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Distributed consensus of heterogeneous multi-agent systems with fixed and switching topologies

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In this article, we study distributed consensus of heterogeneous multi-agent systems with fixed and switching topologies. The analysis is based on graph theory and nonnegative matrix theory. We propose two kinds of consensus protocols based on the consensus protocol of first-order and second-order multi-agent systems. Some necessary and sufficient conditions that the heterogeneous multi-agent system solves the consensus problems under different consensus protocols are presented with fixed topology. We also give some sufficient conditions for consensus of the heterogeneous multi-agent system with switching topology. Simulation examples are provided to demonstrate the effectiveness of the theoretical results.

Keywords: heterogeneous multi-agent system; consensus; directed network; switching topology; graph theory

1. Introduction

Over recent years, distributed coordination of multi-agent systems has received a major attention of multi-disciplinary researchers coming from system control theory, applied mathematics, statistical physics, biology, communication, computer science, etc. This is partly due to its broad applications of multi-agent systems in many areas, such as the formation control of robotic systems, the cooperative control of unmanned aerial vehicles, the attitude alignment of satellite clusters, the target tracking of sensor networks, the congestion control of communication networks, and so on (Ren and Beard 2008). A critical problem arising from multi-agent systems for coordinated control is how to design appropriate control input based on local information that enables all agents to reach an agreement on consistent quantity of interest, which is known as the consensus problem.

In the literature related to the consensus problem, agents are primarily considered being governed by first-order integrators. Vicsek, Czirok, Jacob, Cohen, and Schochet (1995) proposed a simple model for the phase transition of a group of self-driven particles and demonstrated by simulation that the headings of all agents converge to a common value. By virtue of graph theory and matrix theory, Jadbabaie, Lin, and Morse (2003) provided a theoretical explanation for the observed behaviour of the Vicsek model. It was shown that consensus can be achieved if the union of the interaction graphs for the team are connected frequently enough as the system evolves. Olfati-Saber and Murray (2004) investigated a systematical framework of consensus problem in network of agents with a simple scalar continuous-time integrator. They studied three consensus problems, namely, directed networks with fixed topology, directed networks with switching topology, and undirected networks with time-delay and fixed topology. Ren and Beard (2005) extended the results of Jadbabaie et al. (2003), Olfati-Saber and Murray (2004) and presented some more relaxable conditions for consensus of states under dynamically changing interaction topologies. Moreau (2005) studied the non-linear discrete-time multi-agent systems with time-dependent communication channels, and introduced a novel method based on the notion of convexity. In the past several years, investigation for consensus problem of first-order multi-agent systems has been developed very fast and several research topics have been considered, such as consensus with nonlinear protocol (Arcak 2007; Hui and Haddad 2008), consensus with switching topology and time-delays (Xiao and Wang 2006; Wang and Xiao 2007; Sun, Wang, and Xie 2008), consensus with noises (Huang and Manton 2009), asynchronous consensus (Xiao and Wang 2008), consensus over random networks (Hatano and Mesbahi 2005; Tahbaz-Salehi

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and Jadbabaie 2008), finite-time consensus (Jiang and Wang 2009; Wang and Xiao 2010; Zheng, Chen, and Wang 2011a), group consensus (Yu and Wang 2010). In the meanwhile, there is a growing interest in consensus protocols where all agents are governed by second-order integrators. For example, Xie and Wang (2007) and Ren and Atkins (2007) proposed second-order consensus protocols based on the absolute and relative velocity information, respectively, and provided the sufficient conditions for consensus problem of second-order multi-agent systems with fixed and switching topologies. Ren (2008) studied the consensus problems of second-order multi-agent in four cases: (1) with a bounded control input, (2) without velocity measurements, (3) with a group reference velocity, (4) with a bounded control input when a group reference state is available to only a subset of the team. Other research topics for consensus of second-order multi-agent systems were considered, such as the necessary and sufficient condition of consensus (Jiang and Wang 2010), consensus based on observer (Hong, Chen, and Bushnell 2008), consensus based on sampled-data (Gao, Wang, Xie, and Wu 2009), finite-time consensus (Wang and Hong 2008). Up to now, by virtue of matrix theory, graph theory, frequency-domain analysis method, Lyapunov direct method, etc., consensus problem of first-order/second-order multi-agent systems has been studied. For more details, one can refer to survey papers Ren, Beard, and Atkins (2007b) and Olfati-Saber, Fax, and Murray (2007) and the references therein.

All the aforementioned results were concerned with the consensus of homogeneous multi-agent systems, i.e. all the agents have the same dynamics behaviours. However, the dynamics of the agents coupled with each others are not the same because of various restrictions or the common goals with mixed agents in the practical systems. For example, taking dynamic environments and uncertainty external to the multi-robot system itself into account, heterogeneous systems with robots in different shapes and abilities are more applicable than the homogeneous systems in real world (Wang, Wu, Huang, and Wang 2008). Stationary consensus was studied for discrete-time heterogeneous multi-agent systems composed of first-order and second-order agents with communication delays in Liu and Liu (2011). Consensus of continuous-time heterogeneous multi-agent systems was considered under undirected graph with velocity measurements in Zheng, Zhu, and Wang (2011b) and without velocity measurements in Zheng and Wang (2012a). Finite-time consensus of heterogeneous multi-agent systems with and without velocity measurements was studied under strongly connected graph in Zheng and Wang (2012b). By using Lyapunov method, we obtained some sufficient conditions in Zheng et al. (2011a), Zheng and Wang (2012a, b).

