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Yuanshi Zheng [a] , Weisheng Chen [b] & Long Wang [c]

[a] Center for Complex Systems, School of Mechano-electronic Engineering, Xidian University, Xi'an 710071, China

[b] Department of Mathematics, Xidian University, Xi'an 710071, China

[c] Center for Systems and Control, College of Engineering, Peking University, Beijing 100871, China

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Finite-time consensus for stochastic multi-agent systems

Yuanshi Zheng[a*], Weisheng Chen[b] and Long Wang[c]

[a]*Center for Complex Systems, School of Mechano-electronic Engineering, Xidian University, Xi'an 710071, China;* [b]*Department of Mathematics, Xidian University, Xi'an 710071, China;* [c]*Center for Systems and Control, College of Engineering, Peking University, Beijing 100871, China*

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In this article, we study the finite-time consensus in probability for stochastic multi-agent systems. First, we give the nonlinear consensus protocol for multi-agent systems with Gaussian white noise, and define the concept of finite-time consensus in probability. Second, we prove that multi-agent systems can achieve the finite-time consensus in probability under five different kinds of communication topologies by using graph theory, stochastic Lyapunov theory and probability theory. Finally, some simulation examples are provided to illustrate the effectiveness of the theoretical results.

Keywords: stochastic multi-agent system; finite-time consensus; undirected graph; directed graph

1. Introduction

Recent years have witnessed an enormous growth of research on consensus problem for multi-agent systems. It is fundamental in decentralised coordination and control of networks of dynamic agents and has been applied in many fields, such as swarming, flocking, synchronisation and formation control of social insects, unmanned air vehicles, robots and sensor networks (Tsitsiklis 1984; Vicsek, Czirok, Jacob, Cohen, and Schochet 1995; Jabdabaie, Lin, and Morse 2003; Olfati-Saber and Murray 2004; Moreau 2005; Ren and Beard 2005; Chu, Wang, Chen, and Mu 2006; Wu 2006; Bauso, Giarré, and Pesenti 2009; Chandra and Ladde 2010).

In collective behaviours of multi-agent systems, consensus is one of the most interesting behaviours. An early work of multi-agent model was studied by Vicsek et al. (1995) which contained the noise effects, and it was demonstrated by simulation that the system will synchronise if the population density is large and the noise is small. Jabdabaie et al. (2003) provided a theoretical explanation of the consensus behaviour in Vicsek model, and analysed the alignment of a network of agents with switching topologies that are periodically connected. Olfati-Saber and Murray (2004) discussed the consensus problem for networks of dynamic agents with switching topologies and time delays in a continuous-time model by defining a disagreement function, and obtained some useful results for solving the average-consensus problem. Other theoretical explanations for the consensus behaviour of the Vicsek model were given in Ren and Beard (2005), Xiao and Wang (2006a,b), Lin and Jia (2009), Hatano and Mesbahi (2005), Porfiri and Stilwell (2007), without noise. The robust consensus of discrete-time multi-agent systems with noise was studied under undirected networks (Wang and Liu 2008) and directed networks (Wang and Guo 2008). Li and Zhang (2009) gave the necessary and sufficient condition of mean square average consensus for multi-agent systems with noise. Huang and Manton (2009) considered the coordination and consensus of multi-agent systems where each agent has noisy measurements of its neighbours' states. Chandra and Ladde (2010) investigated the qualitative and quantitative properties by formulating stochastic multi-agent systems.

On the other hand, the finite-time stability problem has been studied for various cases. Bhat and Bernstein (2000) considered the finite-time stability of continuous autonomous systems. Moulay and Perruquetti (2006) extended the results of Bhat and Bernstein (2000) to the non-autonomous continuous systems. Chen and Jiao (2010) gave the finite-time stability (finite-time attractiveness) theorem of stochastic nonlinear systems. The idea of finite-time stability has been applied to the finite-time consensus for multi-agent systems. Based on the non-smooth stability analysis, Cortés (2006) discussed the finite-time consensus problem for multi-agent systems under some discontinuous

*Corresponding author. Email: zhengyuanshi2005@163.com

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consensus protocols. Xiao, Wang and Jia (2008) and Xiao and Wang (2010) showed that the multi-agent systems could achieve the finite-time consensus for both the bidirectional and unidirectional interaction cases. Jiang and Wang (2009) investigated the finite-time consensus for multi-agent systems with fixed and switching topologies under nonlinear protocol. Wang and Guo (2008) considered the finite-time consensus problem for the multi-agent systems with second-order dynamics.

Real networks are often in uncertain communication environments, it is natural to consider the random perturbation in the distributive protocols (consensus protocols) for multi-agent systems (Wang and Guo 2008; Huang and Manton 2009; Li and Zhang 2009; Wang and Liu 2009; Chandra and Ladde 2010). Moreover, convergence rate is an important index to evaluate the consensus protocols in the analysis of consensus problems for multi-agent systems (Cortés 2006; Kim and Mesbahi 2006; Xiao and Boyd 2006). Thus, the finite-time consensus problem has attracted considerable attention of researchers and has become an active area of research in the consensus for multi-agent systems (Wang and Hong 2008; Xiao et al. 2008; Jiang and Wang 2009; Wang and Xiao 2010). To the best of our knowledge, almost all the existing literature on the consensus problem have not considered the finite-time consensus problem for multi-agent systems with noise (stochastic multi-agent systems) which is inevitable and significant in the real world.

