Chapter 10

FIR Digital Filter Design

Part A

FIR Digital Filter Design

FIR Digital Filter Design

- Preliminary Considerations
- FIR Filter Design based on Windowed Fourier Series
- Impulse Response of Ideal Filters
- Gibbs Phenomenon
- Fixed Window Function
- Adjustable Window Function
- Implulse Response of FIR Filter with Smooth Transition

1. Preliminary Considerations

- FIR filter design is based on a direct approximation of the specified magnitude response, with the often added requirement that the phase be linear.
- The design of an FIR filter of order N may be accomplished by finding either the length-(N+1) impulse response samples $\{h[n]\}$ or the (N+1) samples of its frequency response $H(e^{j\omega})$.

1.1 Basic Approaches

- For FIR digital filter: the transfer function is a polynomial in z^{-1} with real coefficients：

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n}$$

- If linear phase is desired, the filter coefficients should satisfy:

$$h[n] = \pm h[N-n]$$

1.1 Basic Approaches

- Three commonly used approaches for FIR filter design:
  - **Windowed Fourier series approach**
    - ✓ Truncating the Fourier series representation of the desired frequency response
  - **Frequency sampling approach**
  - **Computer-aided methods:** optimization

1.2 Estimation of the Filter Order

- For reduced computational complexity, degree N of $H(z)$ must be **as small as possible**.
- Lowpass FIR digital filter design
  - Several authors have advanced formulas for estimating the minimum value of the filter order N directly.

1.2 Estimation of the Filter Order

- The digital filter specifications:
  - ω_p : normalized passband edge angular frequency
  - ω_s : normalized stopband edge angular frequency
  - δ_p : peak passband ripple,
  - δ_s : peak stopband ripple.

1.2 Estimation of the Filter Order

- Kaiser's Formula

$$N \approx \frac{-20\log_{10}(\sqrt{\delta_p \delta_s}) - 13}{14.6(\omega_s - \omega_p)/2\pi}$$

- Bellanger's Formula

$$N \approx \frac{-2\log_{10}(10\delta_p \delta_s)}{3(\omega_s - \omega_p)/2\pi} - 1$$

- Hermann's Formula

$$N \approx \frac{D_\infty(\delta_p, \delta_s) - F(\delta_p, \delta_s)[(\omega_s - \omega_p)/2\pi]^2}{(\omega_s - \omega_p)/2\pi}$$

1.2 Estimation of the Filter Order

- The filter orders computed are all different. Each provides only an estimate of the required filter order.

Filter No.	Actual Order	Kaiser's Formula	Bellanger's Formula	Hermann's Formula
1	159	158	163	151
2	38	34	37	37
3	14	12	13	12

1.2 Estimation of the Filter Order

Comparison(1/3):

- The frequency response based the estimated order may or may not meet the given specifications.
- If the specifications are not met, it is recommended that the filter order be gradually increased until the specifications are met.

1.2 Estimation of the Filter Order

Comparison(2/3):

- The estimated filter order N is inversely proportional to the transition band width and does not depend on the actual location of the transition band.
  - A sharp cutoff FIR filter with a narrow transition band would be of very high order, whereas an FIR filter with a wide transition band will have a very low order.

1.2 Estimation of the Filter Order

Comparison(3/3):

- Another interesting property of Kaiser’s and Bellanger’s formulas is that the order depends on the product $\delta_p \delta_s$.
  - If the values of δ_p and δ_s are interchanged, the order remains the same.

2 FIR Filter Design based on Windowed Fourier Series

A variety of Approaches

- Windowed Fourier series
  - based on truncating the Fourier series representation of the prescribed frequency response.
- Frequency sampling approach
  - based on the observation that for a length-(N+1) FIR digital filter, (N+1) distinct equally spaced frequency samples of its frequency response.

