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In this paper, we consider the containment control problem for a group of autonomous agents modelled by heterogeneous dynamics. The communication networks among the leaders and the followers are directed graphs. When the leaders are first-order integrator agents, we present a linear protocol for heterogeneous multi-agent systems such that the second-order integrator agents converge to the convex hull spanned by the first-order integrator agents if and only if the directed graph contains a directed spanning forest. If the leaders are second-order integrator agents, we propose a nonlinear protocol and obtain a necessary and sufficient condition that the heterogeneous multi-agent system solves the containment control problem in finite time. Simulation examples are also provided to illustrate the effectiveness of the theoretical results.

Keywords: heterogeneous multi-agent system; containment control; consensus; finite time; directed graph

1. Introduction

The analysis of distributed coordination received an increasing interest in the past decades. Unlike centralised coordination, distributed coordination has enormous advantages, such as flexibility, reliability and adaptability, etc. As a fundamental of distributed coordination, the consensus or agreement problem of multi-agent systems has been studied by multi-disciplinary researchers (Chu, Wang, Chen, & Mu, 2006; Ji, Wang, Lin, & Wang, 2009; Olfati-Saber & Murray, 2004; Zheng, Chen, & Wang, 2011a). As an important problem of multi-agent systems, the leader–follower coordination has received widespread attention in recent years. Additional details related to the distributed coordination may be obtained from Olfati-Saber, Fax, and Murray (2007), and references therein.

1.1 Related work

The consensus problem was primarily considered without group reference (Olfati-Saber & Murray, 2004; Xie & Wang, 2007). In reality, a group of agents might have a desired state for the group. Thus, the consensus-tracking problem was also studied in recent years. Hong, Hu, and Gao (2006) and Hong, Chen, and Bushnell (2008) studied a multi-agent consensus-tracking problem with an active leader under variable undirected topology. Qin, Zheng, and Gao (2011) investigated the consensus of second-order multi-agent systems with a time-varying reference velocity under directed topology. Containment control is a consensus-like tracking problem with multiple leaders. The main objective of containment control is to drive the states of the followers into the convex hull spanned by the leaders. Motivated by the numerous natural phenomena and applications in practice, containment control has been studied by a number of researchers till now. Ji, Ferrari-Trecate, Egerstedt, and Buffa (2008) presented a hybrid Stop–Go strategy for the agents by single integrator kinematics under the fixed undirected topology. In Notarstefano, Egerstedt, and Haque (2011), the authors studied the first-order multi-agent containment problem under switching communication topologies. In Meng, Ren, and You (2010), finite-time attitude containment control was addressed for multiple rigid bodies under undirected topology. Cao, Stuart, Ren, and Meng (2011) studied the distributed containment control of second-order multi-agent systems with multiple stationary/dynamic leaders under fixed and switching topologies. Lou and Hong (2012) considered the second-order multi-agent containment control with random switching interconnection topologies. Li, Ren, Liu, and Fu (2012) discussed the containment control problem of continuous-time/discrete-time multi-agent systems with general linear dynamics under fixed directed communication topologies. Liu, Xie and Wang investigated containment control of linear multi-agent systems under general interaction topologies in Liu, Xie, and Wang (2012a) and obtained some necessary and sufficient conditions for the containment control of multi-agent systems under continuous-time and sampled-data based protocols in Liu, Xie, and Wang (2012b).

The dynamics of the agents coupled with each other are not the same because of various restrictions or the common

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goals with mixed agents in the practical systems. However, existing containment control protocols often focus on first-order or second-order multi-agent systems, i.e. all the aforementioned multi-agent systems were homogeneous. Sometimes we want to investigate the containment control problem of heterogeneous multi-agent systems owing to wide applications in practice. For example, a team of heterogeneous agents moves from one location to another when only a portion of the agents is equipped with necessary sensors to detect the hazardous obstacles so that the agents that are equipped will stay in a safety area formed by the equipped agents (Liu et al., 2012b). However, only a little work considers heterogeneous cases of the consensus problem. In particular, the output consensus of heterogeneous multi-agent systems was studied in Kim, Shim, and Seo (2011). Consensus of heterogeneous multi-agent systems composed of first-order and second-order integrator agents was investigated in Liu and Liu (2011), Zheng, Zhu, and Wang (2011b), Zheng and Wang (2012a, 2012b, 2012c). Liu and Liu (2011) studied the stationary consensus of discrete-time heterogeneous multi-agent systems. Zheng et al. considered the consensus of continuous-time heterogeneous multi-agent systems (Zheng et al., 2011b; Zheng & Wang, 2012a, 2012b) and finite-time consensus (Zheng & Wang, 2012c) with and without velocity measurements. To the best of our knowledge, this is the first study about the containment control of heterogeneous multi-agent systems.