In this article, we study the finite-time consensus in probability for stochastic multi-agent systems. The main contribution of this article is threefold. First, we give the nonlinear consensus protocol with Gaussian white noise, which is an extension of stochastic multi-agent systems in Chandra and Ladde (2010). Second, the finite-time stability for stochastic nonlinear system is successfully applied to solving the finite-time consensus problem for stochastic multi-agent systems. The idea is motivated by the works of Xiao et al. (2008), Wang and Xiao (2010) which consider the finite-time consensus for multi-agent systems without noise. Third, we prove the finite-time consensus theorems in probability for multi-agent systems by using the corollary in Chen and Jiao (2010), which is an extension of the work of Wang and Xiao (2010).

This article is organised as follows. In Section 2, we formulate the problem. In Section 3, we establish the finite-time consensus theorems in probability for multi-agent systems. In Section 4, we give some examples to explain our results. Finally, we summarise the main conclusions in Section 5.

Throughout this article, we let $\mathbb{R}$, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ be the set of real number, positive real number and non-negative real number, $\mathbb{R}^n$ be the n-dimensional real vector space and $\mathcal{I}_n = \{1, 2, \dots, n\}$. For a given vector or matrix X, X^T denotes its transpose, $\mathrm{Tr}\{X\}$ denotes its trace when X is square, and $\|X\|$ denotes the Euclidean norm of a vector X. $\mathbf{1}_n$ is a vector with elements being all ones, $\mathrm{diag}\{a_1, a_2, \dots, a_n\}$ defines a diagonal matrix with diagonal elements being $a_1, a_2, \dots, a_n$, $\rho(A)$ represents the spectral radius of matrix A. $A = [a_{ij}]$ is said to be non-negative (resp. positive) if all entries a_{ij} are non-negative (resp. positive), denoted by $A \geq 0$ (resp. $A > 0$). $P(x)$ denotes the probability of stochastic variable x, $E(x)$ denotes the expectation of stochastic variable x. $\mathrm{sig}(x)^\alpha = \mathrm{sign}(x)|x|^\alpha$, where $\mathrm{sign}(\cdot)$ is sign function.

2. Problem formulation

2.1 *Algebraic graph theory preliminaries*

The network formed by multi-agent system can be always represented by a graph. Thus, graph theory is an important tool to analyse consensus problem for multi-agent system. We present some basic definitions in graph theory (Godsil and Royal 2001).

An undirected (directed) graph $G = (V, E)$ consists of a vertex set $V = \{v_1, v_2, \dots, v_n\}$ and an edge set $E = \{e_{ij} = (v_i, v_j)\} \subset V \times V$. Denote the set of neighbours of v_i by $N_i = \{v_j : e_{ji} = (v_j, v_i) \in E\}$. A path that connects v_i and v_j in the graph G is a sequence of distinct vertices $v_{i_0}, v_{i_1}, v_{i_2}, \dots v_{i_m}$, where $v_{i_0} = v_i$, $v_{i_m} = v_j$ and $(v_{i_r}, v_{i_{r+1}}) \in E$, $0 \leq r \leq m-1$. An undirected (directed) graph is said to be connected (strong connected) if there exists a path between any two distinct vertices of the graph. For directed graph, if (v_i, v_j) is an edge of G, v_i is called the parent of v_j and v_j is called the child of v_i. A directed tree is a directed graph, where every vertex, except one special vertex without any parent, which is called the root, has exactly one parent, and the root can be connected to any other vertices through paths. The weighted adjacency matrix $A = [a_{ij}]_{n \times n}$ of a graph G is a non-negative matrix with rows and columns indexed by the vertices, all entries of which are non-negative, where $a_{ij} > 0$ if and only if $e_{ji} = (v_j, v_i) \in E$. The degree matrix $D = [d_{ij}]_{n \times n}$ is a diagonal matrix with $d_{ii} = \sum_{v_j \in N_i} a_{ij}$, and the Laplacian matrix of the graph is defined as $L = [l_{ij}]_{n \times n} = D - A$. It is easy to see that adjacency matrix A is symmetric if G is an undirected graph. It has been shown in Olfati-Saber and Murray (2004) and Godsil and Royal (2001) that $L\mathbf{1}_n = 0$. When G is a connected undirected graph, L is positive semi-definite and has a simple zero eigenvalue, $\xi^T L \xi = \frac{1}{2} \times \sum_{i,j=1}^n a_{ij}(\xi_j - \xi_i)^2$ and $\min_{\xi \neq 0, \mathbf{1}_n^T \xi = 0} \frac{\xi^T L \xi}{\xi^T \xi} = \lambda_2$ for any $\xi = [\xi_1, \xi_2, \dots, \xi_n]^T \in \mathbb{R}^n$, where λ_2 is the second smallest eigenvalue of L.

2.2 *Consensus protocol*

We consider n autonomous agents, labelled 1 through n. Let $x_i \in \mathbb{R}$ denote the the state of agent i and $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$. We assume that each agent obeys a single integrator model:

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{I}_n, \tag{1}$$

where $u_i(t)$ is control input, called consensus protocol. The initial state is $x_i(0) = x_{i0}$, and $x_0 = [x_{10}, x_{20}, \ldots, x_{n0}]^T$.

We give the nonlinear control input with random perturbation as

$$u_i(t) = \sum_{j=1}^{n} \Big[(a_{ij} f(x_j - x_i) + b_{ij}^2 g(x_j - x_i)) + b_{ij} g(x_j - x_i) \dot{w}_{ij}(t) \Big], \quad i \in \mathcal{I}_n, \tag{2}$$

where $A = [a_{ij}]_{n\times n}$ is the weighted adjacency matrix, the noise intensity $b_{ij} \ge 0$ and $b_{ij} > 0$ if and only if agent j is the neighbour of agent i. $\{\dot{w}_{ij}(t), i, j = 1, 2, \ldots, n\}$ are independent standard white noises. We assume that F_t is an increasing family of sub-σ-algebras of F on a complete probability space (Ω, F, P), $w_{ij}(t)$ is F_t measurable for all $t \ge 0$ and is a standard Wiener process with independent increments, $w_{ij}(t) = w_{ji}(t)$, $i, j \in \mathcal{I}_n$.