2.1 Least integral-Squared Error

- Let $H_d(e^{j\omega})$ denote the desired frequency response function. $H_d(e^{j\omega})$ is periodic function of ω with period 2π and can be expressed as a Fourier series

$$H_d\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-j\omega n}$$

- The Fourier coefficients $\{h_d[n]\}$ are the impulse response samples

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right) e^{j\omega n} d\omega \qquad -\infty < n < \infty$$

2.1 Least integral-Squared Error

- Usually, $H_d(e^{j\omega})$ is piecewise constant with ideal (or sharp) transitions between bands

  => $\{h_d[n]\}$ sequence is of infinite length and noncausal
- Objective

  Finding a finite-duration impulse response $\{h_t[n]\}$ of length $2M+1$ with DTFT $H_t(e^{j\omega})$ approximating the desired DTFT $H_d(e^{j\omega})$

2.1 Least integral-Squared Error

- Minimizing the integral squared error

$$\Phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| H_t(e^{j\omega}) - H_d(e^{j\omega}) \right|^2 d\omega$$

where
$$H_t\left(e^{j\omega}\right) = \sum_{n=-M}^{M} h_t[n] e^{-j\omega n}$$

- Using the Parseval's relation

$$\Phi_R = \sum_{n=-\infty}^{\infty} \left| h_t[n] - h_d[n] \right|^2$$

constant term

$$= \sum_{n=-M}^{M} \left| h_t[n] - h_d[n] \right|^2 + \boxed{\sum_{n=-\infty}^{-M-1} \left| h_d[n] \right|^2 + \sum_{n=M+1}^{\infty} \left| h_d[n] \right|^2}$$

Φ is minimum when $h_t[n] = h_d[n]$ for $-M \le n \le M$

2.1 Least integral-Squared Error

- The best finite-length approximation to ideal infinite-length impulse response in the mean-square sense is obtained by truncating the impulse response

2.1 Least integral-Squared Error

- A causal impulse response $h(n)$ can be obtained from $h_t(n)$ by delaying it with M samples
$$h[n] = h_t[n - N]$$
  - $h[n]$ has the same magnitude response as $h_t[n]$ but its phase response has a linear phase shift of ωM radians
  - The group delay of $h(n)$ is M samples where the linear phase response is $-\omega M$
$$\tau(\omega) = -\frac{d}{d\omega}(-\omega M) = M$$

3. Impulse Response of Ideal Lowpass Filters

- The ideal lowpass filter has a zero-phase frequency response
$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi_c \end{cases}$$
- The corresponding impulse response coefficients
$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty$$
is doubly infinite, not absolutely summable, and therefore *unrealizable*

3. Impulse Response of Ideal Lowpass Filters

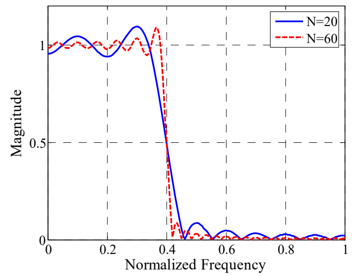
- Truncating to range $-M \le n \le M$ and delaying with M samples yields the causal FIR lowpass filter

$$\hat{h}_{LP}[n] = \begin{cases} \dfrac{\sin[\omega_c (n-M)]}{\pi (n-M)}, & 0 \le n < 2M \\ 0, & \text{otherwise} \end{cases}$$

- The truncation of the impulse response coefficients of the ideal filters exhibit an oscillatory behavior in the respective magnitude responses

4. Gibbs Phenomenon

- *Gibbs phenomenon* - Oscillatory behavior in the magnitude responses of causal FIR filters obtained by truncating the impulse response coefficients of ideal filters



Impact of the length of the window function

(1) Narrower transition band

(2) More ripples

(3) Smaller ripple width

(4) Same largest peak ripple

The performance is better.

How to reduce the highest ripple?