Inspired by the recent developments in heterogeneous multi-agent systems, we try to further investigate the consensus problems with general directed network. The main aim of this article is to obtain the consensus criterions of heterogeneous multi-agent systems in directed network with fixed and switching topologies. One of the challenge is that the heterogeneous feature prevents the application of diagonalisation of Laplacian matrix directly. The main contribution of this article is threefold. For one thing, we propose two kinds of consensus protocols based on the consensus protocols of first-order and second-order multi-agent systems. Secondly, some necessary and sufficient conditions that the heterogeneous multi-agent system solves the consensus problems under different consensus protocols are presented in directed network with fixed topology. Thirdly, we give some sufficient conditions for consensus of the heterogeneous multi-agent system with switching topology. Simulation examples are worked out to illustrate the effectiveness of our theoretical results.

The rest of this article is organised as follows. In Section 2, we present preliminaries and problem formulation. In Section 3, we give the main results. In Section 4, numerical simulations are given to illustrate the effectiveness of theoretical results. Finally, some conclusions are drawn in Section 5.

Notations: Throughout this article, we let $\mathbb{R}$ be the set of real number, $\mathbb{R}^n$ be the n-dimensional real vector space, $\mathbb{R}^{n\times n}$ be the set of $n\times n$ matrix. $\mathcal{I}_m=\{1,2,\dots,m\}$, $\mathcal{I}_n/\mathcal{I}_m=\{m+1, m+2,\dots,n\}$. For a given vector or matrix X, X^Tdenotes its transpose, $\|X\|$ denotes the Euclidean norm of a vector X. I_n is a $n\times n$ identity matrix. $\text{diag}\{a_1,a_2,\dots,a_n\}$ defines a diagonal matrix with diagonal elements being $a_1,a_2,\dots,a_n$. Matrix $A=[a_{ij}]$ is said to be nonnegative (resp. positive) if all entries a_{ij} are nonnegative (resp. positive), denoted by $A\geq 0$ (resp. $A>0$).

2. Preliminaries and problem formulation

In this section, some basic concepts and results about algebraic graph theory are introduced firstly. For more details about algebraic graph theory, one can refer to (Godsil and Royal 2001). Then, we formulate the problem to be studied.

Let $G(t)=(V, E(t), A(t))$ be a weighted directed graph of order n $(n\geq 2)$ with a vertex set $V=\{s_1, s_2,\dots,s_n\}$, an edge set $E(t)=\{e_{ij}=(s_i,s_j)\}\subset V\times V$ and a nonnegative nonsymmetric adjacency matrix $A(t)=[a_{ij}(t)]_{n\times n}$ at time t. $(s_j,s_i)\in E(t)\Leftrightarrow a_{ij}(t)>0$, namely, they are adjacent at time instant t. Moreover, we assume $a_{ii}(t) = 0$ for any time t. The set of neighbours of s_i is denoted by $N_i(t) = \{s_j : e_{ji} = (s_j, s_i) \in E(t)\}$. A directed path that connects s_i and s_j in the graph G is a sequence of distinct vertices $s_{i_0}, s_{i_1}, s_{i_2}, \ldots, s_{i_m}$, where $s_{i_0} = s_i, s_{i_m} = s_j$ and $(s_{i_r}, s_{i_{r+1}}) \in E$, $0 \le r \le m-1$. If a directed graph has the property that $(s_i, s_j) \in E \Leftrightarrow (s_j, s_i) \in E$, the directed graph is called undirected. A directed graph is called strongly connected (connected for undirected graph) if any two distinct nodes of the graph can be connected via a directed path (path for undirected graph) that follows the edges of the graph. For directed graph, if (s_i, s_j) is an edge of G, s_i is called the parent of s_j and s_j is called the child of s_i. A directed tree is a directed graph, where every vertex, except one special vertex without any parent, which is called the root, has exactly one parent, and the root can be connected to any other vertices through paths. A directed spanning tree is a directed tree, which consists of all the nodes and some edges in G. The degree matrix $D(t) = [d_{ij}(t)]_{n\times n}$ is a diagonal matrix with $d_{ii}(t) = \sum_{j: s_j \in N_i(t)} a_{ij}(t)$, and the Laplacian matrix of the graph is defined as $L(t) = [l_{ij}(t)]_{n\times n} = D(t) - A(t)$.

Suppose that the heterogeneous multi-agent system consists of first-order and second-order integrator agents. The number of agents is n, labelled 1 through n, where the number of second-order integrator agents is m $(m < n)$. Each agent has the dynamics as follows:

$$\begin{cases} \dot{x}_i = v_i, \quad \dot{v}_i = u_i, & i \in \mathcal{I}_m, \\ \dot{x}_i = u_i, & i \in \mathcal{I}_n/\mathcal{I}_m. \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the position-like, velocity-like and control input, respectively, of agent i. The initial conditions are $x_i(0) = x_{i0}$, $v_i(0) = v_{i0}$. Let $x(0) = [x_{10}, x_{20}, \ldots, x_{n0}]^T$, $v(0) = [v_{10}, v_{20}, \ldots, v_{m0}]^T$.

Each agent is regarded as a node in a directed graph, $G(A(t))$. Each edge $(s_i, s_j) \in E(t)$ corresponds to an available information link from agent i to agent j at time t. Moreover, each agent updates its current state based on the information received from its neighbours. In this article, we suppose that there exists communication behaviour in multi-agent systems, i.e. there are agent i and agent j which make $a_{ij} > 0$.