Remark 1: In the same communication channel, the noises are often regarded as the same for agent i and j (Nguyen and Shwedyk 2009; Chandra and Ladde 2010). Therefore, the assumption of $w_{ij}(t) = w_{ji}(t)$ is reasonable.

Hence, we can obtain the following Itô type stochastic multi-agent system:

$$\mathrm{d}x_i(t) = \sum_{j=1}^{n} (a_{ij} f(x_j - x_i) + b_{ij}^2 g(x_j - x_i)) \mathrm{d}t + \sum_{j=1}^{n} b_{ij} g(x_j - x_i) \mathrm{d}w_{ij}(t), \quad i \in \mathcal{I}_n. \tag{3}$$

We make the following assumptions on (3):

(i) $f(\cdot): \mathbb{R} \to \mathbb{R}$ and $g(\cdot): \mathbb{R} \to \mathbb{R}$ are continuously differential and odd functions.
(ii) $f(\theta) = g(\theta) = 0$ if only if $\theta = 0$, and $\theta f(\theta) > 0$ when $\theta \ne 0$.
(iii) There exist $\alpha, \beta \in \mathbb{R}$ and $0 < \alpha < 1$, $\beta > 0$ such that $\|f(\theta)\| \ge \beta \|\theta\|^\alpha$.
(iv) $g(\theta)(g(\theta) - \theta) \le 0$.

Remark 2: The existence and uniqueness for the solution of system (3) can be obtained directly from Theorem 170 in Situ (2005) if $f(\cdot)$ and $g(\cdot)$ satisfy a ρ-condition. Compared with stochastic multi-agent system in Chandra and Ladde (2010), deterministic item of the model (3) is related to the stochastic item, i.e. we consider the portion of deterministic item has random perturbation which is actual in real world. In order to move forward to this novel problem, we give the assumptions (i)–(iv) which are achievable, e.g. $f(\theta) = \mathrm{sig}(\theta)^\alpha$ and $g(\theta) = \theta$.

Similar to stability in probability and finite-time stability in probability (globally stochastic finite-time stability) (Chen and Jiao 2010), the definition of finite-time consensus in probability is given through the following definition of consensus with probability one and the stochastic settling time function.

Definition 2.1 (Consensus with probability one)**:** The agents are said to reach consensus with probability one if for any initial states, there exists a constant $\bar{x}$ such that $P(\lim_{t\to\infty} \|x_i(t) - \bar{x}\| = 0$, for all $i \in \mathcal{I}_n) = 1$.

Definition 2.2 (Finite-time consensus in probability)**:** Given the control input u_i, $i \in \mathcal{I}_n$, u_i is said to solve the finite-time consensus in probability if for any initial states, there exists some stochastic settling time function $T(x_0, w)$ in probability and $E[T(x_0, w)] < \infty$, such that $P(\lim_{t\to T(x_0,w)^-} \times \|x_i(t) - \bar{x}\| = 0$, for all $i \in \mathcal{I}_n) = 1$.

Lemma 2.3 (Xiao et al. 2008)**:** *Let* $\xi_1, \xi_2, \ldots, \xi_n \ge 0$ *and* $0 < p \le 1$. *Then*

$$\left(\sum_{i=1}^{n} \xi_i \right)^p \le \sum_{i=1}^{n} \xi_i^p.$$

Lemma 2.4 (Xiao et al. 2008)**:** *Let* $b = [b_1, b_2, \ldots, b_n]^T \ge 0$, $b \ne 0$, *and let* G *be undirected and connected with adjacency matrix* A. *Then* $L(A) + \mathrm{diag}(b)$ *is positive definite.*

Lemma 2.5 (Chen and Jiao 2010)**:** *Consider stochastic nonlinear system*

$$\mathrm{d}x = f(x)\mathrm{d}t + g(x)\mathrm{d}w,$$

which has the unique global solution. If there exists a positive definite, twice continuously differentiable and radially unbounded Lyapunov function $V: \mathbb{R}^n \to \mathbb{R}_{>0}$ *and real numbers* $K > 0$ *and* $0 < \alpha < 1$, *such that*

$$\mathcal{L}V(x) \le -K(V(x))^\alpha,$$

where $f: \mathbb{R}^n \to \mathbb{R}^n$ *and* $g: \mathbb{R}^n \to \mathbb{R}^{n\times r}$ *are continuous,* $\mathcal{L}V(x) = \frac{\partial V}{\partial x} f(x) + \frac{1}{2}\mathrm{Tr}\{g^T(x) \frac{\partial^2 V}{\partial x^2} g(x)\}$. *Then the origin of the stochastic nonlinear system is globally stochastic finite-time stable, and stochastic settling time* $T(x_0, w)$ *satisfies* $E[T(x_0, w)] \le \frac{(V(x_0))^{1-\alpha}}{K(1-\alpha)}$, *which implies* $T(x_0, w) < \infty$ *a.s.*

3. Main results

In this section, we will study the finite-time consensus in probability for the multi-agent system (3) under undirected and directed graphs, respectively.

3.1 *Finite-time consensus under undirected graph*

For undirected graph, we consider the communication topology G with two cases, one is undirected and connected, the other has a vertex which is a leader in the multi-agent system (1) and the network among the followers is undirected and connected.

