Designs, Codes and Cryptography (2023) 91:373–389
https://doi.org/10.1007/s10623-022-01113-4

A design and flexible assignment of orthogonal binary sequence sets for (QS)-CDMA systems

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Received: 27 April 2022 / Revised: 30 August 2022 / Accepted: 5 September 2022 /
Published online: 17 September 2022

Abstract
Boolean functions naturally induce binary sequences of length 2^m and a large number of such orthogonal sequences is required in the design of code-division multiple-access (CDMA) systems. In this paper, Boolean functions are used to construct nonlinear phase orthogonal sequence sets for CDMA communications. For even m, employing carefully designed an m-variable Boolean function with five-valued Walsh spectra, one can get 16 different orthogonal sequence sets with sequence length 2^m. These sequence sets are assigned to a lattice of regular hexagonal cells, and we can ensure the orthogonality of adjacent cells. Moreover, the cross-correlation values between the sequences in a given cell and the sequences in non-neighbouring cells belong to $\{0, \pm 2^{\frac{m}{2}}, \pm 2^{\frac{m}{2}+1}\}$. On the other hand, the cardinality of the sequences sets is 2^{m-3} thus implying a trade-off between the quality of communication and the number of users assigned to each cell. This method can be improved so that the number of users is increased to 2^{m-2} in one half of the network while preserving the

Communicated by K.-U. Schmidt.

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orthogonality between adjacent cells and the same level of low cross-correlation values to the non-neighbouring cells.

Keywords Boolean functions · CDMA systems · Cross-correlation · Five-valued Walsh spectra · Orthogonal sequences

Mathematics Subject Classification 05B20 · 94B99

1 Introduction

Walsh sequences [7] are a set of perfectly orthogonal sequences used to separate users on the downlink channel in CDMA systems (for example in CDMA2000, UMTS cellular standards etc.). In practice, the design of CDMA systems is commonly based on the use of (binary) orthogonal sequences of length 2^m which are called codewords. Even though the whole space comprises 2^{2^m} many sequences, finding a subset of these sequences of large cardinality such that these sequences are mutually orthogonal to each other is difficult. Such a subset of sequences is then allocated to a single cell of users where each user is assigned a unique sequence from this set. To prevent interference from the neighbouring cells, using as a model regular tessellation of hexagonal cells, a standard requirement is that the sequences within any cell are also orthogonal to the sequences in the neighbouring cells (for hexagonal networks there are six neighbouring cells). In addition, the so-called cross-correlation of the sequences in a given cell to non-neighbouring cells should be sufficiently small. A common way of constructing spreading codes (sequences) in these systems is to employ correlation-constrained sets of Hadamard matrices as it has been done in [4, 6, 8]. A particular choice of these matrices that satisfy the imposed orthogonality and cross-correlation conditions was considered in [4, 6, 8].

The problem of allocating a set of sequences satisfying the above criteria to a regular tessellation of hexagonal cells is also governed by some practical requirements. The so-called re-use distance D reflects the ability to use the same codewords in non-adjacent cells that are at sufficient distance D (so that the signal is sufficiently attenuated and does not affect the user employing the same sequence) from the cell where these codewords have originally been placed. In practice, this re-use distance is commonly $D = 4$ which ensures that the use of same sequences only causes acceptable interference levels.

The problem of designing such orthogonal sets was addressed in [5] using cosets of a certain subcode of the first order Reed-Muller code to construct a set of Hadamard matrices with suitable properties. In addition, a suitable assignment of these sets into octants or quadrants (see [5] for definitions) with $D = 4$ was specified, which gave a significant improvement over other methods due to the specific properties of the Hadamard matrices specified in [5]. Furthermore, it was shown in [1] that the same method could be adjusted to provide orthogonal spreading codes of variable length.

In [11], vectorial semi-bent functions were employed for the purpose of constructing large sets of orthogonal sequences and their efficient assignment to a regular tessellation of hexagonal cells was demonstrated. In brief, the number of users per cell achieved by this method was 2^{m-2} which actually provided double as many users compared to the best known methods, in particular compared to the method in [5]. The use of semi-bent functions then implies that the cross-correlation value between the sequences in non-adjacent cells equals to 0 or $\pm 2^{\frac{m}{2}+1}$, for even m. This is a direct consequence of the fact that semi-bent functions

Table 1 A comparison of the main parameters

Methods	$\mathcal{N}$	D	Cross-correlation
[1, 5] ($m = 3, 9$)	2^{m-2}	4	$\{0, \pm 2^{\frac{m}{2}+1}\}$
[11] ($m \geq 6$ even)	2^{m-2}	≥ 4	$\{0, \pm 2^{\frac{m}{2}+1}\}$
[12]($m = 6$)	2^{m-2} or 2^{m-1}	$\sqrt{12}$	$\{0, \pm 2^{\frac{m}{2}+1}\}$
[12]($m \geq 8$ even)	2^{m-2} or 2^{m-1}	$\geq \sqrt{21}$	$\{0, \pm 2^{\frac{m}{2}+1}\}$
Construction 1	2^{m-3}	4	$\{0, \pm 2^{\frac{m}{2}}, \pm 2^{\frac{m}{2}+1}\}$
Construction 2	2^{m-3} or 2^{m-2}	4	$\{0, \pm 2^{\frac{m}{2}}, \pm 2^{\frac{m}{2}+1}\}$

have so-called three-valued Walsh spectra so that any such function is orthogonal to exactly $2^m - 2^{m-2}$ distinct linear functions and its cross-correlation to the remaining 2^{m-2} linear functions is $2^{\frac{m}{2}+1}$. In [12], an efficient design of binary orthogonal sequences of length 2^m, based on vectorial semi-bent functions, is provided so that one third of the network contains cells that comprise 2^{m-1} users whereas two thirds of the network have cells that accommodate 2^{m-2} users. p-phase spreading sequences of length p^m (p is an odd prime number) were also designed for synchronous (QS)-CDMA applications in [11]. This design ensures that the number of orthogonal sequences in any three adjacent cells (referring to regular hexagonal networks) is equal to p^m for the sequences of length p^m. Additionally, we mention two designs of sequences with low correlation proposed in [13, 14] and a recent article [9] which also identifies new applications of these sequences in joint communication and radar systems.

In this paper, we consider the problem of improving the quality of communication by reducing the cross-correlation value between the sequences in non-adjacent cells. More precisely, instead of using sequences that purely stem from semi-bent functions whose cross-correlation value for even m corresponds to $2^{\frac{m}{2}+1}$, we essentially design suitable sets of functions/sequences with spectral values in the set $\{0, \pm 2^{\frac{m}{2}}, \pm 2^{\frac{m}{2}+1}\}$. The main idea is that using such a function we can allocate a subset of linear functions, whose cross-correlation value to this function is $2^{\frac{m}{2}}$, to the first layer of non-adjacent cells. Therefore, the interference between the users in nonorthogonal cells is $2^{\frac{m}{2}}$ or $2^{\frac{m}{2}+1}$ [11]. On the other hand, the improved quality of communication using this method implies a reduced number of users per cell so that each cell can accommodate 2^{m-3} users instead of 2^{m-2} achieved in [11]. We successfully solve the problem of efficient assignment of these sets of sequences within regular hexagonal networks for any even m, thus also ensuring the orthogonality between adjacent cells. In Table 1, we give a comparison of the main parameters with [1, 5, 11, 12].