Definition 2.1: The heterogeneous multi-agent system (1) is said to reach consensus if for any initial conditions, we have

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \quad \text{for } i,j \in \mathcal{I}_n, \quad \text{and}$$
$$\lim_{t\to\infty} \|v_i(t) - v_j(t)\| = 0, \quad \text{for } i,j \in \mathcal{I}_m.$$

To solve the consensus problem of heterogeneous multi-agent system (1) is a challenging work. It needs to find suitable distributed state feedback consensus protocol for each agent not only to solve the consensus of the position-like but also to solve the consensus of velocity-like. On the other hand, the linear consensus protocol has been widely applied for homogeneous multi-agent systems. For first-order multi-agent systems, Olfati-Saber and Murray (2004) proposed a linear consensus protocol as follows:

$$u_i = \sum_{j=1}^{n} a_{ij}(t)(x_j - x_i). \tag{2}$$

And for second-order multi-agent systems, Xie and Wang (2007) proposed a linear consensus protocol as follows:

$$u_i = \sum_{j=1}^{n} a_{ij}(t)(x_j - x_i) - kv_i. \tag{3}$$

Ren (2008) gave another consensus protocol for second-order multi-agent systems with a group reference velocity, i.e.

$$u_i = \sum_{j=1}^{n} a_{ij}(t)(x_j - x_i) - k(v_i - v^d) + \dot{v}^d. \tag{4}$$

Based on the aforementioned analysis, we propose the consensus protocol (control input) for heterogeneous multi-agent system (1) as follows:

$$u_i = \begin{cases} \sum_{j=1}^{n} a_{ij}(t)(x_j - x_i) - k_1 v_i, & i \in \mathcal{I}_m, \\ k_2 \sum_{j=1}^{n} a_{ij}(t)(x_j - x_i), & i \in \mathcal{I}_n/\mathcal{I}_m, \end{cases} \tag{5}$$

or

$$u_i = \begin{cases} \sum_{j=1}^{n} a_{ij}(t)(x_j - x_i) - k_1(v_i - v^d) + \dot{v}^d, & i \in \mathcal{I}_m, \\ k_2 \sum_{j=1}^{n} a_{ij}(t)(x_j - x_i) + v^d, & i \in \mathcal{I}_n/\mathcal{I}_m, \end{cases} \tag{6}$$

where $A = [a_{ij}(t)]_{n\times n}$ is the aforementioned weighted adjacency matrix associated with the graph $G(t)$ at time instant t, $k_1 > 0$, $k_2 > 0$ are the feedback gains, v^d is the time-varying group reference velocity.

3. Main results

In this section, the consensus problem of heterogeneous multi-agent system (1) under protocols (5) and (6) will be considered for networks with fixed and switching topologies, respectively. We first discuss the consensus problem in directed network with fixed topology and obtain some necessary and sufficient conditions in Section 3.1. Then, some sufficient conditions for consensus of the heterogeneous multi-agent system with switching topology is considered in Section 3.2.

3.1 *Networks with fixed topology*

In this subsection, we will focus on analysis the consensus problem under protocols (5) and (6) in directed network with fixed topology, i.e. $G(t) \equiv G$ for any time t. First, a lemmas is given which is a summary of the work in Ren and Beard (2005).

Lemma 3.1: *Suppose that $\xi = [\xi_1, \ldots, \xi_n]^T$ and L is the corresponding Laplacian matrix of directed network G. Then, the first-order multi-agent system $\dot{\xi} = -L\xi$ can solve the consensus problem if and only if the directed network G has a directed spanning tree.*

Theorem 3.2: *Consider a directed network with fixed topology. Assume that the feedback gain $k_1 > 2\sqrt{\max_{i \in \mathcal{I}_m} d_{ii}}$. Then, the heterogeneous multi-agent system* (1) *with consensus protocol* (5) *reaches consensus asymptotically if and only if the fixed topology G has a directed spanning tree.*

Proof: This theorem is proved through the following three steps.

Step 1: Let $y_{i'} = k_{3i} v_i + x_i$ $(i, i' \in \mathcal{I}_m)$. Thus, for $i \in \mathcal{I}_m$,

$$\dot{x}_i = \frac{1}{k_{3i}}(y_{i'} - x_i)$$

and

$$\begin{aligned} \dot{y}_{i'} &= k_{3i}\dot{v}_i + \dot{x}_i \\ &= k_{3i}\left(\sum_{j=1}^{n} a_{ij}(x_j - x_i) - k_1 v_i\right) + \frac{1}{k_{3i}}(y_{i'} - x_i) \\ &= k_{3i}\sum_{j=1}^{n} a_{ij}(x_j - x_i) + \left(k_1 - \frac{1}{k_{3i}}\right)(x_i - y_{i'}) \\ &= k_{3i}\sum_{j=1}^{n} a_{ij}(x_j - y_{i'}) + \left(k_1 - \frac{1}{k_{3i}} - k_{3i}d_{ii}\right)(x_i - y_{i'}). \end{aligned}$$

Let $b_i = (k_1 - \frac{1}{k_{3i}} - k_{3i}d_{ii}), i \in \mathcal{I}_m$. If $d_{ii} = 0$, let $k_{3i} \geq \frac{1}{k_1}$. Otherwise, let $\frac{k_1 - \sqrt{k_1^2 - 4d_{ii}}}{2d_{ii}} \leq k_{3i} \leq \frac{k_1 + \sqrt{k_1^2 - 4d_{ii}}}{2d_{ii}}$. Due to $k_1 > 2\sqrt{\max_{i \in \mathcal{I}_m} d_{ii}} \geq 0$, it is easy to get $\frac{k_1 - \sqrt{k_1^2 - 4d_{ii}}}{2d_{ii}} > 0$, $k_{3i} > 0$ and $b_i \geq 0$ for $i \in \mathcal{I}_m$.

For $i \in \mathcal{I}_n / \mathcal{I}_m$, we have

$$\dot{x}_i = k_2 \sum_{j=1}^{n} a_{ij}(x_j - x_i).$$

Based on the aforementioned analysis, we get a first-order multi-agent system with $n+m$ agents as follows:

$$\begin{cases} \dot{x}_i = \dfrac{1}{k_{3i}}(y_{i'} - x_i), & i \in \mathcal{I}_m, \\ \dot{y}_{i'} = k_{3i}\displaystyle\sum_{j=1}^{n} a_{ij}(x_j - y_{i'}) + b_i(x_i - y_{i'}), & i' \in \mathcal{I}_m, \\ \dot{x}_i = k_2 \displaystyle\sum_{j=1}^{n} a_{ij}(x_j - x_i), & i \in \mathcal{I}_n/\mathcal{I}_m, \end{cases} \tag{7}$$

where $x_i \in \mathbb{R}$ and $y_{i'} \in \mathbb{R}$ are the states of the ith and i'th agents, respectively.