Theorem 3.1: *Consider the network of the multi-agent system* (1) *with fixed topology G. Suppose G is undirected and connected, i.e. $a_{ij}=a_{ji}$ and $b_{ij}=b_{ji}$ for all $i,j\in\mathcal{I}_n$, and the assumptions* (i)–(iv) *hold. Then the multi-agent system* (3) *can achieve the finite-time consensus in probability.*

Proof: Let $y(t)=\frac{1}{n}\sum_{i=1}^{n}x_i(t)$. Since $a_{ij}=a_{ji}$, $b_{ij}=b_{ji}$ and $w_{ij}=w_{ji}$ for all $i,j\in\mathcal{I}_n$, $f(\cdot)$ and $g(\cdot)$ are odd functions, we have

$$\begin{aligned}
\mathrm{d}y(t) &= \frac{1}{n}\sum_{i=1}^{n}\mathrm{d}x_i(t)\\
&= \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}(a_{ij}f(x_j-x_i)+b_{ij}^2 g(x_j-x_i))\mathrm{d}t\right.\\
&\quad \left.+\sum_{j=1}^{n}b_{ij}g(x_j-x_i)\mathrm{d}w_{ij}(t)\right)\\
&= \frac{1}{2n}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}(a_{ij}f(x_j-x_i)+b_{ij}^2 g(x_j-x_i))\mathrm{d}t\right.\right.\\
&\quad \left.+\sum_{j=1}^{n}b_{ij}g(x_j-x_i)\mathrm{d}w_{ij}(t)\right)\\
&\quad +\sum_{j=1}^{n}\left(\sum_{i=1}^{n}(a_{ji}f(x_i-x_j)+b_{ji}^2 g(x_i-x_j))\mathrm{d}t\right.\\
&\quad \left.\left.+\sum_{i=1}^{n}b_{ji}g(x_i-x_j)\mathrm{d}w_{ji}(t)\right)\right)\\
&= \frac{1}{2n}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}(a_{ij}f(x_j-x_i)+b_{ij}^2 g(x_j-x_i))\mathrm{d}t\right.\right.\\
&\quad \left.+\sum_{j=1}^{n}b_{ij}g(x_j-x_i)\mathrm{d}w_{ij}(t)\right)\\
&\quad -\sum_{i=1}^{n}\left(\sum_{j=1}^{n}(a_{ij}f(x_j-x_i)+b_{ij}^2 g(x_j-x_i))\mathrm{d}t\right.\\
&\quad \left.\left.+\sum_{j=1}^{n}b_{ij}g(x_j-x_i)\mathrm{d}w_{ij}(t)\right)\right)\\
&= 0. \qquad (4)
\end{aligned}$$

Therefore, $y(t)$ is time-invariant, i.e. $y(t)=y(0)$. Let $\delta_i(t)=x_i(t)-y(t)$ and $\delta(t)=[\delta_1(t),\delta_2(t),\ldots,\delta_n(t)]^T$. Then we have $\mathbf{1}_n^T\delta(t)=0$, $\frac{\mathrm{d}\delta_i(t)}{\mathrm{d}t}=\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}-\frac{\mathrm{d}y(t)}{\mathrm{d}t}=\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}$ and $\delta_i(t)-\delta_j(t)=x_i(t)-x_j(t)$. Hence

$$\begin{aligned}
\mathrm{d}\delta_i(t) &= \sum_{j=1}^{n}(a_{ij}f(\delta_j-\delta_i)+b_{ij}^2 g(\delta_j-\delta_i))\mathrm{d}t\\
&\quad +\sum_{j=1}^{n}b_{ij}g(\delta_j-\delta_i)\mathrm{d}w_{ij}(t), \quad i\in\mathcal{I}_n. \qquad (5)
\end{aligned}$$

We consider candidate Lyapunov function as

$$V(\delta(t))=\sum_{i=1}^{n}\delta_i^2(t).$$

According to the Itô formula, we have

$$\begin{aligned}
\mathrm{d}V(\delta(t)) &= \sum_{i=1}^{n}2\delta_i(t)\left(\sum_{j=1}^{n}(a_{ij}f(\delta_j-\delta_i)+b_{ij}^2 g(\delta_j-\delta_i))\mathrm{d}t\right.\\
&\quad \left.+\sum_{i=1}^{n}2\delta_i(t)\sum_{j=1}^{n}b_{ij}g(\delta_j-\delta_i)\mathrm{d}w_{ij}(t)\right)\\
&\quad +\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}^2 g^2(\delta_j-\delta_i)\mathrm{d}t. \qquad (6)
\end{aligned}$$

By virtue of the assumption (i), we have

$$\begin{aligned}
\mathcal{L}V(\delta(t)) &= \sum_{i=1}^{n}2\delta_i(t)\sum_{j=1}^{n}(a_{ij}f(\delta_j-\delta_i)+b_{ij}^2 g(\delta_j-\delta_i))\\
&\quad +\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}^2 g^2(\delta_j-\delta_i)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}\delta_i(t)f(\delta_j-\delta_i)+a_{ji}\delta_j(t)f(\delta_i-\delta_j))\\
&\quad +\sum_{i=1}^{n}\sum_{j=1}^{n}(b_{ij}^2\delta_i(t)g(\delta_j-\delta_i)+b_{ji}^2\delta_j(t)g(\delta_i-\delta_j))\\
&\quad +\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}^2 g^2(\delta_j-\delta_i)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(\delta_i(t)-\delta_j(t))f(\delta_j-\delta_i)\\
&\quad +\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}^2 g(\delta_j-\delta_i)\big(g(\delta_j-\delta_i)-(\delta_j(t)-\delta_i(t))\big). \qquad (7)
\end{aligned}$$