4. Gibbs Phenomenon

- Similar oscillatory behavior observed in the magnitude responses of the truncated versions of other types of ideal filters

4. Gibbs Phenomenon

- Gibbs phenomenon can be explained by treating truncation of $h_d[n]$ as *windowing operation*, i.e., by multiplying the $h_d[n]$ sequence with a finite-length sequence $w[n]$

$$h_t[n] = h_d[n] \cdot w[n]$$

where $w[n]$ is a window function

4. Gibbs Phenomenon

- For a rectangular window

$$w_R[n] = \begin{cases} 1, & -M \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

- The Gibbs phenomenon can be explained in the frequency domain by the convolution theorem

4. Gibbs Phenomenon

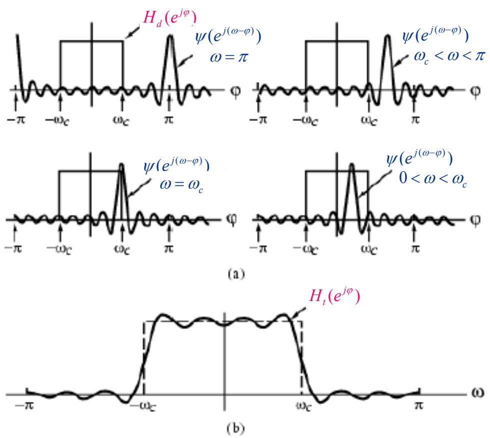
- Multiplication in the time domain corresponds to convolution in the frequency domain

$$H_t(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\varphi})\Psi(e^{j(\omega-\varphi)})d\varphi$$

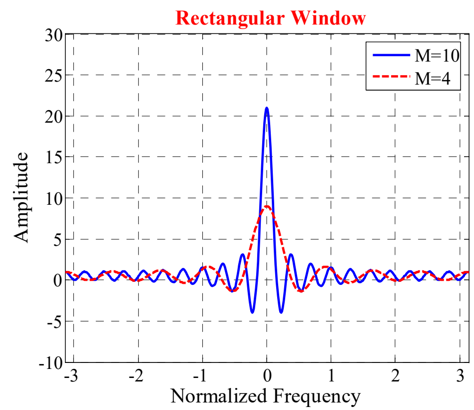
where $H_d(e^{j\omega}) = \mathcal{F}\{h_d[n]\}$ $\Psi(e^{j\omega}) = \mathcal{F}\{w[n]\}$

- $H_t(e^{j\omega})$ is obtained by a periodic continuous convolution of the frequency response $H_d(e^{j\omega})$ with the Fourier transform $\Psi(e^{j\omega})$ of the window

4. Gibbs Phenomenon



4. Gibbs Phenomenon



4. Gibbs Phenomenon

- The frequency response $\Psi(e^{j\omega})$ has a narrow *mainlobe* centered at $\omega=0$
- All the other ripples in the frequency response are called *sidelobes*
- The main lobe is characterized by its width $4\pi/(2M+1)$ defined by the first zero crossings on both sides of $\omega=0$
- As M increases the width of the main lobe decreases
- The area under each lobe remains constant, while the width of each lobe decreases with increasing M

4. Gibbs Phenomenon

- Rectangular window has an abrupt transition to zero outside the range $-M \le n \le M$, which results in Gibbs phenomenon in $H_t(e^{j\omega})$
- Gibbs phenomenon can be reduced either:

  (a) Using a window that tapers smoothly to zero at each end, or

  (b) Providing a smooth transition from passband to stopband in the magnitude specifications

5. Fixed Window Functions

- Using a tapered window causes the height of the sidelobes to diminish, with a corresponding increase in the main lobe width resulting in a wider transition at the discontinuity

5. Fixed Window Functions

- Various window functions: (raised cosine)

Hann:

$$w(n) = \frac{1}{2}\left[1 + \cos\left(\frac{2\pi n}{2M+1}\right)\right], \quad -M \le n \le M$$