The Laplacian matrix of directed network G can be rewritten as $L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$, where $L_{11} \in \mathbb{R}^{m \times m}$, $L_{12} \in \mathbb{R}^{m \times (n-m)}$, $L_{21} \in \mathbb{R}^{(n-m) \times m}$ and $L_{22} \in \mathbb{R}^{(n-m) \times (n-m)}$. Let $X = [x_1, \ldots, x_m, \ y_1, \ldots, y_m, \ x_{m+1}, \ldots, x_n]^T \in \mathbb{R}^{n+m}$, the first-order multi-agent system (7) can be rewritten as

$$\dot{X} = -L'X, \tag{8}$$

where

$$L' = \begin{pmatrix} \frac{1}{K_3} I_m & -\frac{1}{K_3} I_m & \mathbf{0} \\ K_3 L_{11} - B & B & K_3 L_{12} \\ k_2 L_{21} & \mathbf{0} & k_2 L_{22} \end{pmatrix},$$

$K_3 = \mathrm{diag}\{k_{31}, \ldots, k_{3m}\}$, $\frac{1}{K_3} = \mathrm{diag}\{\frac{1}{k_{31}}, \ldots, \frac{1}{k_{3m}}\}$ and $B = \mathrm{diag}\{b_1, \ldots, b_m\}$.

In fact, the velocities of all second-order integrator agents converge to zero if heterogeneous multi-agent system (1) can solve a consensus problem with protocol (5). Because, we have $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$ for i, $j \in \mathcal{I}_n$, which implies that $\lim_{t\to\infty} \|\dot{x}_i(t) - \dot{x}_j(t)\| = 0$ $(i \in \mathcal{I}_m, j \in \mathcal{I}_n/\mathcal{I}_m)$ and $\lim_{t\to\infty} \|\sum_{j=1}^{n} a_{ij}(x_j - x_i)\| = 0$ $(i \in \mathcal{I}_n/\mathcal{I}_m)$, which in turn implies that $\lim_{t\to\infty} \|\dot{x}_i(t) - \dot{x}_j(t)\| = \lim_{t\to\infty} \|v_i(t)\| = 0$ $(i \in \mathcal{I}_m, j \in \mathcal{I}_n/\mathcal{I}_m)$. Then, it is easy to prove that the heterogeneous multi-agent system (1) with consensus protocol (5) reaches consensus asymptotically if and only if the first-order multi-agent system (8) reaches consensus asymptotically.

Step 2: Let $V_1 = \{s_1, \ldots, s_m\}$, $V_2 = \{s_{1'}, \ldots, s_{m'}\}$ and $V_3 = \{s_{m+1}, \ldots, s_n\}$. Let $G' = (V', E')$ be a directed fixed topology of the first-order multi-agent system (7) with a vertex set $V' = V_1 \cup V_2 \cup V_3$.

Suppose that the fixed topology G has directed a spanning tree T_G. For each edge $(s_j, s_i) \in T_G$, we consider the following four cases:

(1) If $s_j \in V_3$ and $s_i \in V_3$, we have $(s_j, s_i) \in E'$;
(2) If $s_j \in V_3$ and $s_i \in V_1$, we have $(s_j, s_{i'}) \in E'$, $(s_{i'}, s_i) \in E'$;

(3) If $s_j \in V_1$ and $s_i \in V_3$, we have $(s_{j'}, s_j) \in E'$, $(s_j, s_i) \in E'$;
(4) If $s_j \in V_1$ and $s_i \in V_1$, we have $(s_{j'}, s_j) \in E'$, $(s_j, s_{i'}) \in E'$, $(s_{i'}, s_i) \in E'$.

Adding these edges to T_G, we get a directed spanning tree for G'.

Suppose that the fixed topology G' has a directed spanning tree $T_{G'}$. For each vertex $s_i \in V_2$, if there exist $s_j, s_k \in V_1 \cup V_3$ which make $(s_j, s_i), (s_i, s_k) \in T_{G'}$, we delete the vertex s_i and add the edge $(s_j, s_k) \in E$. Otherwise, we delete the vertex s_i. Thus, we get a directed spanning tree for G.

Hence, the fixed topology G has a directed spanning tree if and only if the fixed topology G' has a directed spanning tree.

Step 3: From Lemma 3.1, it is easy to know that the first-order multi-agent system (8) reaches consensus asymptotically if and only if the fixed topology G' has a directed spanning tree.

Combining Step 1, Step 2 and Step 3, we get that if the feedback gain $k_1 > 2\sqrt{\max_{i\in\mathcal{I}_m} d_{ii}}$, the heterogeneous multi-agent system (1) with consensus protocol (5) reaches consensus asymptotically if and only if the fixed topology G has a directed spanning tree. □

When $m = n$, the heterogeneous multi-agent system (1) becomes a second-order multi-agent system. Similar to the analysis of Theorem 3.2, we get a necessary and sufficient condition for the second-order multi-agent system (1) as follows.

Corollary 3.3: *Consider a directed network with fixed topology. Assume that the feedback gain* $k_1 \geq 2\sqrt{\max_{i\in\mathcal{I}_n} d_{ii}}$. *Then, the second-order multi-agent system* (1) *with consensus protocol* (3) *reaches consensus asymptotically if and only if the fixed topology G has a directed spanning tree.*

Proof: Note that $\sqrt{\max_{i\in\mathcal{I}_n} d_{ii}} > 0$ for the second-order multi-agent system (1). Hence, we have $k_1 \geq 2\sqrt{\max_{i\in\mathcal{I}_n} d_{ii}} > 0$. Similar to the analysis of Theorem 3.2, we get that if the feedback gain $k_1 \geq 2\sqrt{\max_{i\in\mathcal{I}_n} d_{ii}}$, the second-order multi-agent system (1) with consensus protocol (3) reaches consensus asymptotically if and only if the fixed topology G has a directed spanning tree. □

Remark 1: Compared with the necessary and sufficient condition that the second-order multi-agent system (1) solves the consensus problem given in Jiang and Wang (2010), the condition of Corollary 3.3 has more merits. For example, the feedback gain k_1 is directly estimated through the Laplacian matrix L instead of computation the eigenvalue of the Laplacian matrix L.