Nevertheless, it turns out that there is a space for further improvements regarding the number of orthogonal sequences in a certain portion of the network. Namely, a subtle modification of the original technique proposed above enables us to construct even larger families of sets of orthogonal sequences, also satisfying the necessary orthogonality conditions. More precisely, the proposed method increases the number of sequences (users) to be 2^{m-2} in one half of the network whereas for the remaining half of the network (identifying the network as a regular tessellation of hexagonal cells) the number of sequences remains the same 2^{m-3}. This leads to an uneven distribution of the number of users in different cells which from the application point of view is still well-motivated. For instance, such a scenario is quite reasonable in urban areas where there might exist a need for providing larger number of users in a half of the network (namely 2^{m-2} which is the largest known) and at the same time having a better quality of communication than the known methods due to a lower interference from non-neighbouring cells. The problem of allocating these cells of orthogonal sequences to a regular tessellation of hexagonal cells is also addressed, thus allowing an efficient practical implementation of our method for any even $m \geq 6$.

This paper is organized as follows. Some basic notions and definitions related to sequences are introduced in Sect. 2. In Sect. 3, the design of suitable sets of orthogonal sequences with a decreased level of interference between non-adjacent cells is addressed. In Sect. 4, an improved construction is presented which doubles the number of users locally. Some concluding remarks are given in Sect. 5.

2 Preliminaries

Let $\mathbb{F}_{2^m}$ and $\mathbb{F}_2^m$ denote the finite field $GF(2^m)$ and the corresponding vector space, respectively. An m-variable Boolean function f is a function from $\mathbb{F}_2^m$ to $\mathbb{F}_2$, thus for any fixed $X = (x_1, \ldots, x_m) \in \mathbb{F}_2^m$ we have $X \overset{f}{\mapsto} f(X) \in \mathbb{F}_2$. The set of all Boolean functions in m-variables is denoted by $\mathcal{B}_m$. For simplicity, we use "+" and $\sum_i$ to denote the addition operations over $\mathbb{F}_2^m$ and $\mathbb{F}_{2^m}$. A Boolean function $f \in \mathcal{B}_m$ is generally represented by its *algebraic normal form* (ANF):

$$f(X) = \sum_{I \subseteq \{1, \cdots, n\}} \lambda_I \prod_{i \in I} x_i, \quad \lambda_I \in \mathbb{F}_2.$$

Especially, any linear Boolean function on $\mathbb{F}_2^m$ can be expressed as the *inner product*

$$\omega \cdot X = \sum_{i=1}^{m} \omega_i x_i,$$

where $\omega = (\omega_1, \ldots, \omega_m)$, $X = (x_1, \ldots, x_m) \in \mathbb{F}_2^m$, and the addition is performed modulo two. The Walsh-Hadamard transform of $f \in \mathcal{B}_m$ at point $\omega \in \mathbb{F}_2^m$ is denoted by $W_f(\omega)$ and computed as

$$W_f(\omega) = \sum_{X \in \mathbb{F}_2^m} (-1)^{f(X) + \omega \cdot X}.$$

Let $supp(f) = \{x \in \mathbb{F}_2^m \mid f(x) = 1\}$ denote the support of f. Then, $f \in \mathcal{B}_m$ is said to be *balanced* if its truth table contains equal number of 0's and 1's, i.e., $\#supp(f) = 2^{m-1}$, or equivalently

$$W_f(\mathbf{0}_m) = 0,$$

where $\mathbf{0}_m$ denotes the all-zero vector of length m. The Parseval's identity [2] states that

$$\sum_{\omega \in \mathbb{F}_2^m} W_f(\omega)^2 = 2^{2m},$$

which implies that $\max_{\omega \in \mathbb{F}_2^m} |W_f(\omega)| \geq 2^{m/2}$. The equality occurs if and only if $f \in B_m$ is a *bent function* [3], i.e.,

$$W_f(\omega) \in \{\pm 2^{m/2}\}, \quad \text{for all } \omega \in \mathbb{F}_2^m.$$

The *sequence* of $f \in \mathcal{B}_m$ is a $(1, -1)$-sequence of length $N = 2^m$ defined as

$$\overline{f} = \left((-1)^{f(0,\ldots,0,0)}, (-1)^{f(0,\ldots,0,1)}, \ldots, (-1)^{f(1,\ldots,1,1)}\right).$$

Let $f_1, f_2 \in \mathcal{B}_m$. Then we have

$$\overline{f_1} \cdot \overline{f_2} = \sum_{x \in \mathbb{F}_2^m} (-1)^{f_1(x)+f_2(x)}. \tag{1}$$

The absolute value of $\overline{f_1} \cdot \overline{f_2}$ is called the *cross-correlation* of the sequences $\overline{f_1}$ and $\overline{f_2}$. If the cross-correlation of $\overline{f_1}$ and $\overline{f_2}$ is equal to zero, i.e.

$$\overline{f_1} \cdot \overline{f_2} = 0,$$

$\overline{f_1}$ and $\overline{f_2}$ are said to be *orthogonal*, denoted by $\overline{f_1} \perp \overline{f_2}$. Furthermore, if $f_1 + f_2 = f + \omega \cdot x$, $\omega \in \mathbb{F}_2^m$, then we clearly have

$$\overline{f_1} \cdot \overline{f_2} = W_f(\omega). \tag{2}$$

By (1) and (2), we can easily get the following result.

Lemma 1 *Let $f_1, f_2 \in \mathcal{B}_m$, and $f_1 + f_2 = f + \omega \cdot x$, $\omega \in \mathbb{F}_2^m$. $\overline{f_1} \perp \overline{f_2}$ if and only if $f_1 + f_2$ is balanced, i.e.,*

$$W_{f_1+f_2}(\boldsymbol{0}_m) = W_f(\omega) = 0.$$

Especially, $\overline{f_1} \perp \overline{f_2}$ if $f_1 + f_2$ is a nonzero linear Boolean function.

Definition 1 We call

$$S = \{\overline{f_i} \mid f_i \in \mathcal{B}_m, i = 1, 2, \ldots, \kappa\} \tag{3}$$

a set of orthogonal sequences of cardinality κ if the sequences in S are pairwise orthogonal. Let S and S' be two sets of orthogonal sequences. S and S' are orthogonal to each other, denoted by $S \perp S'$, if $s \cdot s' = 0$ always holds for any $s \in S$ and $s' \in S'$.

