2\lambda_i \end{pmatrix} \\ &\quad = \prod_{i=1}^{m} \left(\mu^2 - (2 - k_2\lambda_i)\mu + 1 + k_1\lambda_i - k_2\lambda_i\right) \\ &\quad = 0. \end{aligned}$$

Let $a_i(\mu) = \mu^2 - (2 - k_2\lambda_i)\mu + 1 + k_1\lambda_i - k_2\lambda_i$. By applying the bilinear transformation $\mu = \frac{\sigma+1}{\sigma-1}$, we get a new polynomial

$$\begin{aligned} r_i(\sigma) = (\sigma - 1)^2 a_i\left(\frac{\sigma+1}{\sigma-1}\right) &= k_1\lambda_i\sigma^2 \\ &\quad + 2\lambda_i(k_2 - k_1)\sigma + k_1\lambda_i - 2k_2\lambda_i + 4. \end{aligned}$$

Define $r_i'(\sigma) = \frac{r_i(\sigma)}{k_1\lambda_i} = \sigma^2 + \frac{2(k_2-k_1)}{k_1}\sigma + \frac{k_1\lambda_i - 2k_2\lambda_i + 4}{k_1\lambda_i}$. Then, the Schur stable of $a_i(\mu)$ is equivalent to the Hurwitz stable of $r_i'(\sigma)$. Using Lemma 2.5, we get that $r_i'(\sigma)$ is Hurwitz stable if k_1 and k_2 satisfy the condition (14). Then, all the eigenvalues of H_2 are in the unit circle.

From $0 < k_1 < k_2$ in (14), we obtain $(k_2 - k_1)^2 < k_2^2$. Thus, $\left((k_1 - 2k_2)|\lambda_i|^2 + 4Re(\lambda_i)\right)(k_2 - k_1)^2|\lambda_i|^2 < 4k_2^2 Re(\lambda_i)|\lambda_i|^2$. Since $\left((k_1 - 2k_2)|\lambda_i|^2 + 4Re(\lambda_i)\right)(k_2 - k_1)^2 > \frac{4k_1 Im^2(\lambda_i)}{|\lambda_i|^2}$ in (14), we get $4k_1 Im^2(\lambda_i) < 4k_2^2 Re(\lambda_i)|\lambda_i|^2$, that is, $\frac{k_2^2}{k_1} > \frac{Im^2(\lambda_i)}{Re(\lambda_i)|\lambda_i|^2}$. From the proof of theorem 2 in Liu, Xie, et al. (2012), we know if $\frac{k_2^2}{k_1} > \frac{Im^2(\lambda_i)}{Re(\lambda_i)|\lambda_i|^2}$, $i = 1, 2, \ldots, n$, all the eigenvalues of H_1 have negative real parts.

Thus, from (17) we obtain $\lim_{t\to\infty}\delta(t) = 0$, that is $\lim_{t\to\infty} x_{\mathcal{F}}(t) = -\mathcal{L}_{\mathcal{FF}}^{-1}\mathcal{L}_{\mathcal{FR}}x_{\mathcal{R}}(t)$ and $\lim_{t\to\infty} v_{\mathcal{F}}(t) = -\mathcal{L}_{\mathcal{FF}}^{-1}\mathcal{L}_{\mathcal{FR}}x_{\mathcal{R}}(t)$. Therefore, the containment control problem of the switched multi-agent system (12)–(13) is solved.

Necessity is proved as follows.

When the switched multi-agent system can solve the containment control problem under arbitrary switching, the continuous-time multi-agent system (12) and the discrete-time multi-agent system (13) can achieve containment control asymptotically, respectively. From theorem 2 in Liu, Xie, et al. (2012), we know that $\frac{k_2^2}{k_1} > \frac{Im^2(\lambda_i)}{Re(\lambda_i)|\lambda_i|^2}$ is necessary for system (12) to achieve containment control. If we rewrite system (13) as $z(t+1) = M_2 z(t)$, where $z(t) = [z_{\mathcal{F}}^T(t), z_{\mathcal{R}}^T(t)]^T$, $M_2 = \begin{pmatrix} H_2 & -\mathcal{L}_{\mathcal{FR}} \otimes F \\ 0 & I_{n-m} \otimes G \end{pmatrix}$, similar to the proof of necessity in Theorem 3.1, we get that all the eigenvalues of H_2 are in the unit circle is necessary for system (13) to achieve the containment control, which implies the necessity of condition (14) from the above proof.