Remark 2: Note that $k_1 > 2\sqrt{\max_{i\in\mathcal{I}_m} d_{ii}}$ presents a unified viewpoint to solve the consensus problem of both the homogeneous multi-agent system (the first-order multi-agent system and the second-order multi-agent system) and the heterogeneous multi-agent system, where m is the number of second-order integrator agents.

Theorem 3.4: *Consider a directed network with fixed topology. Assume that the feedback gain* $k_1 > 2\sqrt{\max_{i\in\mathcal{I}_m} d_{ii}}$. *Then, the heterogeneous multi-agent system* (1) *with consensus protocol* (6) *reaches consensus asymptotically with the time-varying group reference velocity* v^d *if and only if the fixed topology G has a directed spanning tree.*

Proof: Let $\tilde{v}_i = v_i - v^d$ $(i \in \mathcal{I}_m)$ and $\tilde{x}_i = x_i - x^d$ $(i \in \mathcal{I}_n)$, where $x^d = \int_0^t v^d(\tau)\mathrm{d}\tau$. The heterogeneous multi-agent system (1) with consensus protocol (6) can be written as follows:

$$\begin{cases} \dot{x}_i - \dot{x}^d = v_i - v^d, & i \in \mathcal{I}_m, \\ \dot{v}_i - \dot{v}^d = \sum_{j=1}^{n} a_{ij}[(x_j - x^d) - (x_i - x^d)] & \\ \qquad - k_1(v_i - v^d), & i \in \mathcal{I}_m, \\ \dot{x}_i - \dot{x}^d = k_2 \sum_{j=1}^{n} a_{ij}[(x_j - x^d) - (x_i - x^d)], & i \in \mathcal{I}_n/\mathcal{I}_m, \end{cases}$$

which is equivalent to

$$\begin{cases} \dot{\tilde{x}}_i = \tilde{v}_i, & i \in \mathcal{I}_m, \\ \dot{\tilde{v}}_i = \sum_{j=1}^{n} a_{ij}(\tilde{x}_j - \tilde{x}_i) - k_1\tilde{v}_i, & i \in \mathcal{I}_m, \\ \dot{\tilde{x}}_i = k_2 \sum_{j=1}^{n} a_{ij}(\tilde{x}_j - \tilde{x}_i), & i \in \mathcal{I}_n/\mathcal{I}_m. \end{cases} \tag{9}$$

From Theorem 3.2, we know that the heterogeneous multi-agent system (9) reaches consensus asymptotically (i.e. $\lim_{t\to\infty} \|\tilde{x}_i(t) - \tilde{x}_j(t)\| = 0$ $(i, j \in \mathcal{I}_n)$ and $\lim_{t\to\infty} \|\tilde{v}_i(t)\| = 0$ $(i \in \mathcal{I}_m)$) if and only if the fixed topology G has a directed spanning tree, which implies that $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0$ $(i, j \in \mathcal{I}_n)$ and $\lim_{t\to\infty}\|v_i(t) - v^d(t)\| = 0$ $(i \in \mathcal{I}_m)$ if and only if the fixed topology G has a directed spanning tree. □

Remark 3: Olfati-Saber and Murray (2004) found that the convergence speed can be influenced by the weights of networks . From the proof of Theorem 3.2, it is easy to know that the chosen of k_1 and k_2 have an effect on the convergence speed of the heterogeneous multi-agent system. And the types of group reference velocities have an effect on the consensus state. The consensus state of all the agents can reach the specified object through designing the time-varying group reference velocity v^d.

When $m=n$, we get a necessary and sufficient condition for the second-order multi-agent system (1) with the time-varying group reference velocity.

Corollary 3.5: *Consider a directed network with fixed topology. Assume that the feedback gain* $k_1 \geq 2\sqrt{\max_{i\in\mathcal{I}_n} d_{ii}}$. *Then, the second-order multi-agent system* (1) *with consensus protocol* (4) *reaches consensus asymptotically with the time-varying group reference velocity* v^d *if and only if the fixed topology* G *has a directed spanning tree.*

Proof: The proof is similar to the analysis of Corollary 3.3 and Theorem 3.4. □

Remark 4: Note that the stationary consensus of the heterogeneous multi-agent system with consensus protocol (5) is a special case of the dynamic consensus of the heterogeneous multi-agent system with consensus protocol (6) when $v^d=0$. The consensus state of all the agents can reach the specified object through designing the time-varying group reference velocity v^d.

3.2 *Networks with switching topology*

In this subsection, we consider the consensus of the heterogeneous multi-agent system (9) under protocols (5) and (6) with switching topology. The interactions among agents are modelled by the directed networks.

Let $\bar{G}=\{G_1, G_2, \ldots, G_M\}$ denote the set of all possible directed interaction graphs defined for V. Obviously, $\bar{G}$ has finite elements. The union of a group of directed graphs $\{G_{i_1}, G_{i_2}, \ldots, G_{i_m}\} \subset \bar{G}$ is a directed graph with the vertex set V and the edge set given by the union of the edge sets of G_{i_j} $(j=1,\ldots,m)$. We apply the dwell time (Ren and Beard 2005) to the heterogeneous multi-agent system (9) under protocols (5) and (6), which implies that the interaction graph and weighting factors are constrained to change only at discrete times. Let $\tau_i=t_{i+1}-t_i$ $(i=0,1,\ldots)$ be the dwell time. Let $\bar{\tau}$ be a finite set of arbitrary positive numbers. Let Υ be a infinite set generated from $\bar{\tau}$, which is closed under addition and multiplications by positive integers. By choosing the set $\bar{\tau}$ properly, the dwell time can be chosen from the infinite set Υ. In order to facilitate our analysis, we give a lemma and let $d_m=\sqrt{\max_{i\in\mathcal{I}_m}\{d_{ii}(t): t\geq 0\}}$, $\bar{a}$ is finite set of arbitrary positive numbers.