1.2 Our results

In this paper, we consider the containment control problem for continuous-time heterogeneous multi-agent systems that is composed of first-order and second-order integrator agents. First, we present a linear protocol for the heterogeneous multi-agent system when the leaders are first-order integrator agents. Utilising the method of variable substitution and previous results of the first-order multi-agent system, we obtain that the second-order integrator agents (the followers) converge to the convex hull spanned by the first-order integrator agents (the leaders) if and only if the directed graph contains a directed spanning forest. Then, we propose a nonlinear protocol for the heterogeneous multi-agent system if the second-order integrator agents are leaders. By using the Lyapunov theory, Lasalle's invariance principle and the homogeneous domination method, we get that the first-order integrator agents (the followers) converge to the convex hull spanned by the second-order integrator agents (the leaders) in finite time if and only if the directed graph contains a directed spanning forest. Finally, some simulation examples are presented to show the effectiveness of our proposed protocols.

This paper is organised as follows. In Section 2, we present some notions in graph theory and formulate the model to be studied, and assemble some key lemmas. In Section 3, we give the main results. And in Section 4, numerical simulations are given to illustrate the effectiveness of the theoretical results. Some conclusions are drawn in Section 5.

Notation: Throughout this paper, we let $\mathbb{R}$, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ be the sets of real numbers, positive real numbers and non-negative real numbers, $\mathbb{R}^n$ is the $n-$dimensional real vector space, $\mathcal{I}_n = \{1, 2, \ldots, n\}$. For a given vector or matrix X, X^T denotes its transpose. $\mathbf{1}_n$ is a vector with elements being all ones. I_n is the $n \times n$ identity matrix. 0 ($0_{m\times n}$) denotes an all-zero vector or matrix with compatible dimension (dimension $m \times n$). A is said to be non-negative (respectively positive) if all entries a_{ij} are non-negative (respectively positive), denoted by $A \geq 0$ (respectively $A > 0$). $sig(x)^\alpha = sign(x)|x|^\alpha$, where $sign(\cdot)$ is a sign function. $A \otimes B$ denotes the Kronecker product of matrices A and B.

2. Preliminaries

2.1 Graph theory

The network formed by multi-agent systems can always be represented by a graph. Thus, graph theory is an important tool to analyse the coordination problem for multi-agent systems. In this subsection, some basic concepts and properties are presented in the graph theory. For more details, please refer to Godsil and Royal (2001).

A weighted directed graph $\mathscr{G}(\mathscr{A}) = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ of order n consists of a vertex set $\mathscr{V} = \{s_1, s_2, \ldots, s_n\}$, an edge set $\mathscr{E} = \{e_{ij} = (s_i, s_j)\} \subset \mathscr{V} \times \mathscr{V}$ and a non-negative matrix $\mathscr{A} = [a_{ij}]_{n\times n}$. $(s_j, s_i) \in \mathscr{E} \Leftrightarrow a_{ij} > 0 \Leftrightarrow$ agent i and j can communicate with each other, namely, they are adjacent. Moreover, we assume $a_{ii} = 0$. A is called the weighted matrix and a_{ij} is the weight of $e_{ij} = (s_i, s_j)$. The set of neighbours of s_i is denoted by $\mathscr{N}_i = \{s_j : e_{ji} = (s_j, s_i) \in \mathscr{E}\}$. A path that connects s_i and s_j in the directed graph $\mathscr{G}$ is a sequence of distinct vertices $s_{i_0}, s_{i_1}, s_{i_2}, \ldots s_{i_m}$, where $s_{i_0} = s_i$, $s_{i_m} = s_j$ and $(s_{i_r}, s_{i_{r+1}}) \in \mathscr{E}$, $0 \leq r \leq m-1$. For a directed graph, if (s_i, s_j) is an edge of $\mathscr{G}$, s_i is called the parent of s_j and s_j is called the child of s_i. A directed tree is a directed graph, where every vertex, except one special vertex without any parent, which is called the root, has exactly one parent, and the root can be connected to any other vertex through paths. A directed forest is a directed graph consisting of one or more directed trees, no two of which have a vertex in common. A directed spanning tree (directed spanning forest) is a directed tree (directed forest), which consists of all the nodes and some edges in $\mathscr{G}$. The degree matrix $\mathscr{D} = [d_{ij}]_{n\times n}$ is a diagonal matrix with $d_{ii} = \sum_{j: s_j \in \mathscr{N}_i} a_{ij}$ and the Laplacian matrix of the graph is defined as $\mathscr{L} = [l_{ij}]_{n\times n} = \mathscr{D} - \mathscr{A}$. It has been shown that $\mathscr{L}\mathbf{1}_n = 0$. If $\mathscr{G}(\mathscr{A})$ is strongly connected, then there exists a positive column vector $\omega \in \mathbb{R}^n$ such that $\omega^T \mathscr{L} = 0$. For multi-agent systems, an agent is called a leader if the agent has no neighbour, and an agent is called a follower if the agent has at least one neighbour.