Hamming:

$$w(n) = 0.54 + 0.46\cos\left(\frac{2\pi n}{2M+1}\right), \quad -M \le n \le M$$

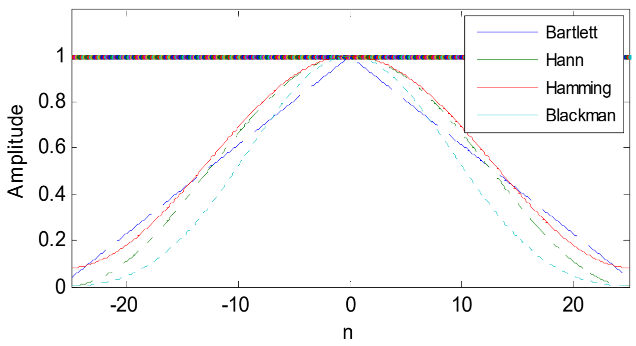
Blackman:

$$w(n) = 0.42 + 0.5\cos\left(\frac{2\pi n}{2M+1}\right) + 0.08\cos\left(\frac{4\pi n}{2M+1}\right)$$

$$-M \le n \le M$$

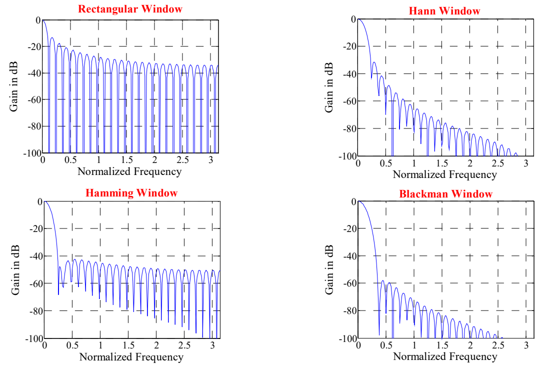
5. Fixed Window Functions

- Various window functions: (raised cosine)



5. Fixed Window Functions

- Plots of magnitudes of the DTFTs of these windows for $M = 25$ are shown below:



5. Fixed Window Functions

- Magnitude spectrum of each window characterized by a main lobe centered at $\omega=0$ followed by a series of sidelobes with decreasing amplitudes
- Parameters predicting the performance of a window in filter design are:

  1) Main lobe width

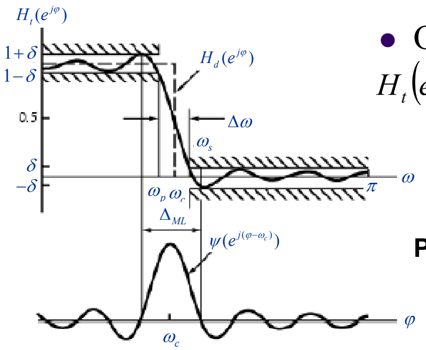
  2) Relative sidelobe level

5. Fixed Window Functions

- Main lobe width - Δ_{ML} given by the distance between zero crossings on both sides of main lobe
- Relative sidelobe level - A_{sl} given by the difference in dB between amplitudes of largest sidelobe and main lobe

5. Fixed Window Functions

Lowpass Filter Design by Windowing



- Observe
$$H_t\left(e^{j(\omega_c+\omega)}\right)+H_t\left(e^{j(\omega_c-\omega)}\right)\approx 1$$
  - Thus
$$H_t\left(e^{j\omega_c}\right)\cong 0.5$$

Passband and stopband ripples are the same.

5. Fixed Window Functions

- Distance between the locations of the maximum passband deviation and minimum stopband value $\approx\Delta_{ML}$
- Width of transition band $\Delta\omega = \omega_s - \omega_p < \Delta_{ML}$

5. Fixed Window Functions

- To ensure a fast transition from passband to stopband, window should have a very small main-lobe width
- To reduce the passband and stopband ripple δ, the area under the sidelobes should be very small
- Unfortunately, these two requirements are **contradictory**

5. Fixed Window Functions

- In the case of rectangular, Hann, Hamming, and Blackman windows, the value of ripple does not depend on filter length or cutoff frequency ω_c, and is essentially constant
- In addition, $\Delta\omega \approx c/M$

  where c is a constant for most practical purposes

5. Fixed Window Functions

Properties of fixed window functions

Type of Window	Main Lobe Width	Relative Sidelobe Level	Mini. Stopband Attenuation	Transition Bandwidth
Rectangular	4π/(2M+1)	13.3dB	20.9dB	0.92π/M
Barlett	4π/(2M+1)	26.5dB		
Hann	8π/(2M+1)	31.5dB	43.9dB	3.11π/M
Hamming	8π/(2M+1)	42.7dB	54.5dB	3.32π/M
Blackman	12π/(2M+1)	58.1dB	75.3dB	5.56π/M