Lemma 3.6 (Ren and Beard 2005)**:** *Let* $t_1, t_2, \ldots$ *be an infinite time sequence at which the interaction graph or weighting factors switch and* $\tau_i=t_{i+1}-t_i\in\Upsilon$ $(i=0,1,\ldots)$. *Let* $G(t_i)\in\bar{G}$ *be a switching interaction graph at time* $t=t_i$ *and* $a_{ij}(t_i)\in\bar{a}$, *where* $\bar{a}$ *is a finite set of arbitrary nonnegative numbers. The first-order multi-agent system* $\dot{\xi}=-L(t)\xi$ *reaches consensus asymptotically if there exists an infinite sequence of contiguous, nonempty, uniformly bounded time intervals* $[t_{i_j}, t_{i_{j+1}})$ $(j=1,2,\ldots)$, *starting at* $t_{i_1}=0$, *with the property that the union of the directed graphs across each interval has a directed spanning tree.*

Theorem 3.7: *Let* $t_1, t_2, \ldots$ *be an infinite time sequence at which the interaction graph or weighting factors switch and* $\tau_i=t_{i+1}-t_i\in\Upsilon$ $(i=0,1,\ldots)$. *Let* $G(t_i)\in\bar{G}$ *be a switching interaction graph at time* $t=t_i$ *and* $a_{ij}(t_i)\in\bar{a}$, *where* $\bar{a}$ *is a finite set of arbitrary nonnegative numbers. Assume that the feedback gain* $k_1>2d_m$. *Then, the heterogeneous multi-agent system* (1) *with consensus protocol* (5) *reaches consensus asymptotically if there exists an infinite sequence of contiguous, nonempty, uniformly bounded time intervals* $[t_{i_j}, t_{i_{j+1}})$ $(j=1,2,\ldots)$, *starting at* $t_{i_1}=0$, *with the property that the union of the directed graphs across each interval has a directed spanning tree.*

Proof: Let $y_{i'}=k_{3i}(t)v_i+x_i$ $(i\in\mathcal{I}_m)$. Similar to the step 1 of Theorem 3.2, we get a first-order multi-agent system with $n+m$ agents as follows:

$$\begin{cases} \dot{x}_i=\dfrac{1}{k_{3i}(t)}(y_{i'}-x_i), & i\in\mathcal{I}_m, \\ \dot{y}_{i'}=k_{3i}(t)\displaystyle\sum_{j=1}^{n}a_{ij}(t)(x_j-y_{i'})+b_i(t)(x_i-y_{i'}), & i'\in\mathcal{I}_m, \\ \dot{x}_i=k_2\displaystyle\sum_{j=1}^{n}a_{ij}(t)(x_j-x_i), & i\in\mathcal{I}_n/\mathcal{I}_m, \end{cases} \tag{10}$$

where $b_i(t)=k_1-\frac{1}{k_{3i}(t)}-k_{3i}(t)d_{ii}(t)$, $k_{3i}(t)\geq\frac{1}{k_1}$ if $d_{ii}(t)=0$, otherwise $\frac{k_1-\sqrt{k_1^2-4d_{ii}(t)}}{2d_{ii}(t)}\leq k_{3i}(t)\leq\frac{k_1+\sqrt{k_1^2-4d_{ii}(t)}}{2d_{ii}(t)}$. Due to $k_1>2d_m>0$, we have $\frac{k_1-\sqrt{k_1^2-4d_{ii}(t)}}{2d_{ii}(t)}>0$, $k_{3i}(t)>0$ and $b_i(t)\geq 0$ for $i\in\mathcal{I}_m$. Thus, the heterogeneous multi-agent system (1) with consensus protocol (5) reaches consensus asymptotically if and only if the first-order multi-agent system (10) reaches consensus asymptotically.

Let $G'(t)$ be a directed switching topology of the first-order multi-agent system (10). Due to $k_{3i}(t)>0$ for $t>0$, we have $a_{ii'}(t)>0$ for $t>0$. Then, the argument of the rest is similar to steps 2 and 3 of Theorem 3.2. Combining the above statement and Lemma 3.6, we complete the proof. □

When employing the consensus protocol (6) or (and) $m=n$, we have the following results which can be proved using similar techniques as Theorem 3.4, Corollary 3.3 and Theorem 3.7.

Theorem 3.8: *Let* $t_1, t_2, \ldots$ *be an infinite time sequence at which the interaction graph or weighting factors switch and* $\tau_i=t_{i+1}-t_i\in\Upsilon$ $(i=0,1,\ldots)$. *Let* $G(t_i)\in\bar{G}$

be a switching interaction graph at time $t = t_i$ and $a_{ij}(t_i) \in \bar{a}$, where $\bar{a}$ is a finite set of arbitrary nonnegative numbers. Assume that the feedback gain $k_1 > 2d_m$. Then, the heterogeneous multi-agent system (1) *with consensus protocol* (6) *reaches consensus asymptotically with the time-varying group reference velocity v^d if there exists an infinite sequence of contiguous, nonempty, uniformly bounded time intervals $[t_{i_j}, t_{i_{j+1}})$ $(j = 1, 2, \ldots)$, starting at $t_{i_1} = 0$, with the property that the union of the directed graphs across each interval has a directed spanning tree.*

