Ch7 LTI-Discrete-Time Systems in the Transform Domain

7.1 Let $G_L(z)$ and $G_H(z)$ represent ideal lowpass and highpass filters magnitude responses as sketched in Fig.1 a). Determine the transfer function $H_k(z) = Y_k(z)/X(z)$ of the discrete-time system of Fig.1 b), $k = 0, 1, 2, 3$, and sketch their magnitude response.

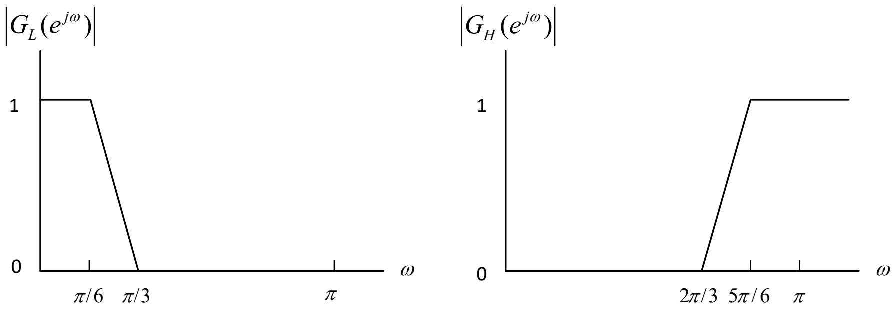

Fig. 1 a)

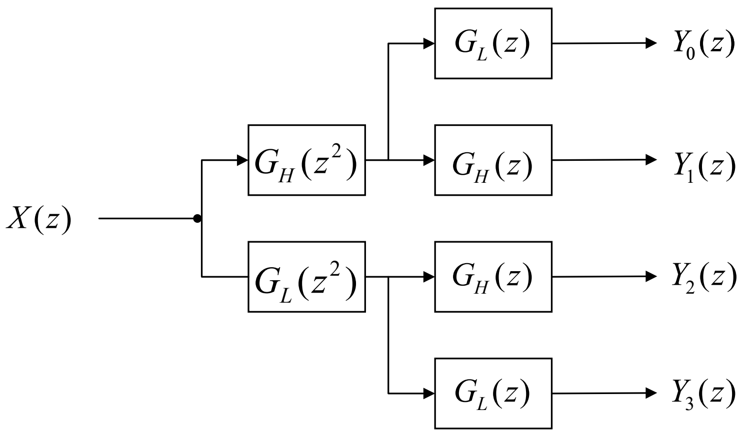

Fig. 1 b)

7.2 Let $H_{LP}(z)$ denote the transfer function of a real coefficient lowpass filter with a passband edge at ω_p, stopband edge at ω_s, passband ripple δ_p, and stopband ripple δ_s. Sketch the magnitude response of $G(z) = H_{LP}(-z)$, for $-\pi \le \omega \le \pi$. What type of filter is $G(z)$? Determine its frequency response $g[n]$ in terms of the impulse response $h_{\mathrm{LP}}[n]$ of $H_{LP}(z)$. Determine the band edges and ripples of $G(z)$ in terms of the band edges and ripples of $H_{LP}(z)$.

7.3 Show that the structure shown in Fig.2 implements the filter $G(z)$ of Problem 7.2.

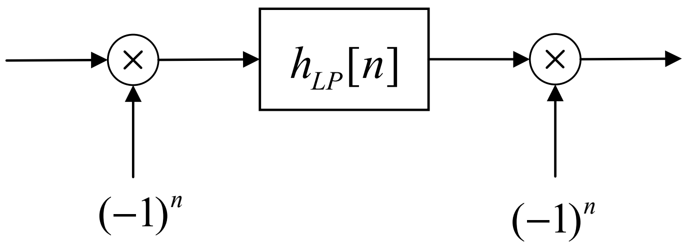

Fig. 2

- (Optional) Let $H_{LP}(z)$ denote the transfer function of an ideal real-coefficient lowpass filter having a cutoff frequency of ω_p, with $\omega_p < \pi/2$. Consider the complex coefficient transfer function $H_{LP}(e^{j\omega_0}z)$, where $\omega_p < \omega_0 < \pi - \omega_p$.
  - a) Sketch its magnitude response for $-\pi < \omega < \pi$. What type of filter does it represent?
  - b) Now consider the transfer function $G(z) = H_{LP}(e^{j\omega_0}z) + H_{LP}(e^{-j\omega_0}z)$. Sketch its magnitude response for $-\pi < \omega < \pi$.
  - c) Show that $G(z)$ is a real-coefficient bandpass filter with a passband centered at ω_0.
  - d) Determine the width of its passband in terms of ω_p and its impulse response $g[n]$ in terms of the impulse response $h_{LP}[n]$ of the parent lowpass filter.