2.2 Heterogeneous multi-agent systems

We consider a system of n agents, comprising m ($m < n$) second-order integrator agents and $n - m$ first-order integrator agents, which update their states based on communication over a directed network. In this paper, we denote the set of leaders as $\mathscr{R}$ and the set of followers as $\mathscr{F}$.

Each second-order agent dynamics is given as follows:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = u_i(t) \quad i \in \mathcal{I}_m, \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}^N$, $v_i \in \mathbb{R}^N$ and $u_i \in \mathbb{R}^N$ are the position, velocity and control input, respectively, of agent i. Each first-order agent dynamics is given as follows:

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{I}_n/\mathcal{I}_m, \tag{2}$$

where $x_i \in \mathbb{R}^N$ and $u_i \in \mathbb{R}^N$ are the position and control input, respectively, of agent i.

Definition 2.1 (Rockafellar, 1972)**:** A subset $\mathcal{C}$ of $\mathbb{R}^m$ is said to be convex if $(1 - \lambda)x + \lambda y \in \mathcal{C}$ whenever $x \in \mathcal{C}$, $y \in \mathcal{C}$ and $0 < \lambda < 1$. The convex hull of a finite set of points $X = \{x_1, \dots, x_n\}$ in $\mathbb{R}^m$ is the minimal convex set containing all points in X, denoted by $Co\{X\}$. Particularly, $Co\{X\} \triangleq \{\sum_{i=1}^{n} \alpha_i x_i | x_i \in X, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1\}$.

Definition 2.2: The heterogeneous multi-agent system (1–2) is said to solve the containment control problem if for any initial conditions, the position of the followers converges to the convex hull spanned by those of the leaders under a certain control input.

2.3 Key lemmas

In this subsection, some key lemmas are given to be used to prove our main results.

Lemma 2.3 (Liu et al., 2012b)**:** *Consider the first-order multi-agent system*

$$\begin{cases} \dot{x}_i(t) = \sum\limits_{j \in \mathscr{F} \cup \mathscr{R}} a_{ij}(x_j(t) - x_i(t)), & i \in \mathscr{F}, \\ \dot{x}_i(t) = 0, & i \in \mathscr{R}, \end{cases} \tag{3}$$

under a fixed directed network for $i = 1, 2, \dots, n$. All followers will converge to the stationary convex hull spanned by the leaders for arbitrary initial conditions if and only if the directed graph $\mathscr{G}$ contains a directed spanning forest.

Lemma 2.4 (Liu et al., 2012b)**:** *Suppose that the multi-agent system has m leaders and $n - m$ followers. Then, $\mathscr{L}$ can be partitioned as $\mathscr{L} = \begin{pmatrix} \mathscr{L}_{\mathscr{F}\mathscr{F}} & \mathscr{L}_{\mathscr{F}\mathscr{R}} \\ 0_{m\times(n-m)} & 0_{m\times m} \end{pmatrix}$ and $\mathscr{L}_{\mathscr{F}\mathscr{F}}$ is invertible if and only if the directed graph $\mathscr{G}$ has a directed spanning forest.*

Consider the autonomous system

$$\dot{x} = f(x), \tag{4}$$

where $f : D \to \mathbb{R}^n$ is a continuous function with $D \subset \mathbb{R}^n$.

Lemma 2.5 (Lasalle's Invariance Principle)**:** *Let $\Omega \subset D$ be a compact set that is positively invariant with respect to (4). Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.*

A function $V(x)$ is homogeneous of degree $\sigma > 0$ with dilation $(r_1, r_2, \dots, r_n)$, $r_i > 0 (i \in \mathcal{I}_n)$, if

$$V(\varepsilon^{r_1}x_1, \varepsilon^{r_2}x_2, \dots, \varepsilon^{r_n}x_n) = \varepsilon^{\sigma}V(x), \varepsilon > 0.$$

A vector field $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$ is homogenous of degree $\sigma > 0$ with dilation $(r_1, r_2, \dots, r_n)$, $r_i > 0 (i \in \mathcal{I}_n)$, if

$$f_i(\varepsilon^{r_1}x_1, \varepsilon^{r_2}x_2, \dots, \varepsilon^{r_n}x_n) = \varepsilon^{\sigma + r_i}f_i(x), i \in \mathcal{I}_n, \varepsilon > 0.$$