Using the assumptions (ii)–(iv) and Lemma 2.3, we get

$$\begin{aligned}
\mathcal{L}V(\delta(t)) &\le -\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(\delta_j(t)-\delta_i(t))f(\delta_j-\delta_i)\\
&\le -\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\beta\|\delta_j(t)-\delta_i(t)\|^{1+\alpha}
\end{aligned}$$

$$= -\beta \sum_{i=1}^{n}\sum_{j=1}^{n}\left[a_{ij}^{\frac{2}{1+\alpha}}(\delta_j(t)-\delta_i(t))^2\right]^{\frac{1+\alpha}{2}}$$
$$\leq -\beta\left[\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}^{\frac{2}{1+\alpha}}(\delta_j(t)-\delta_i(t))^2\right]^{\frac{1+\alpha}{2}}. \quad (8)$$

Let $C=[a_{ij}^{\frac{2}{1+\alpha}}]_{n\times n}$, then

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}^{\frac{2}{1+\alpha}}(\delta_j(t)-\delta_i(t))^2 = 2\delta^T(t)L(C)\delta(t),$$

and because of $\mathbf{1}_n^T\delta(t)=0$, then

$$2\delta^T(t)L(C)\delta(t) \geq 2\lambda_2(L(C))\delta^T\delta = 2\lambda_2(L(C))V(\delta(t)).$$

Therefore,

$$\mathcal{L}V(\delta(t)) \leq -\beta[2\lambda_2(L(C))V(\delta(t))]^{\frac{1+\alpha}{2}}$$
$$= -\beta(2\lambda_2(L(C)))^{\frac{1+\alpha}{2}}V(\delta(t))^{\frac{1+\alpha}{2}}. \quad (9)$$

Thus, by Lemma 2.5, the multi-agent system (3) achieves the finite-time consensus in probability. Let $K_1=\beta(2\lambda_2(L(C)))^{\frac{1+\alpha}{2}}$, we have $E[T(x_0,w)] \leq \frac{2(V(x_0))^{\frac{1-\alpha}{2}}}{K_1(1-\alpha)}$. □

Remark 3: In the assumption $\|f(\theta)\| \geq \beta\|\theta\|^\alpha$, if the parameter α is affected by θ, i.e. $\|f(x_j-x_i)\| \geq \beta\|x_j-x_i\|^{\alpha_{ij}}$, the proof is similar to Theorem 3.1 and Wang and Xiao (2010). For limitation of the space, it is left to the interested readers as an exercise.

The network studied in Theorem 3.1 is undirected. To some extension, we consider the multi-agent system (1) with a leader.

Theorem 3.2: *Under the assumptions* (i)–(iv), *Suppose that the multi-agent system* (1) *has a leader and* $n-1$ *followers, and the communication topology among the followers is undirected and connected. Then the multi-agent system* (3) *achieves the finite-time consensus in probability*.

Proof: Without loss of generality, we assume agents $1,2,\ldots,n-1$ are the followers and agent n is the leader. Thus, we have $\bar{A}=[a_{ij}]_{1\leq i,j\leq n-1}=\bar{A}^T$, $a_{n1}=a_{n2}=\cdots=a_{nn}=0$ and $b_{n1}=b_{n2}=\cdots=b_{nn}=0$, $\bar{a}\geq 0$, where $\bar{a}=[a_{1n},a_{2n},\ldots,a_{(n-1)n}]^T$. We rewrite the protocol (2) as follows:

$$u_i(t)=\sum_{j=1}^{n-1}[(a_{ij}f(x_j-x_i)+b_{ij}^2g(x_j-x_i))$$
$$+b_{ij}g(x_j-x_i)\dot{w}_{ij}(t)]$$
$$+a_{in}f(x_n-x_i)+b_{in}^2g(x_n-x_i)$$
$$+b_{in}g(x_n-x_i)\dot{w}_{in}(t), \quad i\in\mathcal{I}_{n-1} \quad (10)$$

and

$$u_n(t)=0.$$

Let $\delta_i'(t)=x_i(t)-x_n(t)$, $i\in\mathcal{I}_n$, $\bar{\delta}(t)=[\delta_1'(t),\delta_2'(t),\ldots,\delta_{n-1}'(t)]^T$. Then, we have

$$\mathrm{d}\delta_i'(t)=\mathrm{d}x_i(t)=\sum_{j=1}^{n-1}[(a_{ij}f(\delta_j'-\delta_i')+b_{ij}^2g(\delta_j'-\delta_i'))\mathrm{d}t$$
$$+b_{ij}g(\delta_j'-\delta_i')\mathrm{d}w_{ij}(t)]-a_{in}f(\delta_i')\mathrm{d}t$$
$$-b_{in}^2g(\delta_i')\mathrm{d}t-b_{in}g(\delta_i')\mathrm{d}w_{in}(t), \quad i\in\mathcal{I}_{n-1} \quad (11)$$

and

$$\delta_n'(t)=0.$$

Similar to Theorem 3.1, the Lyapunov function is defined as

$$V_1(t)=\sum_{i=1}^{n}\delta_i'^2(t).$$

Then,

$$\mathcal{L}V_1(t)=\sum_{i=1}^{n-1}2\delta_i'(t)\left[\left(\sum_{j=1}^{n-1}(a_{ij}f(\delta_j'-\delta_i')+b_{ij}^2g(\delta_j'-\delta_i'))\right)\right.$$
$$\left.-a_{in}f(\delta_i')-b_{in}^2g(\delta_i')\right]$$
$$+\sum_{i=1}^{n-1}\left[\sum_{j=1}^{n-1}b_{ij}^2g^2(\delta_j-\delta_i)+b_{in}^2g^2(\delta_i')\right]$$
$$\leq -\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}a_{ij}(\delta_j'-\delta_i')f(\delta_j'-\delta_i')-\sum_{i=1}^{n-1}2a_{in}\delta_i'(t)f(\delta_i')$$
$$\leq -\beta\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}a_{ij}\|\delta_j'-\delta_i'\|^{1+\alpha}-2\beta\sum_{i=1}^{n-1}a_{in}\|\delta_i'\|^{1+\alpha}$$
$$\leq -\beta\left[\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}a_{ij}^{\frac{2}{1+\alpha}}(\delta_j'-\delta_i')^2+2\sum_{i=1}^{n-1}a_{in}^{\frac{2}{1+\alpha}}(\delta_i')^2\right]^{\frac{1+\alpha}{2}}. \quad (12)$$