- (Optional) Show that the structure shown in Fig.3 implements the bandpss filter of optional problem above.

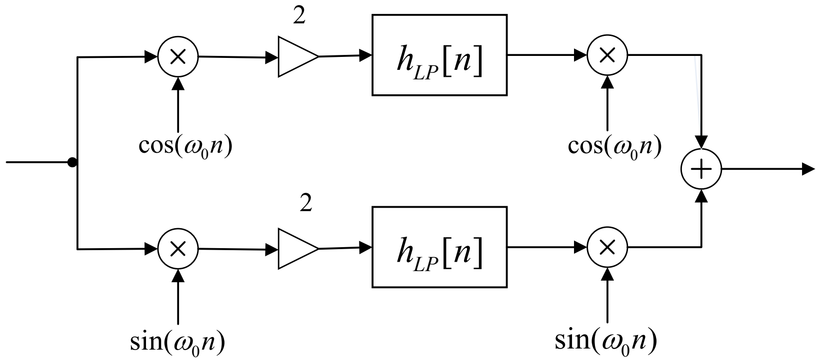

Fig. 3

7.4 Let $H(z)$ be a lowpass filter with unity passband magnitude, a passband edge at ω_p, and a stopband edge at ω_s, as shown in Fig. 4.

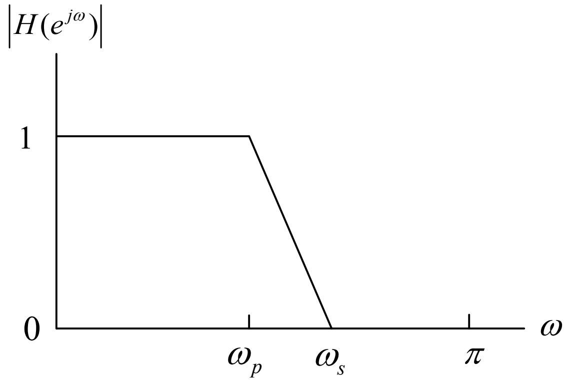

Fig. 4

a) Sketch the magnitude response of the digital $G_1(z) = H(z^M)F_1(z)$, where $F_1(z)$ is a lowpass filter with unity passband magnitude, a passband edge at ω_p / M, and a stopband edge at $(2\pi - \omega_s)/M$. What are the band edges of $G_1(z)$?

b) Sketch the magnitude response of the digital $G_2(z) = H(z^M)F_2(z)$, where $F_2(z)$ is a bandpass filter with unity passband magnitude, and with passband edges at $(2\pi - \omega_p)/M$ and $(2\pi + \omega_p)/M$, and stopband edges at $(2\pi - \omega_s)/M$ and $(2\pi + \omega_s)/M$, respectively. What are the band edges of $G_2(z)$?

7.5 Show analytically that the following causal FIR or IIR transfer functions are BR functions:

a) $H_1(z) = \dfrac{1}{5}(1 + 4z^{-1})$

b) $H_2(z) = \dfrac{(1+\alpha z^{-1})(1-\beta z^{-1})}{(1+\alpha)(1+\beta)}$, $\alpha > 0$, $\beta > 0$

c) $H_3(z) = \dfrac{3.9 + 3.9z^{-1}}{3.4 + z^{-1}}$

d) $H_4(z) = \dfrac{1.9 + 1.2z^{-1} + 1.9z^{-2}}{2 + 1.2z^{-1} + 1.8z^{-2}}$

7.6 Consider the first-order causal and stable allpass transfer function given by

$$A_1(z) = \frac{1 - d_1^* z}{z - d_1}$$

Determine the expression for $1 - \left|A_1(z)\right|^2 \begin{cases} < 0, & \text{for } |z|^2 < 1, \\ = 0, & \text{for } |z|^2 = 1, \\ > 0, & \text{for } |z|^2 > 1. \end{cases}$

Now, using the above approach, show the Property 2 given by:

$$\left|A_M(z)\right| \begin{cases} < 1, & \text{for } |z| > 1, \\ = 1, & \text{for } |z| = 1, \\ > 1, & \text{for } |z| < 1. \end{cases}$$

holds for any arbitrary causal stable allpass transfer function.

7.7 Consider the transfer function given by

$$H(z) = \frac{1}{M} \sum_{k=0}^{M-1} \mathcal{A}_k(z)$$

where $\{\mathcal{A}_k(z)\}$ are stable real-coefficient allpass function. Show that $H(z)$ is a BR function.