Corollary 3.9: *Let $t_1, t_2, \ldots$ be an infinite time sequence at which the interaction graph or weighting factors switch and $\tau_i = t_{i+1} - t_i \in \Upsilon$ $(i = 0, 1, \ldots)$. Let $G(t_i) \in \bar{G}$ be a switching interaction graph at time $t = t_i$ and $a_{ij}(t_i) \in \bar{a}$, where $\bar{a}$ is a finite set of arbitrary nonnegative numbers. Assume that the feedback gain $k_1 \geq 2d_n$. Then, the second-order multi-agent system* (1) *(when $m = n$) with consensus protocol* (3) *(or* (4)*) reaches consensus asymptotically if there exists an infinite sequence of contiguous, nonempty, uniformly bounded time intervals $[t_{i_j}, t_{i_{j+1}})$ $(j = 1, 2, \ldots)$, starting at $t_{i_1} = 0$, with the property that the union of the directed graphs across each interval has a directed spanning tree.*

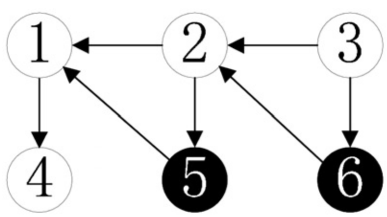

Figure 1. A directed fixed topology.

4. Simulations

In this section, two examples are given to illustrate the effectiveness of the theoretical results. In the following, all directed graphs with 0-1 weights will be needed. Let $k_1 = 3$, $k_2 = 1$, $x(0) = [8, 5, 2, -4, 1, -5]$ and $v(0) = [1, -5, 5, 3]$.

Example 4.1: The multi-agent system is composed of six agents with a directed fixed topology in Figure 1. The vertices 1–4 denote the second-order integrator agents and the vertices 5–6 denote the first-order integrator agents. Then, the fixed topology has a directed spanning tree and $k_1 > 2\sqrt{\max_{i \in \mathcal{I}_m} d_{ii}}$. By using consensus protocol (5), the state trajectories of all the agents reach consensus as shown in Figure 2, which is consistent with the sufficiency of Theorem 3.2. Suppose that the time-varying group reference velocity $v^d(t) = \sin t$, Figure 3 shows that all the agents reach consensus with consensus protocol (6), which is consistent with the sufficiency of Theorem 3.4.

Example 4.2: The multi-agent system is composed of six agents, where the agents 1–4 are governed by second-order integrators and the agents 5–6 are governed by first-order integrators. The directed topology

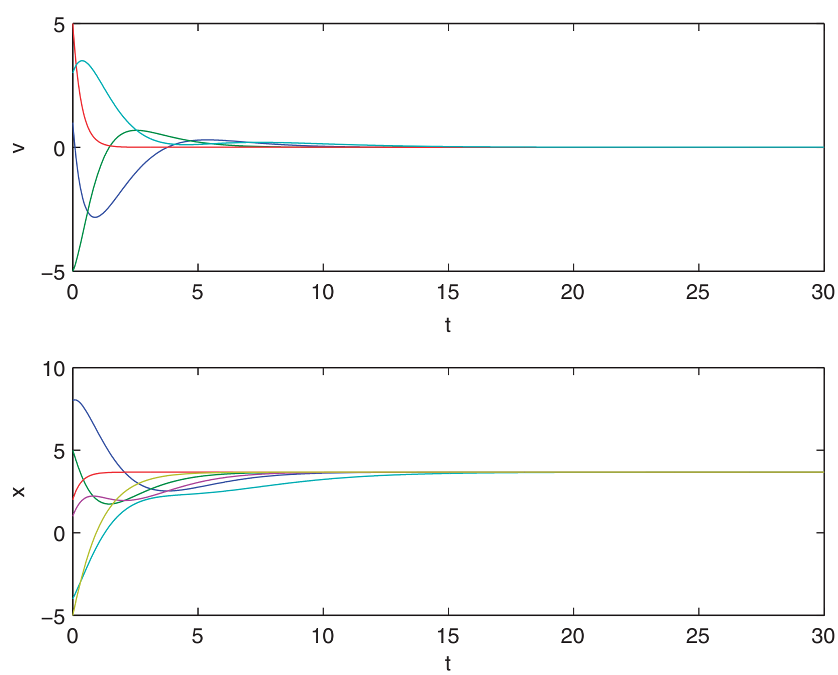

Figure 2. Simulation results with the fixed topology depicted in Figure 1 and consensus protocol (5).

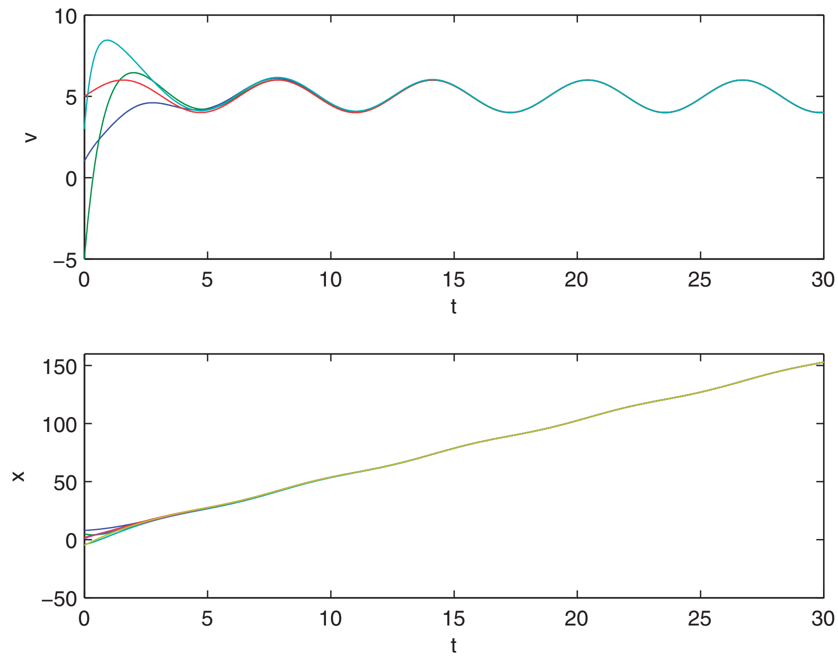

Figure 3. Simulation results with the fixed topology depicted in Figure 1 and consensus protocol (6), where the time-varying group reference velocity $v^d(t) = \sin t$.