Let $\bar{C}=[a_{ij}^{\frac{2}{1+\alpha}}]_{1\leq i,j\leq n-1}$, $\tilde{a}=[a_{1n}^{\frac{2}{1+\alpha}},a_{2n}^{\frac{2}{1+\alpha}},\ldots,a_{(n-1)n}^{\frac{2}{1+\alpha}}]^T$. Since the communication topology among the followers is undirected and connected, $L(\bar{C})+\mathrm{diag}(\tilde{a})$ is positive definite by Lemma 2.4. We denote the smallest eigenvalue by $\lambda_1(L(\bar{C})+\mathrm{diag}(\tilde{a}))$. Then

$$\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}a_{ij}^{\frac{2}{1+\alpha}}(\delta_j'-\delta_i')^2+2\sum_{i=1}^{n-1}a_{in}^{\frac{2}{1+\alpha}}(\delta_i')^2$$
$$=2\bar{\delta}^T(L(\bar{C})+\mathrm{diag}(\tilde{a}))\bar{\delta}$$
$$\geq 2\lambda_1(L(\bar{C})+\mathrm{diag}(\tilde{a}))\bar{\delta}^T\bar{\delta}. \quad (13)$$

Therefore,

$$\mathcal{L}V_1(t) \le -\beta[2\lambda_1(L(\bar{C}) + \mathrm{diag}(\tilde{a}))V_1(t)]^{\frac{1+\alpha}{2}}$$
$$= -\beta(2\lambda_1(L(\bar{C}) + \mathrm{diag}(\tilde{a})))^{\frac{1+\alpha}{2}} V_1(t)^{\frac{1+\alpha}{2}}. \quad (14)$$

Thus, by Lemma 2.5, the multi-agent system (3) achieves the finite-time consensus in probability. Let $K_2 = \beta(2\lambda_1(L(\bar{C}) + \mathrm{diag}(\tilde{a})))^{\frac{1+\alpha}{2}}$, we have $E[T(x_0, w)] \le \frac{2(V_1(x_0))^{\frac{1-\alpha}{2}}}{K_2(1-\alpha)}$. □

3.2 *Finite-time consensus under directed graph*

For directed graph, we first give a corollary of Theorem 3.2. Then, we discuss the finite-time consensus in probability when the communication topology G is a directed tree, and we extend the result to a special directed graph.

Corollary 3.3: *Under the assumptions* (i)–(iv), *suppose that the multi-agent system* (1) *has a leader and $n-1$ followers, and the followers receive information only from the leader. Then the multi-agent system* (3) *achieves the finite-time consensus in probability.*

Proof: Without loss of generality, we assume agents $1, 2, \ldots, n-1$ are the followers and agent n is the leader. Thus, we have $[a_{ij}]_{1\le i,j\le n-1} = [b_{ij}]_{1\le i,j\le n-1} = 0$, $a_{n1} = a_{n2} = \cdots = a_{nn} = 0$ and $b_{n1} = b_{n2} = \cdots = b_{nn} = 0$, $\bar{a} > 0$, where $\bar{a} = [a_{1n}, a_{2n}, \ldots, a_{(n-1)n}]^T$. Therefore, it is easy to get the conclusion of Corollary 3.3 from Theorem 3.2. □

In Theorem 3.1, the network is undirected and connected, and in Theorem 3.2, the network among the followers is undirected and connected. Motivated by Corollary 3.3, we extend the previous results to a class of special directed network – directed tree as follows.

Theorem 3.4: *Under the assumptions* (i)–(iv), *suppose that the network G is a directed tree. Then the multi-agent system* (3) *can achieve the finite-time consensus in probability.*

Proof: Without loss of generality, we assume agent 1 is the leader. Assume further that all vertices of the directed tree G can be classified into the following subsets: $V_0 = \{v_1\}$, $V_1 = \{v_j \in V : v_j$ only receives information from v_1 at any time $t\}, \ldots, V_q = \{v_j \in V : v_j$ only receives information from vertex in V_{q-1} at any time $t\}$. Moreover, $\bigcup_{p=1}^{q} V_p = V$.

Let the event A: the agents corresponding to the vertices in subset V_1, achieve the finite-time consensus. From Corollary 3.3, we have $P(A) = 1$. Let the event B: the agents corresponding to the vertices in subset V_2, achieve the finite-time consensus. If the event A holds, i.e. there exists a settling time T_1, $\lim_{t\to T_1^-} \|x_j(t) - x_1\| = 0$ and $x_j(t) = x_1$ when $t \ge T_1$ for all $j \in \{j : v_j \in V_1\}$. Hence,

$$\mathrm{d}x_i(t) = a_{ij} f(x_j - x_i) + b_{ij}^2 g(x_j - x_i) + b_{ij} g(x_j - x_i)\dot{w}_{ij}(t)$$
$$= a_{ij} f(x_1 - x_i) + b_{ij}^2 g(x_1 - x_i)$$
$$+ b_{ij} g(x_1 - x_i)\dot{w}_{ij}(t), \quad v_i \in V_2, \ v_j \in V_1 \quad (15)$$

when $t \ge T_1$.