3 Reducing the level of interference in non-adjacent cells

The most efficient approaches for designing large sets of orthogonal sequences are based on the use of (vectorial) semi-bent functions [5, 11]. More precisely, when m is even, the Walsh spectrum of any semi-bent function $f \in \mathcal{B}_m$ is three-valued, so that $W_f(\omega) \in \{0, \pm 2^{\frac{m}{2}+1}\}$. Then, by Parseval's equality, we have that there are $2^m - 2^{m-2}$ linear functions $\omega \cdot x$ which are orthogonal to f, i.e., those linear functions $\omega \cdot x$ for which $W_f(\omega) = 0$. The remaining 2^{m-2} linear functions are not orthogonal to f and the cross-correlation between f and these linear functions is $2^{\frac{m}{2}+1}$. This implies that the cross-correlation value between the users in non-adjacent cells equals to $2^{\frac{m}{2}+1}$.

The above discussion indicates that for the purpose of lowering the interference (cross-correlation value) from non-adjacent cells one may consider a class of Boolean functions (sequences) whose Walsh spectrum is five-valued and given by $\{0, \pm 2^{\frac{m}{2}}, \pm 2^{\frac{m}{2}+1}\}$. The main idea is then to place those linear functions corresponding to spectral values $\pm 2^{\frac{m}{2}}$ (at bent distance from f) in the first layer of non-adjacent cells referred to the allocation of f, whereas the linear functions that are orthogonal to f (Walsh spectral value is 0) can be placed in the same cell as f. The remaining linear functions are then placed further apart thus neutralizing the effect of having the standard cross-correlation value $2^{\frac{m}{2}+1}$. Nevertheless, a lower interference from non-adjacent cells compared to the methods that use semi-bent functions is achieved at the price of a reduced number of users per cell which decreases to 2^{m-3} compared to the standard approach which accommodates 2^{m-2} users [11].

In the remainder of this section, we derive a suitable class of five-valued spectra functions and specify allocation of these functions and the set of linear functions within regular hexagonal networks. We notice that our method of designing these functions belongs to the generalized Maiorana-McFarland (GMM) construction technique, which was introduced in [10] for specifying (resilient) Boolean functions with the highest nonlinearity known.

Construction 1 *Let $m = 2k+6$, where $k \geq 0$ is an integer. Let $X = (X_0, X_0') = (X_1, X_1') = (x_1, \ldots, x_m) \in \mathbb{F}_2^m$, where*

$$\begin{aligned} X_0 &= (x_1, \ldots, x_{k+2}) \in \mathbb{F}_2^{k+2}, \\ X_1 &= (x_1, \ldots, x_{k+3}) \in \mathbb{F}_2^{k+3}, \\ X_0' &= (x_{k+3}, \ldots, x_m) \in \mathbb{F}_2^{k+4}, \ \textit{and} \\ X_1' &= (x_{k+4}, \ldots, x_m) \in \mathbb{F}_2^{k+3}. \end{aligned}$$

Let

$$\begin{aligned} E_0 &= \{00, 01\} \times \mathbb{F}_2^{k}, \\ E_1 &= \{10, 11\} \times \mathbb{F}_2^{k+1}, \\ T_0 &= \mathbb{F}_2^{k+1} \times \{000\}, \ \textit{and} \\ T_1 &= \mathbb{F}_2^{k} \times \{001, 010, 011, 100\}. \end{aligned}$$

For $i = 0, 1$, let $\Phi_i : E_i \mapsto T_i$ be a bijective mapping. A Boolean function $f \in \mathcal{B}_m$ is constructed as follows:

$$f(X) = \begin{cases} \Phi_0(X_0) \cdot X_0', \ \textit{if}\ X_0 \in E_0 \\ \Phi_1(X_1) \cdot X_1', \ \textit{if}\ X_1 \in E_1. \end{cases}$$

We construct 16 mutually orthogonal sets of sequences as follows. For any fixed $\alpha \in \mathbb{F}_2^3$, let

$$H_\alpha = \{\bar{l} \mid l = (\beta, \alpha) \cdot X, \ \beta \in \mathbb{F}_2^{m-3}\} \tag{4}$$

and

$$H_\alpha^f = \{\overline{f + l} \mid l = (\beta, \alpha) \cdot X, \ \beta \in \mathbb{F}_2^{m-3}\}. \tag{5}$$

Theorem 1 *For $\alpha \in \mathbb{F}_2^3$, let H_α and H_α^f be as defined in Construction 1. Then:*

(i) $|H_\alpha| = |H_\alpha^f| = 2^{m-3}$;
(ii) H_α *is a set of orthogonal sequences, and* $H_\alpha \perp H_{\alpha'}$ *for any* $\alpha \neq \alpha'$;
(iii) H_α^f *is a set of orthogonal sequences, and* $H_\alpha^f \perp H_{\alpha'}^f$ *for any* $\alpha \neq \alpha'$;
(iv) $H_\alpha^f \perp H_{\alpha'}$ *if* $\alpha + \alpha' \in \{101, 110, 111\}$;
(v) Let $s \in H_\alpha^f$ *and* $s' \in H_{\alpha'}$. *Then*

$$s \cdot s' = \begin{cases} \pm 2^{m/2+1}, & \textit{if}\ \alpha + \alpha' = (000), \textit{i.e.}, \alpha = \alpha' \\ \pm 2^{m/2}, & \textit{if}\ \alpha + \alpha' \in \{001, 010, 011, 100\}. \end{cases}$$

Proof (i) This follows immediately from (4) and (5).

(ii) Let $\overline{l}, \overline{l'}$ be two different sequences in H_α with $l = (\beta, \alpha) \cdot X$ and $l' = (\beta', \alpha) \cdot X$, $\beta \neq \beta'$. By (1),

$$\overline{l} \cdot \overline{l'} = \sum_{x \in \mathbb{F}_2^m} (-1)^{(\beta+\beta', \mathbf{0}_3)\cdot X}.$$

Note that $(\beta + \beta', \mathbf{0}_3) \cdot X$ is a nonzero linear Boolean function. By Lemma 1, $\overline{l} \perp \overline{l'}$ always holds which implies that H_α is a set of orthogonal sequences. For any $\alpha \neq \alpha'$, let $\overline{l} \in H_\alpha$ and $\overline{l'} \in H_{\alpha'}$ with $l = (\beta, \alpha) \cdot X$ and $l' = (\beta', \alpha') \cdot X$. Obviously, $l + l'$ is a nonzero linear Boolean function. So, $\overline{l} \perp \overline{l'}$. By Definition 1, this proves $H_\alpha \perp H_{\alpha'}$.