Corollary 3.4: *Suppose the communication topology among followers is undirected. Then, the switched multi-agent system* (12)–(13) *can solve the containment control problem under arbitrary switching if and only if the communication graph $\mathcal{G}$ contains a directed spanning forest and $k_2 > k_1$, $k_1 - 2k_2 > \frac{-4}{\lambda_m}$.*

Compared with continuous-time control inputs, impulsive control inputs just use the state variables of systems at discrete-time instances, and thus have a relatively simple structure. Next, we consider impulsive protocol for the containment control problem of the switched multi-agent system.

Let $\{t_l\}_{l=0}^{\infty}$ denote the set of impulse instant, which satisfies $0 = t_0 < t_1 < \cdots < t_l < t_{l+1} < \cdots, \lim_{l\to+\infty} t_l = +\infty$. When $t = t_l$,

$$\begin{cases} v_i(t_l^+) = v_i(t_l) + \triangle v_i(t_l), \\ \triangle v_i(t_l) = k_1 \sum\limits_{j\in\mathcal{F}\bigcup\mathcal{R}} a_{ij}(x_j(t_l) - x_i(t_l)) \\ \qquad + k_2 \sum\limits_{j\in\mathcal{F}\bigcup\mathcal{R}} a_{ij}(v_j(t_l) - v_i(t_l)), \qquad i \in \mathcal{F}, \\ \triangle v_i(t_l) = 0, \qquad i \in \mathcal{R}, \end{cases} \tag{18}$$

where $v_i(t_l)$ denotes the value of v_i at t_l before the impulse, $v_i(t_l^+)$ denotes the value of $\boldsymbol{v_i}$ at t_l after the impulse. When $t \in [t_l^+, t_{l+1})$, if the continuous-time dynamics is activated

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = 0, \qquad i \in \{1, \ldots, n\} \end{cases} \tag{19}$$

and if the discrete-time dynamics is activated

$$\begin{cases} x_i(t+1) = x_i(t) + v_i(t), \\ v_i(t+1) = v_i(t), \qquad i \in \{1, \ldots, n\}. \end{cases} \tag{20}$$

The uniform impulsive interval is denoted by h, which means that $t_{l+1} - t_l \equiv h$, $h \in N_+$ always holds for any $l \in N$. In this paper, we assume that if the continuous-time dynamics is activated, the duration time is positive integers. Thus, the dynamics of each agent in the switched multi-agent system can be rewritten as

$$\begin{cases} x_i(t_{l+1}) = x_i(t_l) + hv_i(t_l^+), \qquad i \in \mathcal{F}, l \in N \\ v_i(t_{l+1}) = v_i(t_l^+) = v_i(t_l) + k_1 \sum\limits_{j\in\mathcal{F}\bigcup\mathcal{R}} a_{ij}((x_j(t_l) \\ \qquad - x_i(t_l)) + k_2 \sum\limits_{j\in\mathcal{F}\bigcup\mathcal{R}} a_{ij}(v_j(t_l) - v_i(t_l)) \end{cases} \tag{21}$$

and

$$\begin{cases} x_i(t_{l+1}) = x_i(t_l) + hv_i(t_l), \\ v_i(t_{l+1}) = v_i(t_l) \quad i \in \mathcal{R}, l \in N. \end{cases} \tag{22}$$

Then, it is easy to get that the switched multi-agent system (9)–(10) can solve the containment control problem under impulsive control input (18) if and only if the multi-agent system (21)–(22) can solve the containment control problem, that is, $\lim_{l\to\infty} x_i(t_l) - x_j(t_l) = 0$ and $\lim_{l\to\infty} v_i(t_l) - v_j(t_l) = 0$, $i, j \in \{1, \ldots, n\}$.

The aggregate dynamics of the multi-agent system (21)–(22) is of course represented by

$$\begin{cases} z_{\mathcal{F}}(t_{l+1}) = (I_m \otimes P - \mathcal{L}_{\mathcal{FF}} \otimes Q) z_{\mathcal{F}}(t_l) \\ \qquad - (\mathcal{L}_{\mathcal{FR}} \otimes Q) z_{\mathcal{R}}(t_l), \\ z_{\mathcal{R}}(t_{l+1}) = (I_{n-m} \otimes P) z_{\mathcal{R}}(t_l), \quad l \in N, \end{cases} \tag{23}$$

where $P = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} k_1 h & k_2 h \\ k_1 & k_2 \end{pmatrix}$.