Let $\bar{\delta}_i(t) = x_i(t) - x_1(t)$ and $V_2(t) = \sum_{i\in\{i: v_i \in V_2\}} \bar{\delta}_i^2(t)$. We conclude that the agents corresponding to the vertices in subset V_2 achieve the finite-time consensus in probability when the event A holds, i.e. $P(B \mid A) = 1$.

Based on the conditional probability formula: $P(B \mid A) = \frac{P(AB)}{P(A)} = P(AB)$, the agents corresponding to the vertices in $V_1 \bigcup V_2$ achieve the finite-time consensus in probability. By induction, the multi-agent system (3) achieves the finite-time consensus in probability. □

Here, we consider the other special directed network which is an extension of directed tree. We assume that agent 1 is the leader. Assume further that all vertices of the directed network G can be classified into the following subsets: $V_0 = \{v_1\}$, $V_1 = \{v_j \in V : v_j$ only receives information from v_1 at any time $t\}, \ldots$, $V_q = \{v_j \in V : v_j$ only receives information from vertex in $\bigcup_{p=0}^{q-1} V_p$ at any time $t\}$. Moreover, $\bigcup_{p=1}^{q} V_p = V$. Under this directed network, we get the following result on the finite-time consensus in probability.

Theorem 3.5: *Suppose the directed network G satisfies the above assumption, and the assumptions* (i)–(iv) *hold. Then the multi-agent system* (3) *can achieve the finite-time consensus in probability.* □

4. Simulation examples

In this section, we present some examples to illustrate the theorems and corollary established above.

We consider the network $G = (V, E)$ formed by five autonomous agents with the dynamics (1). It is assumed that $a_{ij} = b_{ij} = 1$ if $(v_j, v_i) \in E$, otherwise $a_{ij} = b_{ij} = 0$, where i, $j \in \mathcal{I}_5$ and $v_i, v_j \in V$. Let $f(\theta) = \mathrm{sig}(\theta)^{\frac{1}{3}}$ and $g(\theta) = \theta$, w_{ij} are equal for all $i, j \in \mathcal{I}_5$, then the assumption holds obviously. Moreover, the initial value of agents are set to be $x_0 = [1, -5, 5, 3, 2]^T$. In the following numerical simulations, we resort to the MATLAB Simulink Toolbox.

Example 4.1: Consider the network G in Figure 1. It can be noted that G is an undirected and connected graph. Then, the motion trajectories of the five autonomous agents are illustrated in Figure 2. It is easy to see that the consensus state of the five agents is

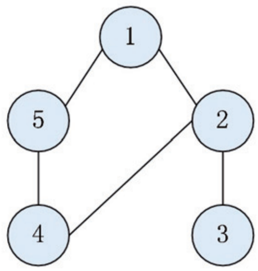

Figure 1. G in Example 4.1.

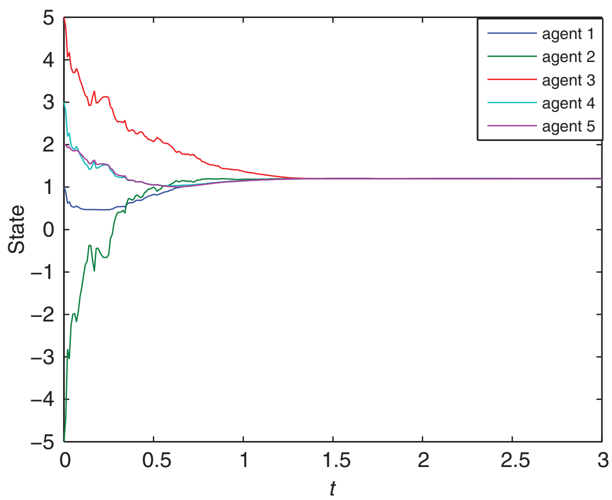

Figure 2. Motion trajectories with network G in Figure 1.

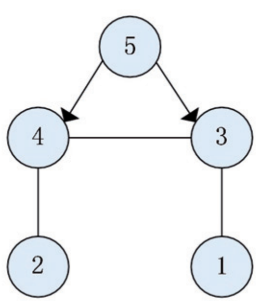

Figure 3. G in Example 4.2.

$\frac{1}{5}\sum_{i=1}^{5} x_{i0} = 1.2$ in probability in Figure 2, which accords with the results established in Theorem 3.1.

Example 4.2: The multi-agent system (1) has a leader 5 and the four followers, and the communication topology among the followers is undirected and connected group (Figure 3). Then, Figure 4 demonstrates the motion trajectories of the five autonomous agents. We can see that the consensus state of the five agents is the leader's state in probability in Figure 4, which accords with the results established in Theorem 3.2.

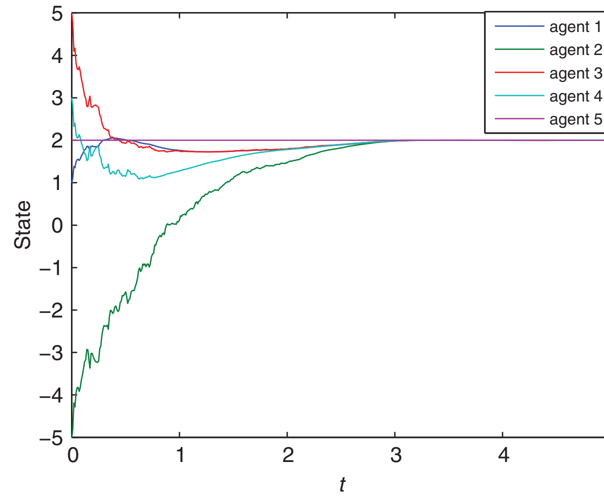

Figure 4. Motion trajectories with network G in Figure 3.