7.8 Consider the cascade of two causal LTI discrete-time systems $h_1[n] = \alpha\delta[n] + \beta\delta[n-1]$ and $h_2[n] = \gamma^n \mu[n]$, $\beta < 1$. Determine the frequency response $H(e^{j\omega})$ of the overall system. For what value of α, β and γ will $\left|H(e^{j\omega})\right| = K$, where K is a **real constant**?

7.9 A noncausal LTI FIR discrete-time system is characterized by an impulse response

$$h[n] = a_1\delta[n-2] - a_2\delta[n-1] - a_3\delta[n] + a_4\delta[n+1] - a_5\delta[n+2]$$

For what values of the impulse response samples will its frequency response $H(e^{j\omega})$ have a **zero phase.**

- (Optional) A causal FIR discrete-time system is characterized by the difference function

$$y[n] = a_1 x[n+k+1] + a_2 x[n+k] + a_2 x[n+k-1] + a_1 x[n+k-2]$$

where $y[n]$ and $x[n]$ denote, respectively, the output and the input sequences. Determine the expression for its frequency response $H(e^{j\omega})$. For what value of the constant k will the system have a frequency response $H(e^{j\omega})$ that is a **real function of ω**?

7.10 Let a causal LTI discrete-time system be characterized by a real impulse response $h[n]$ with a DTFT $H(e^{j\omega})$. Consider the system of Fig.5, where $x[n]$ is a finite-length sequence. Determine the frequency response of the overall system $G(e^{j\omega})$ in terms of $H(e^{j\omega})$ and show that it has a zero-phase response.

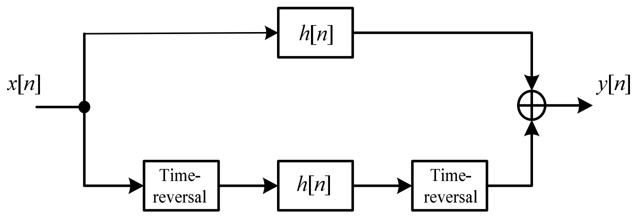

Fig. 5

7.11 Given causal IIR transfer functions:

$$H_1(z)=\frac{z^3+3z^2+2z+7}{(3z-4)(z^2-0.25)}$$

a) Check the stability of the transfer functions.
b) If it is not stable, find a stable transfer function with an identical magnitude function.
c) Are there any other transfer functions having the same magnitude response as those shown above?

7.12 Determine all possible causal stable transfer function $H(z)$ with a squared-magnitude function given by

$$\left|H(e^{j\omega})\right|^2=\frac{4(1.25+\cos\omega)(1.36-1.2\cos\omega)}{(1.36+1.2\cos\omega)(1.64+1.6\cos\omega)}$$

- (Optional) Consider the following FIR transfer functions:

a) $H_1(z)=1-0.52z^{-1}+0.92z^{-2}+0.87z^{-3}+0.92z^{-4}-0.52z^{-5}+z^{-6}$

b) $H_2(z)=0.6+0.652z^{-2}+0.6928z^{-3}+0.4032z^{-4}-0.2z^{-5}-0.12z^{-6}$

c) $H_3(z)=3.12-2.5z^{-1}+0.6z^{-2}+0.5z^{-3}+0.06z^{-4}+z^{-5}$

Using the M-file *zplane*, determine the zero locations of each, and then answer the following questions:

① Is there any FIR filters having a linear-phase response?

② Is there any FIR filters having a minimum-phase response?

③ Is there any FIR filters having a maximum-phase response?

7.13 The transfer function of a type 2 linear-phase FIR filter is given by

$$H(z) = 2.5 + 0.5z^{-1} + 0.35z^{-2} + 5.47z^{-3} + 5.47z^{-4} + 0.35z^{-5} + 0.5z^{-6} + 2.5z^{-7}$$

a) Determine the transfer function of $H_1(z)$ of a minimum-phase FIR filter having the same magnitude as that of $H(z)$.
b) Determine the transfer function of $H_2(z)$ of a maximum-phase FIR filter having the same magnitude as that of $H(z)$.
c) Determine the transfer function of $H_3(z)$ of a mixed-phase FIR filter having the same magnitude as that of $H(z)$.
d) How many other length-8 FIR filters exist that have the same magnitude responses as that of $H(z)$.