5. Fixed Window Functions

Filter Design Steps -

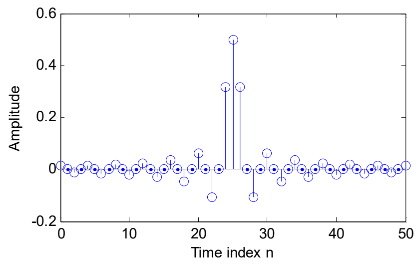
① Set $\omega_c = (\omega_p + \omega_s)/2$

② Choose window based on specified α_s

③ Estimate M using $\Delta\omega \approx c/M$

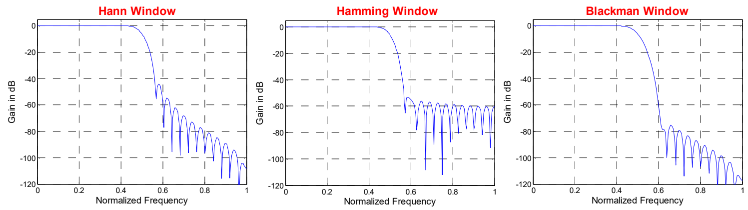
5. Fixed Window Functions

- Lowpass filter of length 51 and $\omega_c = \pi/2$



5. Fixed Window Functions

- Lowpass filter of length 51 and $\omega_c = \pi/2$



- An increase in the main lobe width is associated with an increase in the width of the transition band
- A decrease in the sidelobe amplitude results in an increase in the stopband attenuation

5. Fixed Window Functions

- Step to design a FIR digital Filter
  - Determine the type of the window by the stopband attenuation
  - Determine the length of the window by the width of the transition width

6. Adjustable Window Functions

- Dolph-Chebyshev Window –

$$w(n) = \frac{1}{2M+1}\left[\frac{1}{\gamma} + 2\sum_{k=1}^{M} T_k\left(\beta\cos\frac{k\pi}{2M+1}\right)\cos\left(\frac{2nk\pi}{2M+1}\right)\right] \quad -M \le n \le M$$

where
$$\gamma = \frac{\text{sidelobe amplitude}}{\text{main lobe amplitude}}$$
$$\beta = \cosh\left(\frac{1}{2M}\cosh^{-1}\frac{1}{\gamma}\right)$$

and
$$T_l(x) = \begin{cases} \cos(l\cos^{-1}x), & \text{for } |x| \le 1 \\ \cosh(l\cos^{-1}x), & \text{for } |x| > 1 \end{cases}$$

6. Adjustable Window Functions

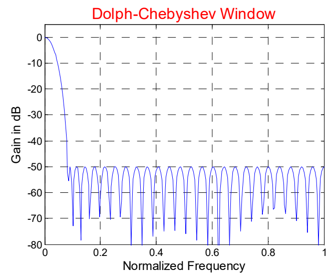
- Dolph-Chebyshev window can be designed with any specified relative sidelobe level while the main lobe width adjusted by choosing length appropriately
- Filter order is estimated using $N = \dfrac{2.056\alpha_s - 16.4}{2.285(\Delta\omega)}$

  where $\Delta\omega$ is the normalized transition bandwidth, e.g, for a lowpass filter

$$\Delta\omega = \omega_s - \omega_p$$

6. Adjustable Window Functions

- Gain response of a Dolph-Chebyshev window of length 51 and relative sidelobe level of 50 dB is shown below



6. Adjustable Window Functions

Properties of Dolph-Chebyshev window:

- All sidelobes are of equal height
- Stopband approximation error of filters designed have essentially equiripple behavior
- For a given window length, it has the smallest main lobe width compared to other windows resulting in filters with the smallest transition band