$iii)$ Let $\overline{f + l}$ and $\overline{f + l'}$ be two different sequences in H_α^f, where $l = (\beta, \alpha) \cdot X$ and $l' = (\beta', \alpha) \cdot X$, $\beta \neq \beta'$. Obviously, $\overline{f + l} \cdot \overline{f + l'} = \overline{l} \cdot \overline{l'} = 0$, which implies H_α^f is a set of orthogonal sequences. For any $\alpha \neq \alpha'$, let $\overline{f + l} \in H_\alpha^f$ and $\overline{f + l'} \in H_{\alpha'}^f$ with $l = (\beta, \alpha) \cdot X$ and $l' = (\beta', \alpha') \cdot X$. Obviously, $(f + l) + (f + l') = l + l'$ is balanced. By Lemma 1, $\overline{f + l} \perp \overline{f + l'}$. By Definition 1, this proves $H_\alpha^f \perp H_{\alpha'}^f$.

$iv)$ For $s \in H_\alpha^f$ and $s' \in H_{\alpha'}$, their inner product is computed as

$$s \cdot s' = \overline{f + l} \cdot \overline{l'} = \sum_{x \in \mathbb{F}_2^m} (-1)^{f+l+l'}, \tag{6}$$

where $l = (\beta, \alpha) \cdot X$, $l' = (\beta', \alpha') \cdot X$. By (2), and noticing $E_0 \times \mathbb{F}_2^{k+4} \cup E_1 \times \mathbb{F}_2^{k+3} = \mathbb{F}_2^m$, (6) can be written as,

$$s \cdot s' = W_f(\beta + \beta', \alpha + \alpha') = S_{E_0} + S_{E_1}, \tag{7}$$

where

$$S_{E_0} = \sum_{X_0 \in E_0} \sum_{X_0' \in \mathbb{F}_2^{k+4}} (-1)^{\Phi_0(X_0)\cdot X_0' + (\beta+\beta', \alpha+\alpha')\cdot X}$$

and

$$S_{E_1} = \sum_{X_1 \in E_1} \sum_{X_1' \in \mathbb{F}_2^{k+3}} (-1)^{\Phi_1(X_1)\cdot X_1' + (\beta+\beta', \alpha+\alpha')\cdot X}.$$

For $i = 0, 1$, let $\beta + \beta' = (\eta_i, \theta_i)$, where $\eta_i \in \mathbb{F}_2^{k+2+i}$ and $\theta_i \in \mathbb{F}_2^{k+1-i}$. Then

$$S_{E_i} = \sum_{X_i \in E_i} (-1)^{\eta_i \cdot X_i} \sum_{X_i' \in \mathbb{F}_2^{k+4-i}} (-1)^{\Phi_i(X_i)\cdot X_i' + (\theta_i, \alpha+\alpha')\cdot X_i'}. \tag{8}$$

Note that $\Phi_i : E_i \mapsto T_i$ is a bijective mapping with $T_0 = \mathbb{F}_2^{k+1} \times \{000\}$, and $T_1 = \mathbb{F}_2^k \times \{001, 010, 011, 100\}$. Thus, when $\alpha + \alpha' \in \{101, 110, 111\}$, $e_i(X_i') = \Phi_i(X_i) \cdot X_i' + (\theta_i, \alpha + \alpha') \cdot X_i'$ must be a nonzero linear Boolean function for any fixed $X_i \in E_i$. More precisely, $e_i(X_i')$ must contain some of the variables x_{m-2}, x_{m-1}, x_m, thus $e_i(X_i')$ is balanced. Hence,

$$\sum_{X_i' \in \mathbb{F}_2^{k+4-i}} (-1)^{\Phi_i(X_i)\cdot X_i' + (\theta_i, \alpha+\alpha')\cdot X_i'} = 0,$$

which implies that $S_{E_i} = 0$. By (7), we have $s \cdot s' = 0$. By Definition 1, $H_\alpha^f \perp H_{\alpha'}$.

v) Let us analyze the case $\alpha+\alpha' = \mathbf{0}_3$ in detail. By the definition of Φ_0, the linear function $\Phi_0(X_0)\cdot X_0'$ does not depend on the variables x_{m-2}, x_{m-1}, x_m for $T_0 = \mathbb{F}_2^{k+1} \times \{000\}$. By (8), we have

$$S_{E_0} = \sum_{X_0\in E_0} (-1)^{\eta_0\cdot X_0} \sum_{X_0'\in\mathbb{F}_2^{k+4}} (-1)^{\Phi_0(X_0)\cdot X_0' + (\theta_0,\mathbf{0}_3)\cdot X_0'}.$$

Since Φ_0 is bijection, then for any $\beta+\beta' = (\eta_0,\theta_0) \in \mathbb{F}_2^{m-3}$, there will exist a unique $X_0 \in E_0$ such that $\Phi_0(X_0) = (\theta_0, \mathbf{0}_3)$. Thus,

$$S_{E_0} = (-1)^{\eta_0\cdot\Phi_0^{-1}(\theta_0,\mathbf{0}_3)} \sum_{X_0'\in\mathbb{F}_2^{k+4}} (-1)^0 = \pm 2^{k+4} = \pm 2^{m/2+1}.$$

By (8), we have

$$S_{E_1} = \sum_{X_1\in E_1} (-1)^{\eta_1\cdot X_1} \sum_{X_1'\in\mathbb{F}_2^{k+3}} (-1)^{\Phi_1(X_1)\cdot X_1' + (\theta_1,\mathbf{0}_3)\cdot X_1'}. \tag{9}$$

Notice that $T_1 = \mathbb{F}_2^k \times \{001, 010, 011, 100\}$, $\Phi_1(X_1)\cdot X_1'$ contains some of the variables x_{m-2}, x_{m-1}, x_m, and so does $\Phi_1(X_1)\cdot X_1' + (\theta_1,\mathbf{0}_3)\cdot X_1'$. So $\Phi_1(X_1)\cdot X_1' + (\theta_1,\mathbf{0}_3)\cdot X_1'$ is balanced. Furthermore,

$$\sum_{X_1'\in\mathbb{F}_2^{k+3}} (-1)^{\Phi_1(X_1)\cdot X_1' + (\theta_1,\mathbf{0}_3)\cdot X_1'} = 0.$$

By (9), $S_{E_1} = 0$. By (7), we have $s\cdot s' = S_{E_0} + S_{E_1} = 2^{m/2+1}$.

The case $\alpha+\alpha' \in \{001, 010, 011, 100\}$ can be analyzed similarly. In this case $S_{E_0} = 0$ due to the presence of x_{m-2}, x_{m-1}, x_m in $e_0(X_0')$, whereas $S_{E_1} = \pm 2^{k+3} = 2^{m/2}$ due to the definition of T_1. In this case $s\cdot s' = S_{E_0} + S_{E_1} = 2^{m/2}$. This proves v). □

Remark 1 It is easily verified that indeed $E_0 \times \mathbb{F}_2^{k+4} \cup E_1 \times \mathbb{F}_2^{k+3}$ equals to $\mathbb{F}_2^m$. For instance, if $k = 1$ then

$$\mathbb{F}_2^8 = \{00\}\times\mathbb{F}_2^6 \cup \{01\}\times\mathbb{F}_2^6 \cup \{100\}\times\mathbb{F}_2^5 \cup \{101\}\times\mathbb{F}_2^5$$

$$\cup\{110\}\times\mathbb{F}_2^5 \cup \{111\}\times\mathbb{F}_2^5.$$

The cross-correlation between H_α and H_α^f is depicted in Table 2, which is then utilized to provide an efficient assignment of these orthogonal sets in a regular hexagonal networks as given in Fig. 2.

