Chapter 9 pre

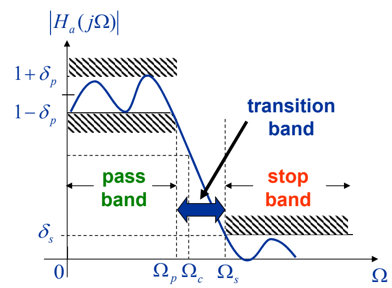
Design of Analog Filters

Design of Analog Filters

- Analog Lowpass Filter Specifications
- Butterworth Approximation
- Design of Other Types of Analog Filters

1. Analog Lowpass Filter Specifications

- Typical magnitude response of an analog lowpass filter may be given as indicated below



1. Analog Lowpass Filter Specifications

- In the **passband**, defined by $0 \le \Omega \le \Omega_p$, we require $1-\delta_p \le |H_a(j\Omega)| \le 1+\delta_p, \quad |\Omega| \le \Omega_p$ i.e., $|H_a(j\Omega)|$ approximates unity within an error of $\pm\delta_p$
- In the **stopband**, defined by $\Omega_s \le \Omega \le \infty$, we require $|H_a(j\Omega)| \le \delta_s, \quad \Omega_s \le \Omega \le \infty$ i.e., $|H_a(j\Omega)|$ approximates zero within an error of δ_s

1. Analog Lowpass Filter Specifications

Ω_p -- passband edge frequency

Ω_s -- stopband edge frequency

δ_p -- peak ripple value in the passband

δ_s -- peak ripple value in the stopband

Peak passband ripple

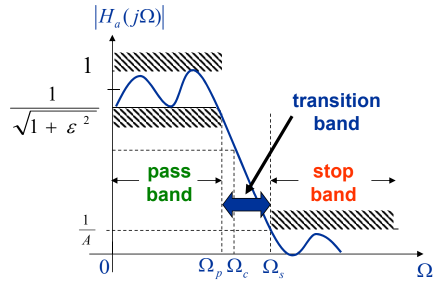
$$\alpha_p = -20\log_{10}(1-\delta_p)\ \text{dB}$$

Minimum stopband attenuation

$$\alpha_s = -20\log_{10}(\delta_s)\ \text{dB}$$

1. Analog Lowpass Filter Specifications

- Normalized Specifications



1. Analog Lowpass Filter Specifications

- Two additional parameters
  - Transition Ratio

$$k = \frac{\Omega_p}{\Omega_s} \qquad \text{Lowpass filter: } k<1 \quad \text{Highpass filter: } k>1$$

  - Discrimination parameter (Usually, $k_1 << 1$)

$$k_1 = \frac{\varepsilon}{\sqrt{A^2-1}}$$

2. Butterworth Approximation

- The magnitude-square response of an N-th order analog lowpass Butterworth Filter is given by

$$|H_a(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

- First $2N-1$ derivatives of $|H_a(j\Omega)|^2$ at $\Omega = 0$ are equal to zero
- The Butterworth lowpass filter thus is said to have a maximally-flat magnitude at $\Omega = 0$

2. Butterworth Approximation

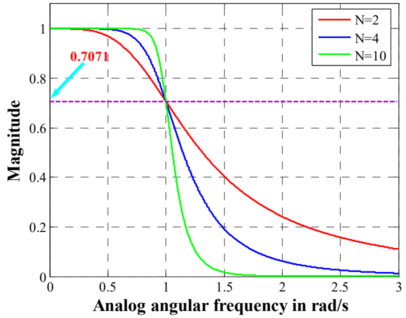
- Gain in dB is $G(\Omega) = 10\log_{10}\left|H_a(j\Omega)\right|^2$ dB
- As $G(0) = 10\log_{10}\left|H_0(j0)\right|^2 = 10\log_{10}(1) = 0$ dB