Theorem 3.5: *The switched multi-agent system* (9)–(10) *with the impulsive protocol* (18) *can solve the containment control problem under arbitrary switching if and only if the communication graph* $\mathcal{G}$ *contains a directed spanning forest and impulsive interval* $h \in N_+$ *satisfies*

$$h < \min_{\lambda\in\Lambda(\mathcal{L}_{\mathcal{FF}})} \frac{4Re(\lambda) \mid \lambda \mid^2 - 2k_2 \mid \lambda \mid^4}{k_1 \mid \lambda \mid^4 + 4k_1 Im^2(\lambda)/k_2^2}. \tag{24}$$

Proof: Sufficiency is proved as follows.

Let $\delta(t_l) = z_{\mathcal{F}}(t_l) + (\mathcal{L}_{\mathcal{FF}}^{-1}\mathcal{L}_{\mathcal{FR}} \otimes I_2) z_{\mathcal{R}}(t_l)$, we can obtain from (23) that $\delta(t)$ satisfies the following dynamics:

$$\delta(t_{l+1}) = (I_m \otimes P - \mathcal{L}_{\mathcal{FF}} \otimes Q)\delta(t) = H_3\delta(t), \tag{25}$$

where $H_3 = (I_m \otimes P - \mathcal{L}_{\mathcal{FF}} \otimes Q)\delta(t)$.

The Hurwitz stability analysis of H_3 is similar to that of H_2 in the deduce of Theorem 3.3. We can obtain that all the eigenvalues of H_3 are in the unit circle if and only if k_1,k_2 and h satisfy the condition (24). Thus, we obtain $\lim_{t\to\infty} \delta(t) = 0$, that is $\lim_{t\to\infty} x_{\mathcal{F}}(t) = -\mathcal{L}_{\mathcal{FF}}^{-1}\mathcal{L}_{\mathcal{FR}} x_{\mathcal{R}}(t)$ and $\lim_{t\to\infty} v_{\mathcal{F}}(t) = -\mathcal{L}_{\mathcal{FF}}^{-1}\mathcal{L}_{\mathcal{FR}} x_{\mathcal{R}}(t)$. Therefore, the containment control problem is solved.

Necessity is proved as follows.

For system (23), it can be equivalently written as $z(t_{l+1}) = M_3 z(t_l)$, where $z(t_l) = [z_{\mathcal{F}}^T(t_l), z_{\mathcal{R}}^T(t_l)]^T$, $M_3 = \begin{pmatrix} H_3 & -\mathcal{L}_{\mathcal{FR}} \otimes Q \\ 0 & I_{n-m} \otimes P \end{pmatrix}$. Similar to the proof of necessity in Theorem 3.1, we get that all the eigenvalues of H_3 are in the unit circle is necessary for system (23) to achieve containment control, which implies the necessity of condition (24) from the above proof. □

Remark 3: Note that when the duration time of discrete-time subsystem equals to zero, the impulsive containment control of the switched multi-agent system becomes the impulsive containment control of continuous-time multi-agent system, which was studied in Liu, Su, et al. (2014). The authors gave a sufficient condition for the case of dynamic leaders in Liu, Su, et al. (2014), while in this paper a necessary and sufficient condition is obtained for solving the containment control problem.

Corollary 3.6: *Suppose the communication topology among followers is undirected. Then, the switched multi-agent system* (9)–(10) *with the impulsive protocol* (18) *can solve can solve the containment control problem under arbitrary switching if and only if the communication graph* $\mathcal{G}$ *contains a directed spanning forest and the impulsive interval* $h \in N_+$ *satisfies* $h \le \frac{4}{\lambda_m k_1} - \frac{2k_2}{k_1}$.

4. Simulations

In this section, we will provide two examples to demonstrate the effectiveness of the theoretical results.

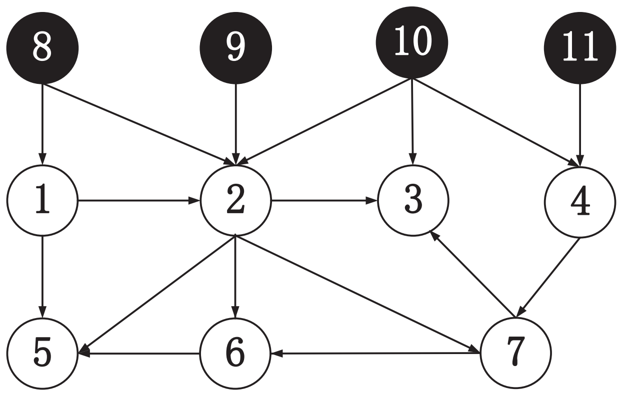

Figure 1. The communication topology among agents.