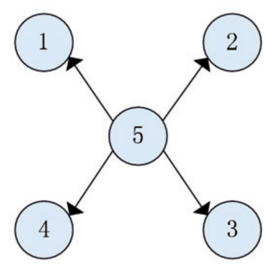

Figure 5. G in Example 4.3.

Example 4.3: The multi-agent system (1) has a leader 5 and four followers, and relevant communication topology is shown in Figure 5. Obviously, these four followers can just receive information from the leader 5. Then, the motion trajectories of the five autonomous agents are given in Figure 6. Note that the consensus state of the five agents is the leader's state in probability in Figure 6, which accords with the results established in Corollary 3.3.

Example 4.4: Consider the network G which is a directed tree in Figure 7. By virtue of numerical simulation, the motion trajectories of the five autonomous agents are illustrated in Figure 8. Note that the consensus state of the five agents is the root vertex's (the agent 1) state in probability in Figure 8, which accords with the results established in Theorem 3.4.

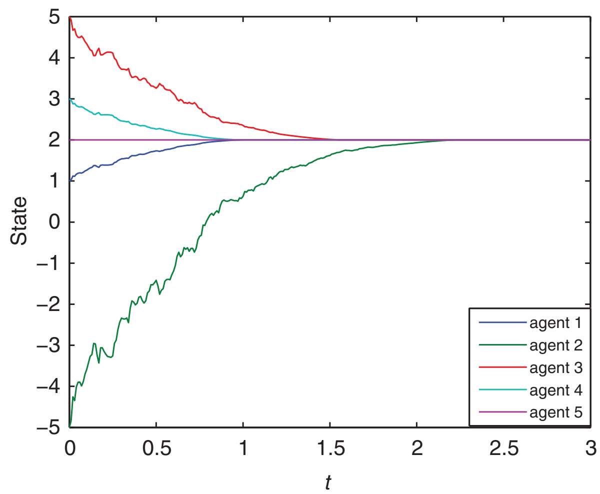

Figure 6. Motion trajectories with network G in Figure 5.

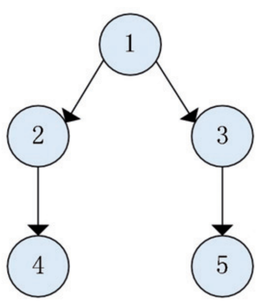

Figure 7. G in Example 4.4.

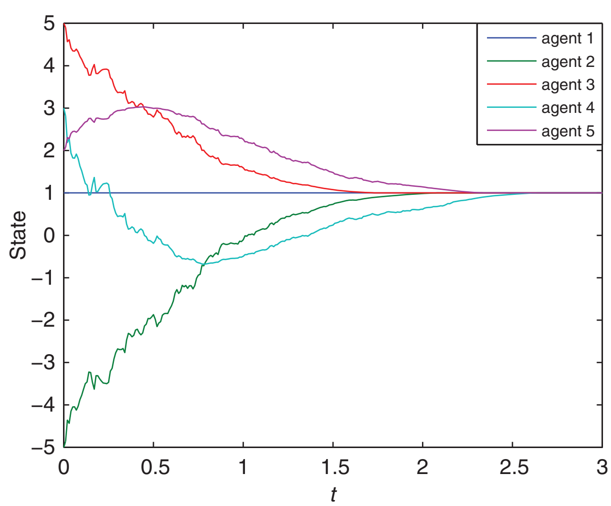

Figure 8. Motion trajectories with network G in Figure 7.

Example 4.5: Consider the network G in Figure 9, then the motion trajectories of the five autonomous agents are given in Figure 10. Note that the consensus state of the five agents is the agent 1's state in probability in Figure 10, which accords with the results established in Theorem 3.5.

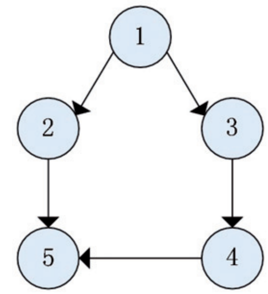

Figure 9. G in Example 4.5.

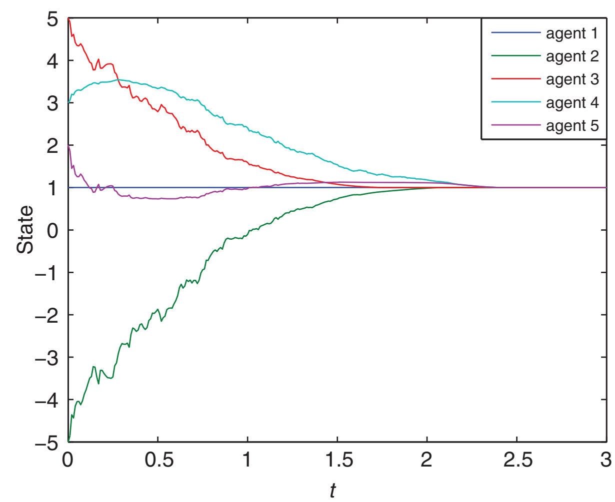

Figure 10. Motion trajectories with network G in Figure 9.

5. Conclusions

In this article, we study the finite-time consensus in probability for stochastic multi-agent system under five different kinds of communication topologies, and give some examples to illustrate the effectiveness of the theoretical results. We will extend the idea in this article to the more complicated multi-agent systems in the future works such as stochastic multi-agent systems with non-symmetric noises or communication delays.

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