Example 1 Let $m = 6$, thus $k = 0$. Let $f \in \mathcal{B}_6$ be a five-valued spectra function given by (13) whose ANF is $f(X) = \overline{x}_1x_2x_3 + x_1\overline{x}_2\overline{x}_3x_6 + x_1\overline{x}_2x_3x_5 + x_1x_2\overline{x}_3(x_5+x_6) + x_1x_2x_3x_4$, where each $\overline{x}_i = x_i + 1$. By using the signs instead of $+1, -1$, we have

$$\begin{aligned}\overline{f} : (&++++++++++++++++\\ &++++++++--------\\ &+-+-+-+-++--++--\\ &+--++--++++++----).\end{aligned}$$

Table 2 The cross-correlation between orthogonal sets of sequences

	H^f_{000}	H^f_{001}	H^f_{010}	H^f_{011}	H^f_{100}	H^f_{101}	H^f_{110}	H^f_{111}
H_{000}	$\pm 2^{m/2+1}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	0	0
H_{001}	$\pm 2^{m/2}$	$\pm 2^{m/2+1}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	$\pm 2^{m/2}$	0	0
H_{010}	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2+1}$	$\pm 2^{m/2}$	0	0	$\pm 2^{m/2}$	0
H_{011}	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2+1}$	0	0	0	$\pm 2^{m/2}$
H_{100}	$\pm 2^{m/2}$	0	0	0	$\pm 2^{m/2+1}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$
H_{101}	0	$\pm 2^{m/2}$	0	0	$\pm 2^{m/2}$	$\pm 2^{m/2+1}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$
H_{110}	0	0	$\pm 2^{m/2}$	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2+1}$	$\pm 2^{m/2}$
H_{111}	0	0	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2+1}$

The sets of orthogonal sequences $\{H_\alpha \mid \alpha \in \mathbb{F}_2^3\}$ divides the Hadamard matrix of size 64×64 into 8 parts, see Appendix for details. The inner products between any sequence in H^f_α and H_α, for $\alpha \in \mathbb{F}_2^3$, are given in Table 2, for $m = 6$. These sets of sequences are then assigned to a lattice of regular hexagonal cells as depicted in Fig. 1.

3.1 Allocation of orthogonal sets to regular hexagonal networks

As already mentioned, for practical applications taking $D = 4$ suffices to keep the effect of the interference caused by reused sequences at acceptable level. Based on this fact one possible assignment of sixteen orthogonal sets of sequences within a regular tessellation of hexagonal cells is given in Fig. 1 below.

In Table 3, the maximum cross-correlation values $R_{\max}$ between the codewords assigned to different cells are given for each distance measured from the cell centre up to the re-use distance.

This relative distance d between two cells, say R and S, is calculated as $\sqrt{i^2 + j^2 + ij}$, where i and j are nonnegative integers that specify the distance on two axis u and v. This calculation is depicted in more detail in Fig. 2, where the distance between R and S is $\sqrt{3^2 + 6^2 + 3 \cdot 6} = 3\sqrt{7}$.

Remark 2 The re-use distance for the assignment in Fig. 2 is $D = 4$ and the sequences in adjacent cells are orthogonal. The cross-correlation value of two sequences is $\pm 2^{m/2+1}$ if and only if the distance between the cells are 2 and $2\sqrt{3}$.

4 Increasing the number of users locally

The previous approach essentially reduces the interference to the first layer of non-adjacent cells by a factor two, though at the price of having "only" 2^{m-3} users per cell. Nevertheless, it is possible to increase the number of users locally. Thus, in what follows we preserve the same benefits in terms of a reduced interference while increasing the number of users in one half of the network to be 2^{m-2}.

Construction 2 *Let $m = 2k + 4$ with $k \geq 2$. Let $Y = (y_1, \ldots, y_{2k}) \in \mathbb{F}_2^{2k}$, and $x = (x_1, x_2, x_3, x_4) \in \mathbb{F}_2^4$. Two Boolean functions $g_1, g_2 \in B_{2k}$ are defined as*

$$g_0(Y) = (y_1, \ldots, y_k) \cdot (y_{k+1}, \ldots, y_{2k}) \tag{10}$$

and

$$g_1(Y) = ((y_1, \ldots, y_k)M) \cdot (y_{k+1}, \ldots, y_{2k}), \tag{11}$$

where

$$M = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 1 \end{pmatrix}.$$

We define two Boolean functions $h_0, h_1 \in \mathbb{F}_2^4$ as follows:

$$h_0(X) = x_1(x_2 + 1)x_4 + x_1x_2(x_3 + x_4), \tag{12}$$

$$h_1(X) = x_1(x_2 + 1)(x_3 + x_4) + x_1x_2x_3. \tag{13}$$

We next construct a vectorial Boolean function $F : \mathbb{F}_2^m \mapsto \mathbb{F}_2^2$ as follows:

$$F = (f_0, f_1), \tag{14}$$

where for $i = 0, 1$,

$$f_i(Y, X) = g_i(Y) + h_i(X). \tag{15}$$

For $c \in \mathbb{F}_2^2$ and $\alpha \in \mathbb{F}_2^3$, let $f_c = c \cdot F$ and

$$H_\alpha^{f_c} = \{\overline{f_c + l} \mid l = (\beta, \alpha) \cdot (Y, X),\ \beta \in \mathbb{F}_2^{m-3}\}. \tag{16}$$

Let

$$S_{10}^{f_c} = H_{010}^{f_c} \cup H_{110}^{f_c} \text{ and } S_{01}^{f_c} = H_{001}^{f_c} \cup H_{101}^{f_c}. \tag{17}$$

Then, we obtain 24 sets of disjoint sequences, where for $c \in \mathbb{F}_2^2$, $|S_{10}^{f_c}| = |S_{01}^{f_c}| = 2^{m-2}$ and $|H_\alpha^{f_c}| = 2^{m-3}$ when $\alpha \in \{000, 100, 011, 111\}$.

Theorem 2 *Let the sets of sequences be defined by (16) as in Construction 2.*

(i) For any $c \in \mathbb{F}_2^2$, $H_\alpha^{f_c}$ is a set of orthogonal sequences, and $H_\alpha^{f_c} \perp H_{\alpha'}^{f_c}$ for any $\alpha \neq \alpha'$.
(ii) Let $s \in H_\alpha^{f_c}$ and $s' \in H_{\alpha'}^{f_{c'}}$, where $c \neq c'$. Let $v = c + c'$ and $\lambda = \alpha + \alpha'$. Then

$$s \cdot s' = \begin{cases} \pm 2^{m/2+1}, & \text{if } \lambda = \mathbf{0}_3 \\ 0, & \text{if } \lambda \in \{(0, v), (1, v), 100\} \\ \pm 2^{m/2}, & \text{otherwise.} \end{cases} \tag{18}$$

Proof (i) The proof uses the same technique as the one in Theorem 1 (iii) and is therefore omitted.