Example 4.1 (Containment control with stationary leaders)**:** Consider a network of first-order dynamic agents, with $x_i = [x_{i1}, x_{i2}]^T$, where x_{i1} and x_{i2} are, respectively, the positions of the agent i along the x and y coordinates. Let the interaction graph $\mathcal{G}$ be given by Figure 1, in which agents 8,9,10 and 11 are leaders (the filled circles) and the others are followers. Assume each edge weight to be 1. It can be noted that $\mathcal{G}$ has a directed spanning forest. For simpleness, we assume the duration time of continuous-time subsystem (4) from the time to be activated to the time to be switched to discrete-time subsystem (5) is always the same and equals to 4. For discrete-time subsystem, we make the same assumption. The duration time equals to 5. By Theorem 3.1, we choose $k = 0.03 < 0.667$, and the simulation results are shown in Figure 2. We can see that positions of the followers converge to the stationary convex hull formed by positions of the leaders.

Example 4.2 (Containment control with dynamic leaders)**:** Consider a network of second-order dynamic agents, with $x_i = [x_{i1}, x_{i2}]^T$. To illustrate Theorem 3.3, assume that the

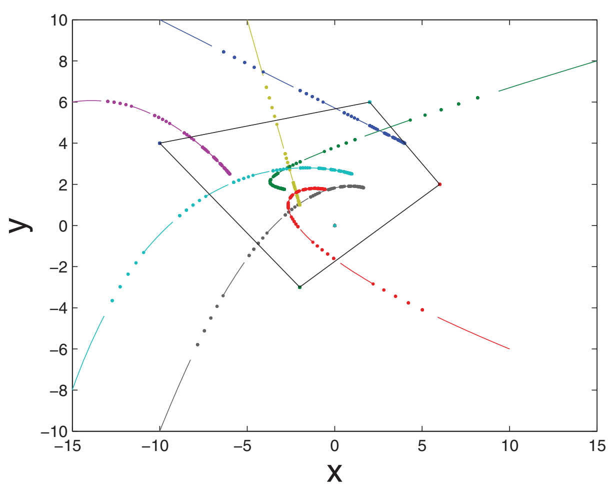

Figure 2. Trajectories of all the agents of switched multi-agent system (4)–(5).

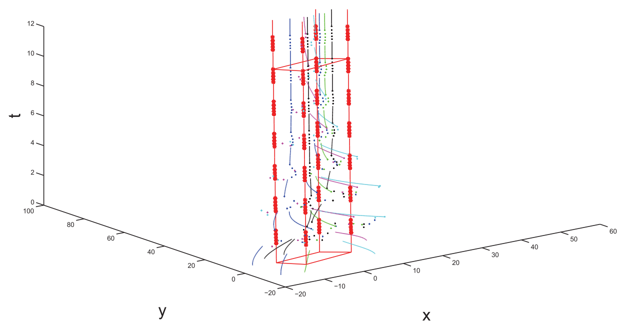

Figure 3. Trajectories of all the agents of switched multi-agent system (12)–(13).

interaction graph $\mathcal{G}$ is shown in Figure 1. We make the same assumption as in Example 4.1. By Theorem 3.3, we choose $k_1 = 0.5$ and $k_2 = 0.6$. The simulation results are shown in Figure 3 . We can see as desired that positions of the followers converge to the dynamic convex hull formed by positions of the leaders.

5. Conclusions

This paper has considered the containment control problem for a novel switched multi-agent system under directed communication topologies. When the leaders are stationary, continuous-time protocols based on the relative state measurements of agents have been designed. We have proved that the containment control problem can be solved under arbitrary switching if and only if the communication graph contains a directed spanning forest and the control gain satisfies that $0 < k < \min_{\lambda \in \Lambda(\mathcal{L}_{\mathcal{FF}})} \frac{2Re(\lambda)}{|\lambda|^2}$. When the leaders are dynamic, continuous-time and impulsive protocols have been proposed, respectively. Based on the graph theory and the impulsive control theory, some sufficient and necessary conditions have been obtained for solving the containment control problem under arbitrary switching. Our future work will consider the containment control problem of switched multi-agent systems with delays and quantised information.

Disclosure statement

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