6. Adjustable Window Functions

- Kaiser Window -

$$w(n) = \frac{I_0\left\{\beta\sqrt{1-(n/M)^2}\right\}}{I_0(\beta)}, \quad -M \le n \le M$$

where β is an adjustable parameter and $I_0(u)$ is the modified zeroth-order Bessel function of the first kind:

$$I_0(u) = 1 + \sum_{r=1}^{\infty}\left[\frac{(u/2)^r}{r!}\right]^2$$

- Note $I_0(u)>0$ for u being real

6. Adjustable Window Functions

- In practice

$$I_0(u) = 1 + \sum_{r=1}^{20}\left[\frac{(u/2)^r}{r!}\right]^2$$

- β controls the minimum stopband attenuation of the windowed filter response
- β is estimated using

$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7), & \text{for } \alpha_s > 50 \\ 0.5824(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21), & \text{for } 21 \le \alpha_s \le 50 \\ 0, & \text{for } \alpha_s < 21 \end{cases}$$

6. Adjustable Window Functions

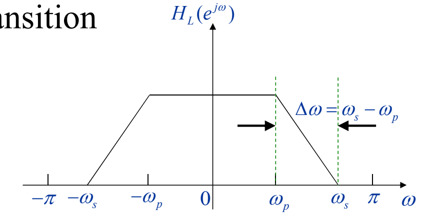
- Filter order is estimated using

$$N = \frac{\alpha_s - 8}{2.285(\Delta\omega)}$$

where $\Delta\omega$ is the normalized transition bandwidth

7. Impulse Responses of FIR Filters with a Smooth Transition

- First-order spline passband-to-stopband transition



$$\omega_c = (\omega_s + \omega_p)/2$$
$$\Delta\omega = \omega_s - \omega_p$$

$$h_{LP}(n) = \begin{cases} \omega_c/\pi, & n = 0 \\ \dfrac{2\sin(\Delta\omega n/2)}{\Delta\omega n} \cdot \dfrac{\sin(\omega_c n)}{\pi n}, & |n| > 0 \end{cases}$$

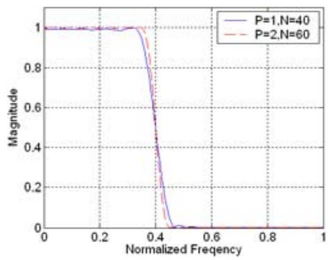
7. Impulse Responses of FIR Filters with a Smooth Transition

- *P*th-order spline passband-to-stopband transition

$$h_{LP}(n) = \begin{cases} \omega_c/\pi, & n = 0 \\ \left(\dfrac{\sin(\Delta\omega n/2P)}{\Delta\omega n/2P}\right)^P \cdot \dfrac{\sin(\omega_c n)}{\pi n}, & |n| > 0 \end{cases}$$

7. Impulse Responses of FIR Filters with a Smooth Transition

Example- $\omega_p = 0.35\pi$ $\omega_s = 0.45\pi$



8. Frequency Sampling Approach

- The specified frequency response $H_d(e^{j\omega})$ is first uniformly sampled at M equally spaced points $\omega_k = 2\pi k/M \quad 0 \le k \le M-1$, providing M frequency samples $H(k) = H_d(e^{j\omega})$
- These M frequency samples constitute an M-point DFT $H(k)$, whose M-point inverse-DFT thus yields the impulse response coefficients $h[n]$ of the FIR filter of length M.

8. Frequency Sampling Approach

- There is a basic assumption that the specified frequency response is uniquely characterized by the *M* frequency samples and, hence, can be fully recovered from these samples.

8. Frequency Sampling Approach

$$H_d(e^{j\omega}) \xrightarrow{\text{采样}} H_d(k) \xrightarrow{\text{IDFT}} h(n) \longrightarrow H(Z)$$

$$H_d(k) = H_d(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} = H_d(e^{j\frac{2\pi k}{N}})$$

$$h[n] = IDFT[H_d(k)] = \frac{1}{N}\sum_{k=0}^{N-1} H_d(k) e^{j\frac{2\pi}{N}kn}$$