7.14 Consider the following IIR transfer function of a causal digital filter:

i. $$H_1(z) = \frac{6 - 5z^{-1} + z^{-2}}{1 + 0.1z^{-1} - 0.56z^{-2}}$$

ii. $$H_2(z) = \frac{3 - 7z^{-1} + 2z^{-2}}{1 + 0.1z^{-1} - 0.56z^{-2}}$$

iii. $$H_3(z) = \frac{2 - 7z^{-1} + 3z^{-2}}{1 + 0.1z^{-1} - 0.56z^{-2}}$$

iv. $$H_4(z) = \frac{1 - 5z^{-1} + 6z^{-2}}{1 + 0.1z^{-1} - 0.56z^{-2}}$$

a) Are the transfer function BIBO stable?
b) What is the relation between their magnitude functions?
c) What is the relation between their phase functions?

7.15 Is the transfer function

$$H(z) = \frac{(3z + 4)(z - 5)}{(z - 0.5)(z + 0.8)}$$

Minimum-phase? If it is not minimum-phase, the construct a minimum-phase transfer function $G(z)$ such that $|G(e^{j\omega})| = |H(e^{j\omega})|$. Determine their corresponding unit sample responses, $g[n]$ and $h[n]$, for $n = 0,\ 1,\ 2,\ 3,\ 4$. For what values of m is $\sum_{n=1}^{m} |g[n]|^2$ bigger than $\sum_{n=1}^{m} |h[n]|^2$.

- (Optional) Consider a causal minimum-phase transfer function $H_m(z)$ and a causal non-minimum-phase transfer function $H(z)$, with identical magnitude response function $\left|H(e^{j\omega})\right| = s\left|H_m(e^{j\omega})\right|$. Let $h_m[n]$ and $h[n]$ denote the impulse responses of $H_m(z)$ and $H(z)$. Prove:

a) $\left|h_m[0]\right| \geq s\left|h[0]\right|$

b) $\sum_{l=1}^{n}\left|h_m[l]\right|^2 \geq \sum_{l=1}^{n}\left|h[l]\right|^2$

(Property of causal minimum-phase transfer function)

7.16 A causal LTI FIR discrete-time system is characterized by an impulse response

$$h[n] = a_1\delta[n] + a_2\delta[n-1] + a_3x[n-2] + a_4x[n-3] + a_5x[n-4] + a_6x[n-5]$$

For what values of the impulse response samples will its frequency response $H(e^{j\omega})$ have a **constant group delay**?

7.17 Show that the amplitude response $\breve{H}(\omega)$ of Type 1 and Type 3 linear-phase FIR transfer functions is a periodic function of ω with a period 2π and the amplitude response $\breve{H}(\omega)$ of Type 2 and Type 4 linear-phase FIR transfer functions is a periodic function of ω with a period 4π.

7.18 A type 2 real-coefficient FIR filter with a transfer function $H(z)$ has the following zeros: $z_1=1$, $z_2=-1$, $z_3=0.5$, $z_4=0.8+j$.

a) Determine the locations of the remaining zeros of $H(z)$ having the lowest order.

b) Determine the transfer function $H(z)$ of the filter.

- (Optional) We have shown that a real-coefficient FIR transfer function $H(z)$ with a symmetric impulse response has linear-phase response. As a result, the all-pole IIR transfer function $G(z) = 1/H(z)$ will also have a linear-phase response. What are the practical difficulties in implementing $G(z)$? Justify your answer.

7.19 The first 5 samples of the impulse response of an FIR filter $H(z)$ are given by $h[0] = 2$, $h[1] = 1.5$, $h[2] = -3.2$, $h[3] = -5.2$, $h[4] = 6.4$.

a) Determine the remaining impulse response samples of $H(z)$ of lowest order for each type of linear-phase filter;

b) Using *zplane*, determine the zero locations for $H(z)$ for each type of linear-phase filter;

c) Does $H(z)$ have a zero at $z = 1$ and/or $z = -1$;

d) Do the zeros on the unit circle appear in complex conjugate pairs? Justify your answer.

e) Do the zeros *not* on the unit circle appear in mirror-image symmetry? Justify your answer.

7.20 The phase response $\theta(\omega)$ of a linear-phase FIR filter is given by

$$\theta(\omega) = \begin{cases} -2\omega - \pi, & -\pi < \omega \leq 0, \\ \pi - 2\omega, & 0 < \omega \leq \pi. \end{cases}$$

What is the length of the impulse response of the filter and is it symmetric or anti-symmetric? What are the values of the magnitude response of the filter at dc and at $\omega = \pi$? Justify your answer.

7.21 Design a first-order lowpass IIR filter for each of the following normalized 3-dB cutoff frequencies:

a) 0.42 rad/samples

b) 0.65π

7.22 Design a first-order highpass IIR filter for each of the following normalized 3-dB cutoff frequencies:

a) 0.5 rad/samples

b) 0.45π