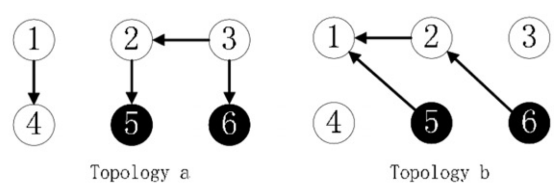

Figure 4. Two directed topologies.

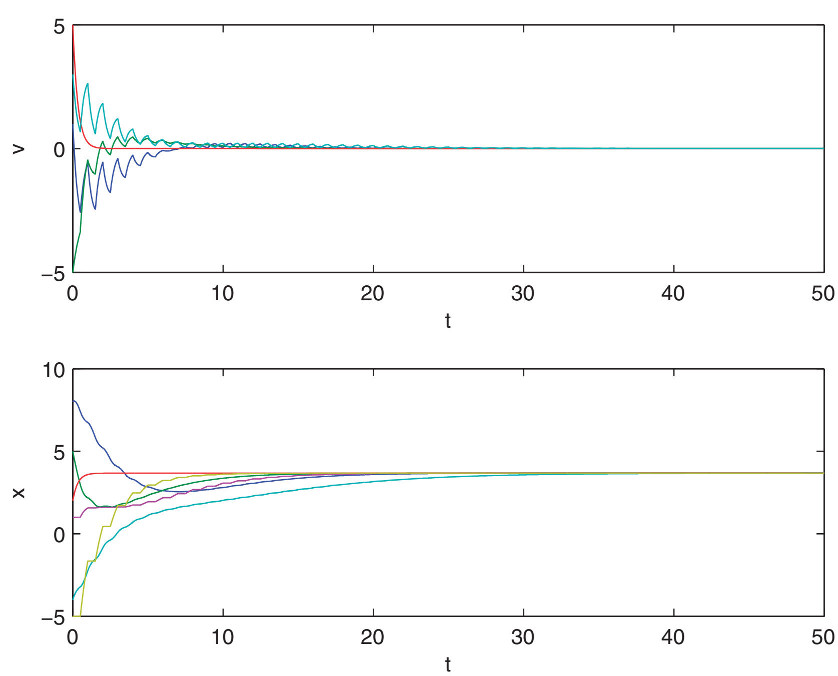

Figure 5. Simulation results with the switching topology depicted in Figure 4 and consensus protocol (5).

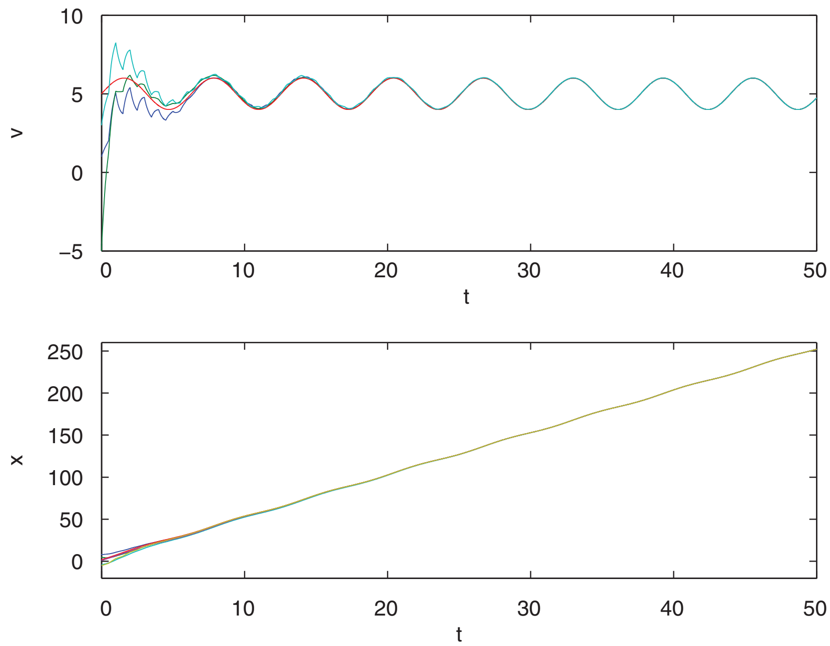

Figure 6. Simulation results with the switching topology depicted in Figure 4 and consensus protocol (6), where the time-varying group reference velocity $v^d(t) = \sin t$.

of the system is switched between topology a and topology b per second in Figure 4. Then, the union of the topology a and topology b has a directed spanning tree and $k_1 > 2d_m$. By using consensus protocol (5), the state trajectories of all the agents reach consensus as shown in Figure 5, which is consistent with the results in Theorem 3.7. Suppose that the time-varying group reference velocity $v^d(t) = \sin t$, Figure 6 shows that all the agents reach consensus with consensus protocol (6), which is consistent with the results in Theorem 3.8.

5. Conclusions

In this article, consensus problem of a heterogeneous multi-agent system with agents modelled by first-order and second-order integrators with fixed or switching topology was presented. Two kinds of consensus protocols based on the consensus protocols of first-order and second-order multi-agent systems were proposed. Some necessary and sufficient conditions were derived for the consensus of the heterogeneous multi-agent system with fixed topology. Under switching topology, some sufficient conditions for the consensus of the heterogeneous multi-agent system were established.

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