Lemma 2.6 (Hong, 2002)**:** *Suppose that the system (4) is homogeneous of degree σ with dilation $(r_1, r_2, \dots, r_n)$, function f(x) is continuous and $x = 0$ is its asymptotically stable equilibrium. If homogeneity degree $\sigma < 0$, the equilibrium of the system (4) is finite-time stable.*

3. Main results

3.1 Leaders with first-order integrator dynamics

Two well-known linear protocols for first-order and second-order multi-agent systems are presented in Olfati-Saber and Murray (2004) and Xie and Wang (2007), respectively. Based on the aforementioned linear protocols, we proposed a linear protocol for the heterogeneous multi-agent system (1–2) in Zheng et al. (2011b) as follows:

$$u_i = \begin{cases} \sum\limits_{j=1}^{n} a_{ij}(x_j - x_i) - k_1 v_i, & i \in \mathcal{I}_m, \\ k_2 \sum\limits_{j=1}^{n} a_{ij}(x_j - x_i), & i \in \mathcal{I}_n/\mathcal{I}_m, \end{cases} \tag{5}$$

where $\mathscr{A} = [a_{ij}]_{n\times n}$ is the weighted adjacency matrix, $k_1 > 0$, $k_2 > 0$ are the feedback gains. In this subsection, we study the containment control problem of heterogeneous multi-agent systems using protocol (5). First, we make an assumption for heterogeneous multi-agent systems with protocol (5).

Assumption 1: Suppose that the first-order integrator agents are the leaders and the second-order integrator agents are the followers.

Theorem 3.1: *Consider a directed network under Assumption 1. Assume that the feedback gain $k_1 > 2\sqrt{\max_{i\in\mathcal{I}_m} d_{ii}}$. Then, the heterogeneous multi-agent system (1–2) with protocol (5) solves the containment control problem if and only if the directed graph $\mathscr{G}$ contains a directed spanning forest.*

Proof: Sufficiency. The sufficiency is proved through the following three steps.
Step 1: From Assumption 1, we know that the first-order integrator agents are leaders, i.e. $a_{ij} = 0$ for $i \in \mathcal{I}_n/\mathcal{I}_m$, $j \in \mathcal{I}_n$. Thus, for $i \in \mathcal{I}_n/\mathcal{I}_m$, we have

$$\dot{x}_i(t) = 0.$$

Let $y_{i'}(t) = k_{3i} v_i(t) + x_i(t)$ for $i, i' \in \mathcal{I}_m$. Thus, for $i \in \mathcal{I}_m$, we have

$$\dot{x}_i(t) = \frac{1}{k_{3i}}(y_{i'}(t) - x_i(t))$$

and

$$\begin{aligned}\dot{y}_{i'}(t) &= k_{3i}\dot{v}_i(t) + \dot{x}_i(t)\\ &= k_{3i}\Big(\sum_{j=1}^{n} a_{ij}(x_j - x_i) - k_1 v_i\Big) + \frac{1}{k_{3i}}(y_{i'}(t) - x_i(t)\big)\\ &= k_{3i}\sum_{j=1}^{n} a_{ij}(x_j - x_i) + \left(k_1 - \frac{1}{k_{3i}}\right)(x_i - y_{i'})\\ &= k_{3i}\sum_{j=1}^{n} a_{ij}(x_j - y_{i'}) + \left(k_1 - \frac{1}{k_{3i}} - k_{3i}d_{ii}\right)(x_i - y_{i'}).\end{aligned}$$

Let $b_i = (k_1 - \frac{1}{k_{3i}} - k_{3i}d_{ii})$, $i \in \mathcal{I}_m$. If $d_{ii} = 0$, let $k_{3i} \geq \frac{1}{k_1}$. Otherwise, let $\frac{k_1 - \sqrt{k_1^2 - 4d_{ii}}}{2d_{ii}} \leq k_{3i} \leq \frac{k_1 + \sqrt{k_1^2 - 4d_{ii}}}{2d_{ii}}$. Due to $k_1 > 2\sqrt{\max_{i\in\mathcal{I}_m} d_{ii}} \geq 0$, we get $\frac{k_1 - \sqrt{k_1^2 - 4d_{ii}}}{2d_{ii}} > 0$, $k_{3i} > 0$ and $b_i \geq 0$ for $i \in \mathcal{I}_m$.