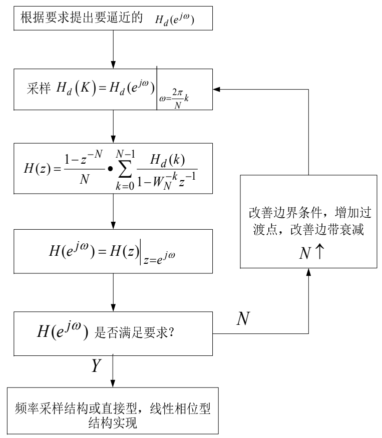
8. Frequency Sampling Approach

$$h[n] = \sum_{n=-\infty}^{\infty} h_d[n+rN] \cdot R_N[n] \qquad n = 0,\ 1, \ldots, N-1$$

$$H(k) = \sum_{n=0}^{N-1} h[n] \cdot z^{-n}$$

$$H(z) = \frac{1-z^{-N}}{N} \sum_{n=0}^{N-1} \frac{H(k)}{1-W_N^{-k} z^{-N}}$$

8. Frequency Sampling Approach



8. Frequency Sampling Approach

- Let $|H_d(e^{j\omega})|$ denote the desired magnitude response of a real linear-phase FIR filter of length N.
- For N odd (Type 1 FIR filter), the DFT samples $H(k)$ needed for a frequency sampling-based design are given by

$$H[k]=\begin{cases} \left|H_d\left(e^{2\pi k/N}\right)\right| e^{-j2\pi k(N-1/N)}, & k=0,1,\cdots,\dfrac{N-1}{2} \\ \left|H_d\left(e^{j2\pi k/N}\right)\right| e^{j2\pi(N-k)(N-1)/N}, & k=\dfrac{N+1}{2},\cdots,N-1 \end{cases}$$

8. Frequency Sampling Approach

$$\begin{cases} H_{N-k}=H_k \\ \theta_{N-k}=-\theta_k=\dfrac{N-1}{N}k\pi \end{cases}$$

$$H_{\frac{N}{2}}=0$$

8. Frequency Sampling Approach

Example

- Determine a linear-phase FIR lowpass filter of length 33 with a passband at $\omega_p=0.5\pi$.

  Realized it by Type 1 FIR filter, i.e $h[n]=h[N-n]$

8. Frequency Sampling Approach

Example

$$H_d(e^{j\omega})=H_d(\omega)\cdot e^{j\theta(\omega)}$$

$$\begin{cases} H_d(e^{j\omega})=\begin{cases} 1 & 0\le\omega\le\dfrac{\pi}{2},\dfrac{3\pi}{2}\le\omega\le 2\pi \\ 0 & \dfrac{\pi}{2}<\omega<\dfrac{3\pi}{2} \end{cases} \\ \theta(\omega)=-\omega\dfrac{N-1}{2} \end{cases}$$

$$\theta_k=-\frac{2\pi k}{N}\bullet\frac{N-1}{2}=-\frac{32}{33}k\pi \qquad k=0,1,\ldots,32$$

8. Frequency Sampling Approach

$$H_k = \begin{cases} 1 & 0 \le k \le 8, 25 \le k \le 32 \\ 0 & 9 \le k \le 24 \end{cases}$$

$$\theta_k = \begin{cases} -\frac{32}{33} k\pi & 0 \le k \le 8 \\ \frac{32}{33}(33-k)\pi & 17 \le k \le 32 \end{cases}$$

8. Frequency Sampling Approach

$$H_d(k) = e^{j\theta_k} = \begin{cases} e^{-j\frac{32}{33}k\pi} & 0 \le k \le 8 \\ e^{-j\frac{32}{33}(33-k)\pi} & 25 \le k \le 32 \\ 0 & 9 \le k \le 24 \end{cases}$$

$$H(z) = \frac{1 - z^{-33}}{33} \bullet \sum_{k=0}^{32} \frac{H_d(k)}{1 - W_{33}^{-k} \bullet z^{-1}}$$

8. Frequency Sampling Approach

- Improvement
  - $N \uparrow$
  - Increase the samples in transition band to improve the attenuation in stopband.
    - ✓ No samples in transition band, -20dB;
    - ✓ One samples, $H_1 = 0.5 \quad -30\,dB$
      $H_2 = 0.3904 \quad -40dB$