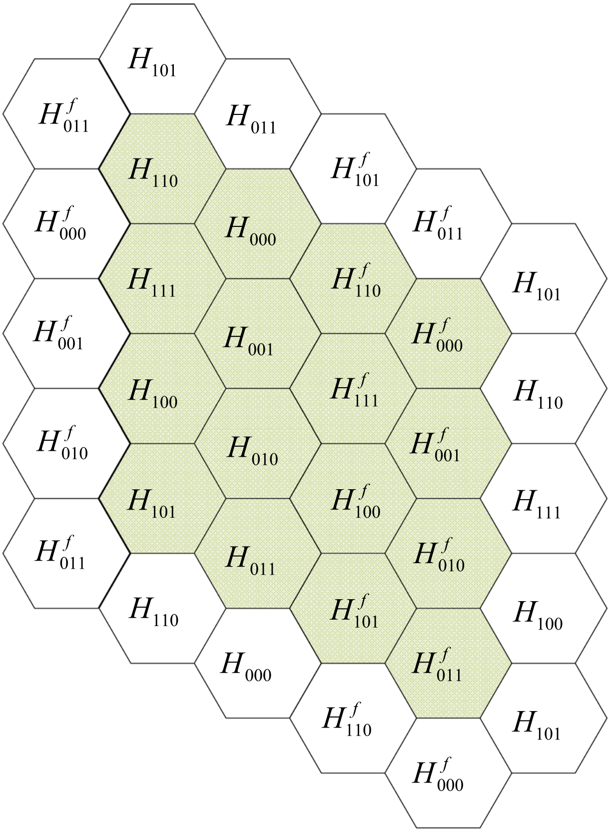

Fig. 1 Assignment of 16 orthogonal sets to a lattice of regular hexagonal cells

(ii) Let $s = \overline{f_c + l}$ and $s' = \overline{f_{c'} + l'}$, where $l = (\beta, \alpha) \cdot (Y, X)$ and $l' = (\beta', \alpha') \cdot (Y, X)$. Let $\beta + \beta' = (\theta, \delta)$, where $\theta \in \mathbb{F}_2^{2k}$ and $\delta \in \mathbb{F}_2$. Then $l + l' = (\theta, \delta, \lambda)$ Note that $f_c + f_{c'} = f_v$, where $v = c + c' \neq \mathbf{0}_2$. We have

$$s \cdot s' = \sum_{(Y,X)\in\mathbb{F}_2^m} (-1)^{f_c + f_{c'} + l + l'} = W_{f_v}(\theta, \delta, \lambda) \tag{19}$$

Let $g_v = v \cdot (g_0, g_1)$ and $h_v = v \cdot (h_0, h_1)$. Then

$$\begin{aligned} W_{f_v}(\theta, \delta, \lambda) &= \sum_{(Y,X)\in\mathbb{F}_2^m} (-1)^{g_v(Y) + h_v(X) + (\theta,\delta,\lambda)\cdot(Y,X)} \\ &= \sum_{Y\in\mathbb{F}_2^{2k}} (-1)^{g_v(Y) + \theta\cdot Y} \sum_{X\in\mathbb{F}_2^4} (-1)^{h_v(X) + (\delta,\lambda)\cdot X} \\ &= W_{g_v}(\theta) W_{h_v}(\delta, \lambda). \end{aligned} \tag{20}$$

It is well known that $g_0(Y)$ is a quadratic bent function [2, pp. 430]. By (10) and (11), we have

$$g_0 + g_1 = ((y_1, \cdots, y_k)(\mathbf{I}_k + M)) \cdot (y_{k+1}, \cdots, y_{2k}),$$

where $\mathbf{I}_k$ is the identity matrix of order k. It is easy to see that $g_1 = g_0(YA)$ and $g_0 + g_1 = g_0(YB)$ where

$$A = \begin{pmatrix} M & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_k \end{pmatrix}$$

and

$$B = \begin{pmatrix} \mathbf{I}_k + M & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_k \end{pmatrix}.$$

Note that M and $\mathbf{I}_k + M$ are two nonsingular $n \times n$ matrices, which implies that g_1 and $g_0 + g_1$ are obtained from g_0 by the linear transformations A and B, respectively. This means that g_1 and $g_0 + g_1$ are also bent functions. Thus, in Eq. (20),

$$W_{g_v}(\theta) = \pm 2^k, \ v \neq \mathbf{0}_2 \tag{21}$$

always holds. Obviously, $h_{10} = h_0$, $h_{01} = h_1$, and $h_{11} = h_0 + h_1$. We have

$$\begin{aligned} W_{h_{10}}(\delta,\lambda) = & \sum_{\substack{x_1=0 \\ (x_2,x_3,x_4)\in\mathbb{F}_2^3}} (-1)^{h_0+(\delta,\lambda)\cdot X} + \sum_{\substack{(x_1,x_2)=(10) \\ (x_3,x_4)\in\mathbb{F}_2^2}} (-1)^{h_0+(\delta,\lambda)\cdot X} + \sum_{\substack{(x_1,x_2)=(11) \\ (x_3,x_4)\in\mathbb{F}_2^2}} (-1)^{h_0+(\delta,\lambda)\cdot X} \\ = & \sum_{(x_2,x_3,x_4)\in\mathbb{F}_2^3} (-1)^{\lambda\cdot(x_2,x_3,x_4)} + (-1)^{\delta} \sum_{(x_3,x_4)\in\mathbb{F}_2^2} (-1)^{\lambda(0,x_3,x_4)+x_4} \\ & + (-1)^{\delta} \sum_{(x_3,x_4)\in\mathbb{F}_2^2} (-1)^{\lambda(1,x_3,x_4)+x_3+x_4}. \end{aligned}$$

Note that

$$\sum_{(x_2,x_3,x_4)\in\mathbb{F}_2^3} (-1)^{\lambda\cdot(x_2,x_3,x_4)} = \begin{cases} 8, & \text{if } \lambda = \mathbf{0}_3 \\ 0, & \text{otherwise,} \end{cases} \tag{22}$$

$$\sum_{(x_3,x_4)\in\mathbb{F}_2^2} (-1)^{\lambda(0,x_3,x_4)+x_4} = \begin{cases} 4, & \text{if } \lambda \in \{001, 101\} \\ 0, & \text{otherwise} \end{cases} \tag{23}$$

and

$$\sum_{(x_3,x_4)\in\mathbb{F}_2^2} (-1)^{\lambda(1,x_3,x_4)+x_3+x_4} = \begin{cases} \pm 4, & \text{if } \lambda \in \{011, 111\} \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