Because the first-order integrator agents are the leaders and the second-order integrator agents are the followers, we get a first-order multi-agent system with $n + m$ agents as follows:

$$\begin{cases}\dot{x}_i(t) = \dfrac{1}{k_{3i}}(y_{i'} - x_i), & i \in \mathscr{F},\\ \dot{y}_{i'}(t) = k_{3i}\displaystyle\sum_{j\in\mathscr{F}\cup\mathscr{R}} a_{ij}(x_j - y_{i'}) + b_i(x_i - y_{i'}), & i' \in \mathscr{F},\\ \dot{x}_i(t) = 0, & i \in \mathscr{R},\end{cases} \tag{6}$$

where $x_i \in \mathbb{R}^N$ and $y_{i'} \in \mathbb{R}^N$ are the states of the ith and i'th agents, respectively.

Thus, it is easy to prove that the heterogeneous multi-agent system (1–2) with protocol (5) solves the containment control problem if the first-order multi-agent system (6) solves the containment control problem.
Step 2: Let $\mathscr{V}_1 = \{s_1, \ldots, s_m\}$, $\mathscr{V}_2 = \{s_{1'}, \ldots, s_{m'}\}$ and $\mathscr{V}_3 = \{s_{m+1}, \ldots, s_n\}$. Let $\mathscr{G}' = (\mathscr{V}', \mathscr{E}')$ be a fixed directed network of the first-order multi-agent system (6) with a vertex set $\mathscr{V}' = \mathscr{V}_1 \cup \mathscr{V}_2 \cup \mathscr{V}_3$. In this step, we will prove that the directed graph $\mathscr{G}'$ contains a directed spanning forest if the directed graph $\mathscr{G}$ contains a directed spanning forest.

Suppose that the directed graph $\mathscr{G}$ has a directed spanning forest $\mathscr{F}(\mathscr{G})$. For each edge $(s_j, s_i) \in \mathscr{F}(\mathscr{G})$, we consider the following two cases:

(1) If $s_j \in \mathscr{V}_3$ and $s_i \in \mathscr{V}_1$, we have $(s_j, s_{i'}) \in \mathscr{E}'$, $(s_{i'}, s_i) \in \mathscr{E}'$;
(2) If $s_j \in \mathscr{V}_1$ and $s_i \in \mathscr{V}_1$, we have $(s_{j'}, s_j) \in \mathscr{E}'$, $(s_j, s_{i'}) \in \mathscr{E}'$, $(s_{i'}, s_i) \in \mathscr{E}'$.

Adding these edges to $\mathscr{F}(\mathscr{G})$, we get a directed spanning tree $\mathscr{F}'(\mathscr{G})$ for $\mathscr{G}'$.
Step 3: From Lemma 2.3, we know that the first-order multi-agent system (6) solves the containment control problem if the directed graph $\mathscr{G}'$ contains a directed spanning forest.

Combining *Step* 1, *Step* 2 with *Step* 3, we get that if the feedback gain $k_1 > 2\sqrt{\max_{i\in\mathcal{I}_m} d_{ii}}$ and Assumption 1 holds, the heterogeneous multi-agent system (1–2) with protocol (5) solves the containment control problem if the directed graph $\mathscr{G}$ contains a directed spanning forest.

Necessity. The proof of necessity is similar to the proof of necessity in Theorem 1 of Liu et al. (2012b). When the directed graph $\mathscr{G}$ does not contain a directed spanning forest, there exists at least one follower (the second-order integrator agent) such that it does not belong to any one of the directed trees. The position of this follower is independent of the position of the leaders. Thus, the containment control problem cannot be solved. □

In fact, the approach in *Step* 1 has also been used when dealing with the consensus problem for second-order multi-agent systems in Qin et al. (2011) and Qin and Gao (2012) and heterogeneous multi-agent systems in Zheng and Wang (2012b). For clarity, we give an example to illustrate *Step* 2 of the sufficiency proof.

Example 3.2: Consider a heterogeneous multi-agent system that is composed of six agents with a directed network $\mathscr{G}$ in the left of Figure 1 . The vertices $1-3$ denote the second-order integrator agents and the vertices $4-6$ denote the first-order integrator agents. The solid lines and the vertices $1-6$ of $\mathscr{G}$ compose a directed spanning forest. Using the method in the *Step* 2 of the aforementioned proof,

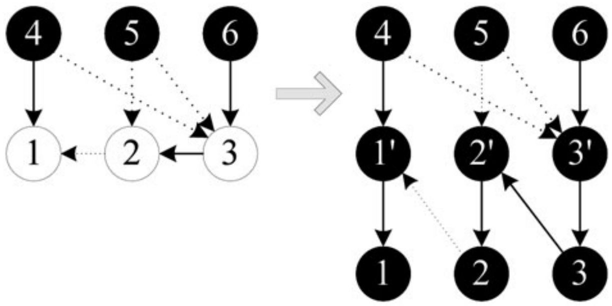

Figure 1. A transformation of directed spanning forest.

we get a directed spanning forest in the right of Figure 1 that is composed of solid lines and the vertices $1-6$, $1'-3'$.