  $G(\Omega_c) = 10\log_{10}(0.5) = -3.0103 \approx -3$ dB

  Ω_c is called the 3-dB cutoff frequency

2. Butterworth Approximation

- Typical magnitude responses with $\Omega_c = 1$



The magnitude response is monotonically decreasing with Ω increasing

2. Butterworth Approximation

- For $\Omega >> \Omega_c$, the squared-magnitude function can be approximated by

$$|H_a(j\Omega)|^2 \approx \frac{1}{(\Omega/\Omega_c)^{2N}}$$

  - The gain $G(\Omega_2)$ in dB at $\Omega_2 = 2\Omega_1$ with $\Omega_1 >> \Omega_c$ is given by

$$G(\Omega_2) = -10\log_{10}\left(\frac{\Omega_2}{\Omega_c}\right)^{2N} = -10\log_{10}\left(\frac{2\Omega_1}{\Omega_c}\right)^{2N}$$
$$= G(\Omega_1) - 6N \quad dB$$

2. Butterworth Approximation

- **Two parameters** completely characterizing a Butterworth lowpass filter are Ω_c and N
- They are determined from
  - the specified bandedges Ω_p and Ω_s
  - minimum passband magnitude $\left|H_a(j\Omega_p)\right|$
  - maximum stopband magnitude $\left|H_a(j\Omega_s)\right|$
- Ω_c and N are thus determined from

$$\left|H_a(j\Omega_p)\right|^2 = \frac{1}{1+(\Omega_p/\Omega_c)^{2N}} \qquad \left|H_a(j\Omega_s)\right|^2 = \frac{1}{1+(\Omega_s/\Omega_c)^{2N}}$$

2. Butterworth Approximation

- According to the definition of the loss function

$$\alpha = 10\log_{10}\frac{1}{|H(j\Omega)|^2}$$

- We know that

$$|H_a(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

then $\boxed{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = 10^{\frac{\alpha_p}{10}}}$ $\boxed{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = 10^{\frac{\alpha_s}{10}}}$

2. Butterworth Approximation

- Suppose

$$\lambda = \frac{\Omega_s}{\Omega_p} \qquad \eta = \sqrt{\frac{10^{\frac{\alpha_p}{10}}-1}{10^{\frac{\alpha_s}{10}}-1}}$$

- Then

$$N = -\frac{\lg \eta}{\lg \lambda}$$

2. Butterworth Approximation

- With
  - minimum passband magnitude $1/\sqrt{1+\varepsilon^2}$
  - maximum stopband ripple $1/A$
- Ω_c and N are thus determined from

$$\boxed{|H_a(j\Omega_p)|^2 = \frac{1}{1+(\Omega_p/\Omega_c)^{2N}} = \frac{1}{1+\varepsilon^2}}$$

$$\boxed{|H_a(j\Omega_s)|^2 = \frac{1}{1+(\Omega_s/\Omega_c)^{2N}} = \frac{1}{A^2}}$$

2. Butterworth Approximation

- Solving the above

$$N = \frac{1}{2}\frac{\log_{10}[(A^2-1)/\varepsilon^2]}{\log_{10}(\Omega_s/\Omega_p)} = \frac{\lg(1/k_1)}{\lg(1/k)}$$

2. Butterworth Approximation

- Since order N must be an integer, value obtained is rounded up to the next highest integer
- This value of N is used next to determine Ω_c by satisfying either the stopband edge or the passband edge specification exactly
- If the stopband edge specification is satisfied, then the passband edge specification is exceeded providing a safety margin

2. Butterworth Approximation

- It is preferable to use the latter since this ensures the smallest ripple in the passband or, in other words, the smallest amplitude distortion to the signal being filtered in its band of interest.

2. Butterworth Approximation

- Since $H_a(s)H_a(-s)$ evaluated at $s = j\Omega$ is simply equal to $|H_a(j\Omega)|^2$, it follows that

$$H_a(s)H_a(-s) = \frac{1}{1+\left(-s^2/\Omega_c^2\right)^N}$$

- The poles of this expression occur on a circle of radius Ω_c at equally spaced points
- Because of the stability and causality, the transfer function itself will be specified by just the poles in the negative real half-plane of s

2. Butterworth Approximation

- The transfer function of an analog Butterworth lowpass filter has the form of

$$H_a(s) = \frac{C}{D_N(s)} = \frac{\Omega_c^N}{s^N + \sum_{l=0}^{N-1} d_l s^l} = \frac{\Omega_c^N}{\prod_{l=1}^{N}(s - s_l)}$$