Combining (22), (23) and (24),

$$W_{h_{10}}(\delta,\lambda) = \begin{cases} 8, & \text{if } \lambda = \mathbf{0}_3 \\ 0, & \text{if } \lambda \in \{010, 110, 100\} \\ \pm 4, & \text{if } \lambda \in \{001, 101, 011, 111\}. \end{cases} \tag{25}$$

Similarly, we have

$$W_{h_{01}}(\delta,\lambda) = \begin{cases} 8, & \text{if } \lambda = \mathbf{0}_3 \\ 0, & \text{if } \lambda \in \{001, 101, 100\} \\ \pm 4, & \text{if } \lambda \in \{011, 111, 010, 110\} \end{cases} \tag{26}$$

Table 3 Maximum cross-correlation value for sequences within cell distance 4

d	0	1	$\sqrt{3}$	2	$\sqrt{7}$	3	$2\sqrt{3}$	$\sqrt{13}$	4
$R_{\max}$	0	0	$\pm 2^{\frac{m}{2}}$	$\pm 2^{m/2+1}$	$\pm 2^{\frac{m}{2}}$	$\pm 2^{\frac{m}{2}}$	$\pm 2^{m/2+1}$	$\pm 2^{\frac{m}{2}}$	2^m

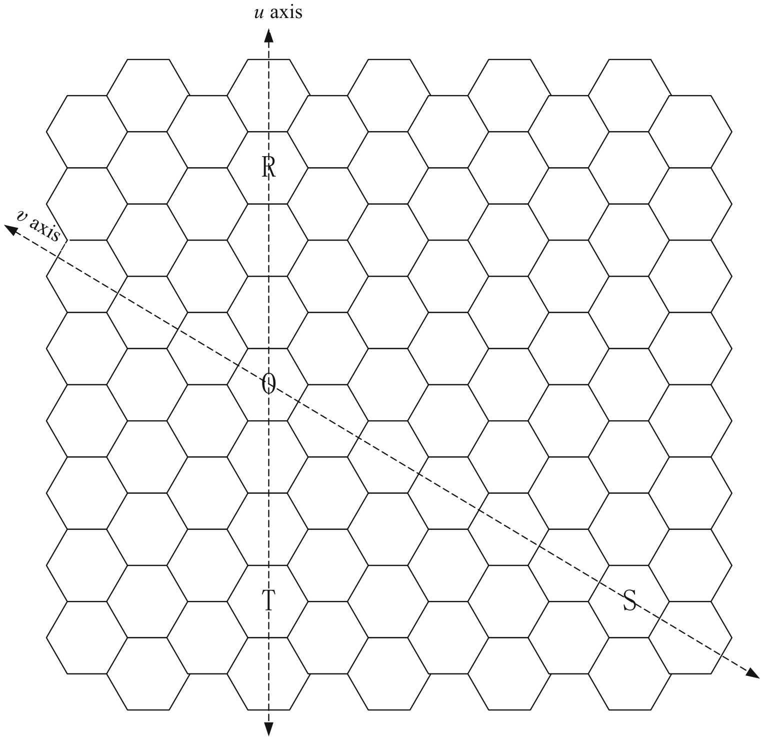

Fig. 2 The distance between the cells

and

$$W_{h_{11}}(\delta,\lambda)=\begin{cases}8, & \text{if } \lambda=\mathbf{0}_3\\ 0, & \text{if } \lambda\in\{011,111,100\}\\ \pm 4, & \text{if } \lambda\in\{010,110,001,101\}.\end{cases} \tag{27}$$

Combining (25), (26) and (27), for $v\in\mathbb{F}_2^{3*}$,

$$W_{h_v}(\delta,\lambda)=\begin{cases}8, & \text{if } \lambda=\mathbf{0}_3\\ 0, & \text{if } \lambda\in\{(0,v),(1,v),100\}\\ \pm 4, & \text{otherwise.}\end{cases} \tag{28}$$

(21) and (28) are substituted in (20), and we get

$$W_{f_v}(\theta,\delta,\lambda)=\begin{cases}\pm 2^{m/2+1}, & \text{if } \lambda=\mathbf{0}_3\\ 0, & \text{if } \lambda\in\{(0,v),(1,v),100\}\\ \pm 2^{m/2}, & \text{otherwise.}\end{cases}$$

Table 4 The cross-correlation between the sets of orthogonal sequences

	H_{000}	H_{100}	H_{001}	H_{101}	H_{010}	H_{110}	H_{011}	H_{111}
H_{000}^{f10}	$\pm 2^{m/2+1}$	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$
H_{100}^{f10}	0	$\pm 2^{m/2+1}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$
H_{001}^{f10}	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2+1}$	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	0
H_{101}^{f10}	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	$\pm 2^{m/2+1}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	0
H_{010}^{f10}	0	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2+1}$	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$
H_{110}^{f10}	0	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	$\pm 2^{m/2+1}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$
H_{011}^{f10}	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2+1}$	0
H_{111}^{f10}	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	$\pm 2^{m/2+1}$
H_{000}^{f01}	$\pm 2^{m/2+1}$	0	0	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$
H_{100}^{f01}	0	$\pm 2^{m/2+1}$	0	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$
H_{001}^{f01}	0	0	0	$\pm 2^{m/2+1}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$
H_{101}^{f01}	0	0	$\pm 2^{m/2+1}$	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$
H_{010}^{f01}	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	$\pm 2^{m/2+1}$	0	0
H_{110}^{f01}	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2+1}$	0	0	0
H_{011}^{f01}	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	0	$\pm 2^{m/2+1}$	0
H_{111}^{f01}	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	0	0	$\pm 2^{m/2+1}$
H_{000}^{f11}	$\pm 2^{m/2+1}$	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	0
H_{100}^{f11}	0	$\pm 2^{m/2+1}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	0
H_{001}^{f11}	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	$\pm 2^{m/2+1}$	0	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$
H_{101}^{f11}	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2+1}$	0	0	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$
H_{010}^{f11}	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	0	0	$\pm 2^{m/2+1}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$
H_{110}^{f11}	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	0	$\pm 2^{m/2+1}$	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$
H_{011}^{f11}	0	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2+1}$	0
H_{111}^{f11}	0	0	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	$\pm 2^{m/2}$	0	$\pm 2^{m/2+1}$

By (19), the desired result follows immediately. □

The cross-correlation between any two sets of sequences is depicted in Table 4. We then assign 16 sets of sequences to a regular hexagonal network as illustrated in Fig. 3. These 16 sequences sets are composed of 8 sets with cardinality 2^{m-2} and 8 sets whose cardinality (of any set) is 2^{m-3}. The reuse distance for the assignment in Fig. 3 is $D = 4$ and this assignment also ensures the orthogonality of sequences in adjacent cells. Note that the notation $H_\alpha^{f_{00}}$ is abbreviated to H_α in Fig. 3.