Remark 1: In this paper, we consider the network of the leaders without communication. In fact, the leaders communicate with each other in many practical systems. Let $u_i(t)=\sum_{j\in\mathscr{R}} a_{ij}(x_j-h_j)-(x_i-h_i)$ for $i\in\mathcal{I}_n/\mathcal{I}_m$. Using the formation control theory, the method in Theorem 3.1 and the result of Theorem 5.2.1 in Liu (2012), we can easily obtain that if the feedback gain $k_1>2\sqrt{\max_{i\in\mathcal{I}_m} d_{ii}}$, the heterogeneous multi-agent system (1–2) solves the containment control problem if and only if the directed graph $\mathscr{G}$ contains a directed spanning tree.

In Qin et al. (2011), the authors studied the consensus problem of second-order multi-agent systems with both absolute and relative velocity information. Thus, we propose a protocol with both absolute and relative velocity information for the heterogeneous multi-agent system (1–2) as follows:

$$u_i=\begin{cases}\sum_{j=1}^{n} a_{ij}\left[(x_j-x_i)+k(v_j-v_i)\right]-k_1 v_i, & i\in\mathcal{I}_m,\\ k_2\sum_{j=1}^{n} a_{ij}(x_j-x_i), & i\in\mathcal{I}_n/\mathcal{I}_m,\end{cases} \tag{7}$$

where $\mathscr{A}=[a_{ij}]_{n\times n}$ is the weighted adjacency matrix, $k>0$, $k_1>0$, $k_2>0$ are the feedback gains. Similar to the proof of Theorem 2 of Qin et al. (2011) and Theorem 3.1, we can get a corollary as follows:

Corollary 3.3: *Consider a directed network under Assumption 1. Assume that the feedback gain $k\geq\frac{1}{k_1}$. Then, the heterogeneous multi-agent system (1–2) with protocol (7) solves the containment control problem if and only if the directed graph $\mathscr{G}$ contains a directed spanning forest.*

Remark 2: In Liu et al. (2012b), the authors also investigated the containment control of first-order multi-agent systems for sampled date-based protocol. Therefore, it is easy to get the necessary and sufficient condition so that discrete-time first-order multi-agent systems solve the containment control problem. By using the method in Theorem 3.1, we can obtain related results about the containment control of discrete-time heterogeneous multi-agent systems, which is left to the interested readers as an exercise.

3.2 Leaders with second-order integrator dynamics

In this subsection, we study the containment control problem of heterogeneous multi-agent systems using a class of nonlinear protocol. First, we propose a class of nonlinear protocol as follows:

$$u_i=\begin{cases} sig\left(\sum_{j=1}^{n} a_{ij}(x_j-x_i)-k_1 v_i\right)^{\alpha_1}, & i\in\mathcal{I}_m,\\ k_2 sig\left(\sum_{j=1}^{n} a_{ij}(x_j-x_i)\right)^{\alpha_2}, & i\in\mathcal{I}_n/\mathcal{I}_m.\end{cases} \tag{8}$$

where $\mathscr{A}=[a_{ij}]_{n\times n}$ is the weighted adjacency matrix, $0<\alpha_1,\alpha_2<1$, $k_1>0$, $k_2>0$ are the feedback gains. Then, we make an assumption for heterogeneous multi-agent systems with protocol (8).

Assumption 2: Suppose that the second-order integrator agents are the leaders and the first-order integrator agents are the followers.

Theorem 3.4: *Consider a directed network under Assumption 2. The heterogeneous multi-agent system (1–2) with protocol (8) solves the containment control problem in finite time if and only if the directed graph $\mathscr{G}$ contains a directed spanning forest.*

Proof: Sufficiency. From Assumption 2, we know that the second-order integrator agents are leaders, i.e. $a_{ij}=0$ for $i\in\mathcal{I}_m$, $j\in\mathcal{I}_n$. Thus, for $i\in\mathcal{I}_m$, we have

$$\begin{cases}\dot{x}_i(t)=v_i(t),\\ \dot{v}_i(t)=sig(-k_1 v_i)^{\alpha_1}.\end{cases} \tag{9}$$

Because $\dot{v}_i(t)=sig(-k_1v_i)^{\alpha_1}=-k_1 sig(v_i)^{\alpha_1}$ is finite-time stable, i.e. there exists a $T_1<\infty$ such that $\lim_{t\to T_1^-} v_i=0$ and $v_i=0$ when $t\geq T_1$ for $i\in\mathcal{I}_m$. Thus, when $t\geq T_1$, we have

$$\begin{cases}\dot{x}_i(t)=0, & i\in\mathscr{R},\\ \dot{x}_i(t)=k_2 sig\left(\sum_{j=1}^{n} a_{ij}(x_j-x_i)\right)^{\alpha_2}, & i\in\mathscr{F}.\end{cases} \tag{10}$$

Without a loss of generality, we consider that the network of the followers is undirected. For the directed network, the analysis method is similar to the finite-time consensus analysis in Wang and Xiao (2010).