Where

$$p_l = \Omega_c e^{j[\pi(N+2l-1)/2N]}, \quad l = 1, 2, \ldots, N$$

- Denominator $D_N(s)$ is known as the Butterworth polynomial of order N

2. Butterworth Approximation

Example:

- Determine the lowest order of a Butterworth lowpass filter with a 1 dB cutoff frequency at 1 kHz and a minimum attenuation of 40 dB at 5 kHz

$$10\log_{10}\left[\frac{1}{1+(2000\pi/\Omega_c)^{2N}}\right]=-1$$

$$10\log_{10}\left[\frac{1}{1+(10000\pi/\Omega_c)^{2N}}\right]=-40$$

$$N=\log_{10}\left\{\frac{10^{0.1}-1}{10^{4}-1}\right\}\bigg/\log_{10}\frac{1}{5}=3.2811$$

We choose $N = 4$

3. Design of Other Types of Analog Filters

- Design of the other three classes of analog filters, namely, the highpass, bandpass, and bandstop filters, can be carried out by simple **spectral transformations** of the frequency variables.

3. Design of Other Types of Analog Filters

Highpass Filter $H_D(s)$

- *Step*1 Develop of specifications of a prototype analog lowpass filter $H_{LP}(s)$ from specifications of desired analog filter $H_D(s)$ using a frequency (spectrum) transformation.
- *Step* 2 Design the prototype analog lowpass filter $H_{LP}(s)$
- *Step* 3 Determine the transfer function $H_D(s)$ of desired analog filter by applying the inverse frequency transformation to $H_{LP}(s)$

3. Design of Other Types of Analog Filters

- Let s denote the Laplace transform variable of prototype analog lowpass filter $H_{LP}(s)$ and $\hat{s}$ denote the Laplace transform variable of desired analog filter $H_D(\hat{s})$
- The mapping from s -domain to $\hat{s}$ -domain is given by the invertible transformation $s = F(\hat{s})$

Then

$$H_D(\hat{s}) = H_{LP}(s)\big|_{s=F(\hat{s})}$$

$$H_{LP}(s) = H_D(\hat{s})\big|_{\hat{s}=F^{-1}(s)}$$

3. Design of Other Types of Analog Filters

- Spectral Transformation (Lowpass to Highpass)

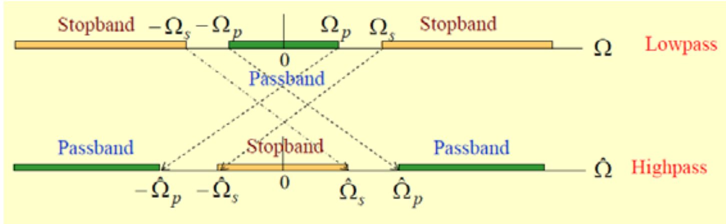
$$s = \frac{\Omega_p \hat{\Omega}_p}{\hat{s}}$$

where Ω_p is the passband edge frequency of $H_{LP}(s)$ and $\hat{\Omega}_p$ is the passband edge frequency of $H_{HP}(\hat{s})$

- On the imaginary axis the transformation is

$$\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$$

3. Design of Other Types of Analog Filters



- **Mapping** $-\Omega_p < \Omega < \Omega_p \Longleftrightarrow |\hat{\Omega}| \geq \hat{\Omega}_p$

$$|\Omega| \geq \Omega_s \Longleftrightarrow |\hat{\Omega}| \leq \hat{\Omega}_s$$

3. Design of Other Types of Analog Filters

Example:

- Design an analog Butterworth highpass filter with the specifications:

$\hat{F}_p = 4kHz$ $\quad \hat{F}_s = 1kHz$ $\quad \alpha_p = 0.1\,\text{dB}$ $\quad \alpha_s = 40\,\text{dB}$

  - Choose $\Omega_p = 1$
  - Then $\Omega_s = \frac{2\pi \hat{F}_p}{2\pi \hat{F}_s} = \frac{4000}{1000} = 4$
  - Analog lowpass filter specifications

$\Omega_p = 1$ $\quad \Omega_s = 4$ $\quad \alpha_p = 0.1\,\text{dB}$ $\quad \alpha_s = 40\,\text{dB}$

3. Design of Other Types of Analog Filters

Example:

- **Gain plots**