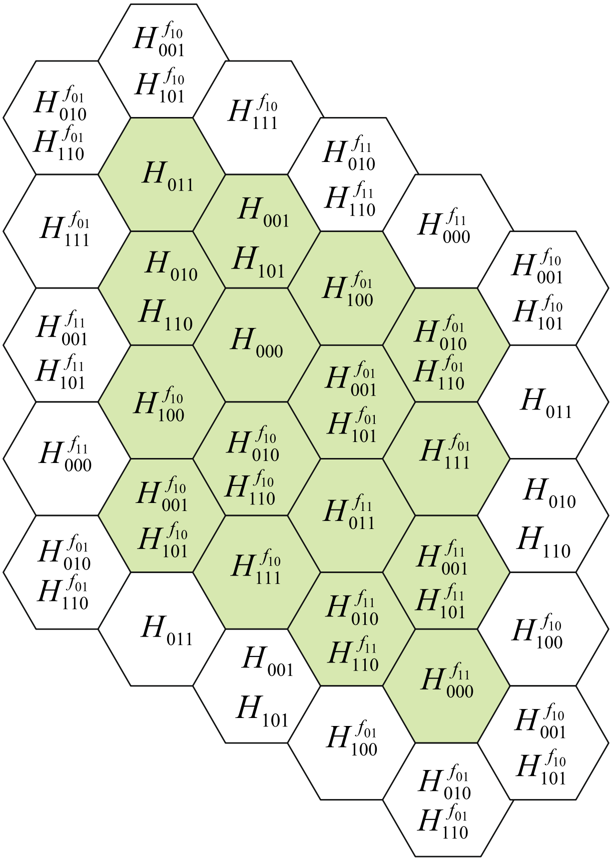

Fig. 3 Assignment of orthogonal sets to a lattice of regular hexagonal cells

5 Conclusions

We have shown that Boolean functions whose Walsh spectra are five-valued are useful in the design of orthogonal sequences for CDMA applications. In particular, the use of such functions leads to an interesting trade-off between the number of users and the quality of communication. An improvement of our basic approach gives an increased number of users in one half of the network and it is an interesting question whether this technique can be optimized further.

Acknowledgements WeiGuo Zhang is partly supported by the National Natural Science Foundation of China (Grant 61972303, 62272360). Enes Pasalic is partly supported by the Slovenian Research Agency (research program P1-0404 and research projects J1-9108, J1-1694).

Data availability All data generated or analysed during this study are included in this published article.

Appendix

The sets of orthogonal sequences $H_{000}, H_{001}, \ldots, H_{111}$ in Example 1 is given as below:

H_{000}:
(++)
(++++++++--------++++++++--------++++++++--------++++++++--------)
(++++++++++++++++----------------++++++++++++++++----------------)
(++++++++----------------++++++++++++++++----------------++++++++)
(++++++++++++++++++++++++++++++++--------------------------------)
(++++++++--------++++++++----------------++++++++--------++++++++)
(++++++++++++++++--------------------------------++++++++++++++++)
(++++++++----------------++++++++--------++++++++++++++++--------)
H_{001}:
(+-)
(+-+-+-+--+-+-+-++-+-+-+--+-+-+-++-+-+-+--+-+-+-++-+-+-+--+-+-+-+)
(+-+-+-+-+-+-+-+--+-+-+-+-+-+-+-++-+-+-+-+-+-+-+--+-+-+-+-+-+-+-+)
(+-+-+-+--+-+-+-+-+-+-+-++-+-+-+-+-+-+-+--+-+-+-+-+-+-+-++-+-+-+-)
(+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+--+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-+)
(+-+-+-+--+-+-+-++-+-+-+--+-+-+-+-+-+-+-++-+-+-+--+-+-+-++-+-+-+-)
(+-+-+-+-+-+-+-+--+-+-+-+-+-+-+-+-+-+-+-+-+-+-+-++-+-+-+-+-+-+-+-)
(+-+-+-+--+-+-+-+-+-+-+-++-+-+-+--+-+-+-++-+-+-+-+-+-+-+--+-+-+-+)
H_{010}:
(++--++--++--++--++--++--++--++--++--++--++--++--++--++--++--++--)
(++--++----++--++++--++----++--++++--++----++--++++--++----++--++)
(++--++--++--++----++--++--++--++++--++--++--++----++--++--++--++)
(++--++----++--++--++--++++--++--++--++----++--++--++--++++--++--)
(++--++--++--++--++--++--++--++----++--++--++--++--++--++--++--++)
(++--++----++--++++--++----++--++--++--++++--++----++--++++--++--)
(++--++--++--++----++--++--++--++--++--++--++--++++--++--++--++--)
(++--++----++--++--++--++++--++----++--++++--++--++--++----++--++)
H_{011}:
(+--++--++--++--++--++--++--++--++--++--++--++--++--++--++--++--+)
(+--++--+-++--++-+--++--+-++--++-+--++--+-++--++-+--++--+-++--++-)
(+--++--++--++--+-++--++--++--++-+--++--++--++--+-++--++--++--++-)
(+--++--+-++--++--++--++-+--++--++--++--+-++--++--++--++-+--++--+)
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H_{100}:
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(++++--------++++----++++++++--------++++++++----++++--------++++)
H_{101}:
(+-+--+-++-+--+-++-+--+-++-+--+-++-+--+-++-+--+-++-+--+-++-+--+-+)
(+-+--+-+-+-++-+-+-+--+-+-+-++-+-+-+--+-+-+-++-+-+-+--+-+-+-++-+-)
(+-+--+-++-+--+-+-+-++-+--+-++-+-+-+--+-++-+--+-+-+-++-+--+-++-+-)
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(+-+--+-+-+-++-+--+-++-+-+-+--+-+-+-++-+-+-+--+-++-+--+-+-+-++-+-)
H_{110}:
(++----++++----++++----++++----++++----++++----++++----++++----++)
(++----++--++++--++----++--++++--++----++--++++--++----++--++++--)
(++----++++----++--++++----++++--++----++++----++--++++----++++--)
(++----++--++++----++++--++----++++----++--++++----++++--++----++)
(++----++++----++++----++++----++--++++----++++----++++----++++--)

(++−−++−++++−++−−++−++++−−++++−++−−++−++++−++−−++)
(++−−++++−−++−++++−−++++−−++++−−++++−++−−++++−−++)
(++−−++−++++−−++++−++−−++−++++−++−−++++−−++−++++−)

H_{111}:

(+−+−++−+−+−++−+−+−++−+−+−++−+−+−++−+−+−++−+−+−++−+−+−++−)
(+−+−++−++−+−++−+−++−++−+−++−+−++−+−+−++−+−++−++−+−+)
(+−+−++−+−+−++−++−+−+−++−+−++−+−++−+−+−++−++−+−+−++−+−+)
(+−+−++−++−+−+−++−+−+−++−+−++−+−+−++−++−+−+−++−+−++−+−++−)
(+−+−++−+−+−++−+−+−++−+−+−++−++−+−+−++−+−+−++−+−+−++−+−+)
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