Let $y_i = \sum_{j=1}^{n} a_{ij}(x_j - x_i)$ and $x = [x_{\mathscr{R}}, x_{\mathscr{F}}]^T = [x_1, \ldots, x_m, x_{m+1}, \ldots, x_n]^T$, $y = [y_1, y_2, \ldots, y_n]^T$. Then, from (10), we have $y = -(\mathscr{L} \otimes I_N)x$ and $\dot{x}_i = k_2 sig(y_i)^{\alpha_2}$ for $i \in \mathcal{I}_n$. Consider a Lyapunov function

$$V(t) = \sum_{i=1}^{n} \frac{1}{\alpha_2 + 1} \mid y_i \mid^{\alpha_2+1},$$

which is positive-definite with respect to y_i for $i \in \mathcal{I}_n$. Differentiating $V(t)$, gives

$$\begin{aligned} \frac{dV(t)}{dt} &= \sum_{i=1}^{n} sig(y_i)^{\alpha_2} \frac{dy_i}{dt} \\ &= \sum_{i=1}^{n} sig(y_i)^{\alpha_2} \sum_{j=1}^{n} a_{ij}\left(\frac{dx_j}{dt} - \frac{dx_i}{dt}\right) \\ &= \sum_{i=1}^{n} sig(y_i)^{\alpha_2} \sum_{j=1}^{n} k_2 a_{ij}\left(sig(y_j)^{\alpha_2} - sig(y_i)^{\alpha_2}\right). \end{aligned}$$

Let $sig(y)^{\alpha_2} = [sig(y_1)^{\alpha_2}, sig(y_2)^{\alpha_2}, \ldots, sig(y_n)^{\alpha_2}]^T$ and $sig(y^T)^{\alpha_2} = (sig(y)^{\alpha_2})^T$. Then, we have

$$\frac{dV(t)}{dt} = -k_2 sig(y^T)^{\alpha_2}(\mathscr{L} \otimes I_N) sig(y)^{\alpha_2} \leq 0.$$

Note that $\frac{dV(t)}{dt} = 0$ implies that $sig(y_i)^{\alpha_2} = sig(y_j)^{\alpha_2}$ for $i, j \in \mathcal{I}_n$. Owing to $sig(y_i)^{\alpha_2} = 0$ when $t \geq T_1$ for $i \in \mathcal{I}_m$, we have $y_i = 0$ if $\frac{dV(t)}{dt} = 0$ for $i \in \mathcal{I}_n$. By Lemma 2.5, we get $y = -(\mathscr{L} \otimes I_N)x \to 0$ as $t \to \infty$.

Note that the system (10) is a homogeneous system of degree $\sigma = 1 - \frac{1}{\alpha_2} < 0$ with dilation $(\alpha_2, \ldots, \alpha_2)$. Therefore, there exists a T_2 $(T_1 < T_2 < \infty)$ such that $y = -(\mathscr{L} \otimes I_N)x \to 0$ in finite time by Lemma 2.6. Thereby, $(\mathscr{L}_{\mathscr{F}\mathscr{F}} \otimes I_N)x_{\mathscr{F}} + (\mathscr{L}_{\mathscr{F}\mathscr{R}} \otimes I_N)x_{\mathscr{R}} = 0$ when $t \geq T_2$. Because the directed graph $\mathscr{G}$ contains a directed spanning forest, we have $x_{\mathscr{F}} = -\left((\mathscr{L}_{\mathscr{F}\mathscr{F}}^{-1}\mathscr{L}_{\mathscr{F}\mathscr{R}}) \otimes I_N\right)x_{\mathscr{R}}$ from Lemma 2.4. Thus, by Definition 2.1, we know that the heterogeneous multi-agent system (1–2) with protocol (8) solves the containment control problem in finite time if the directed graph $\mathscr{G}$ contains a directed spanning forest.

Necessity. When the directed graph $\mathscr{G}$ does not contain a directed spanning forest, there exists at least one follower (the first-order integrator agent) such that it does not belong to any one of the directed trees. The position of this follower is independent of the position of the leaders. Thus, the containment control problem cannot be solved. □

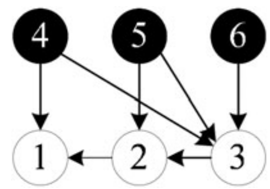

Figure 2. A directed graph $\mathscr{G}$ in which the first-order integrator agents are leaders.

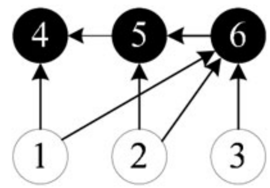

Figure 3. A directed graph $\mathscr{G}$ in which the second-order integrator agents are leaders.

4. Simulations

In this section, we give two numerical simulations to illustrate the effectiveness of the theoretical results in Section 3. Let $N = 2$.

Example 4.1: Consider a directed graph $\mathscr{G}$ depicted in Figure 2 in which the first-order integrator agents (the filled circles) are leaders. It is easy to see that the directed graph $\mathscr{G}$ depicted in Figure 2 has a directed spanning forest. Suppose that the weight of each edge is 1, $k_1 = 4$, $k_2 = 1$. Thus, $k_1 > 2\sqrt{\max_{i \in \mathcal{I}_m} d_{ii}}$. Figure 4 shows the state trajectories of all the agents using protocol (5). We can see that the second-order integrator agents $1 - 3$ converge to the convex hull spanned by the first-order integrator agents $4 - 6$, which is consistent with the sufficiency of Theorem 3.1.

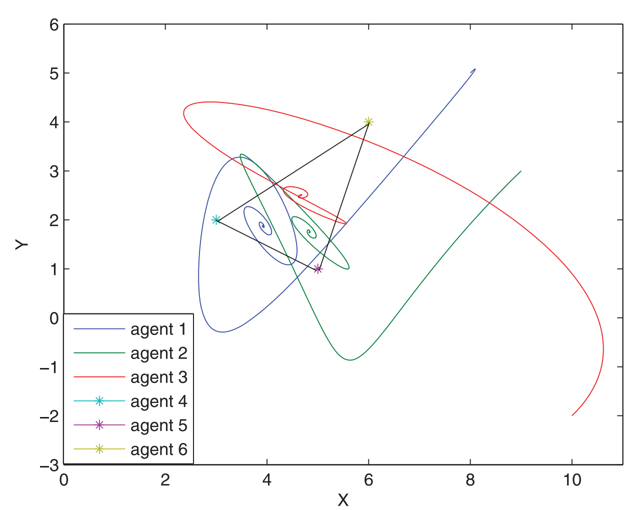

Figure 4. Trajectories of all the agents with the network depicted in Figure 2 and protocol (5).

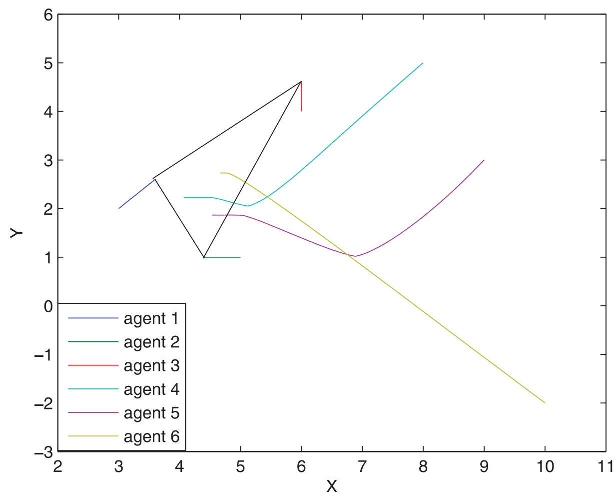

Figure 5. Trajectories of all the agents with the network depicted in Figure 3 and protocol (8).

Example 4.2: Suppose the communication network $\mathscr{G}$ is chosen as in Figure 3 in which the second-order integrator agents (the hollow circles) are leaders. We can see that the directed graph $\mathscr{G}$ depicted in Figure 3 has a directed spanning forest. Assume that the weight of each edge is 1, $k_1 = k_2 = 1$, $\alpha_1 = \alpha_2 = \frac{1}{3}$. Figure 5 shows the state trajectories of all the agents using protocol (8). We can see that the first-order integrator agents $4 - 6$ converge to the convex hull spanned by the second-order integrator agents $1 - 3$, which is consistent with the sufficiency of Theorem 3.4.

5. Conclusion

In this paper, the containment control problem of the heterogeneous multi-agent system with agents modelled by the first-order and second-order integrators was considered. We present the linear and nonlinear protocols for a heterogeneous multi-agent system if the leaders are first-order and second-order integrator agents, respectively. Based on the graph theory, Lyapunov theory and previous results of a homogeneous multi-agent system, we get some necessary and sufficient conditions so that the heterogeneous multi-agent system solves the containment control problem. Future work will focus on the more complex cooperative control problem of heterogeneous multi-agent systems, for example, the containment control of heterogeneous multi-agent systems with switching topologies and so on.

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