Chapter 8

Digital Filter Structures

Digital Filter Structures

- Block Diagram Representation
- Equivalent Structures
- Basic FIR Digital Filter Structures
- Basic IIR Digital Filter Structures

1. Block Diagram Representation

- In the time domain, the input-output relations of an LTI digital filter is given by the convolution sum or, by the linear constant coefficient difference equation

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$y[n] = -\sum_{k=1}^{N} d_k y[n-k] + \sum_{k=0}^{M} p_k x[n-k]$$

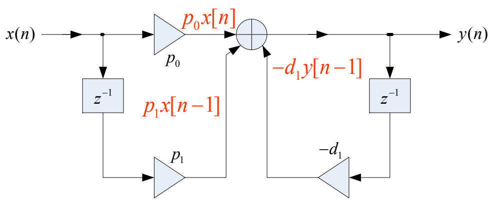
  - For the implementation of an LTI digital filter, the input-output relationship must be described by a valid computational algorithm.

1. Block Diagram Representation

- The convolution sum description of an LTI discrete-time system can, in principle, be used to implement the system.
- For an IIR finite-dimensional system, this approach is not practical as here the impulse response is of infinite length.
- However, a direct implementation of the IIR finite-dimensional system is practical

1. Block Diagram Representation

- To illustrate what we mean by a computational algorithm, consider the causal first-order LTI digital filter shown below



$$y[n] = -d_1 y[n-1] + p_0 x[n] + p_1 x[n-1]$$

1. Block Diagram Representation

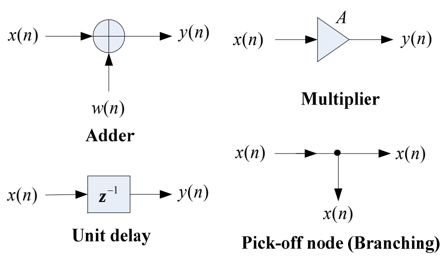
- Using the above equation we can compute $y[n]$ for $n\geq 0$ knowing the initial condition $y[-1]$ and the input $x[n]$ for $n\geq -1$

$$y[0] = -d_1 y[-1] + p_0 x[0] + p_1 x[-1]$$
$$y[1] = -d_1 y[0] + p_0 x[1] + p_1 x[0]$$
$$y[2] = -d_1 y[1] + p_0 x[2] + p_1 x[1]$$

- We can continue this calculation for any value of n we desire (by iterative computation)

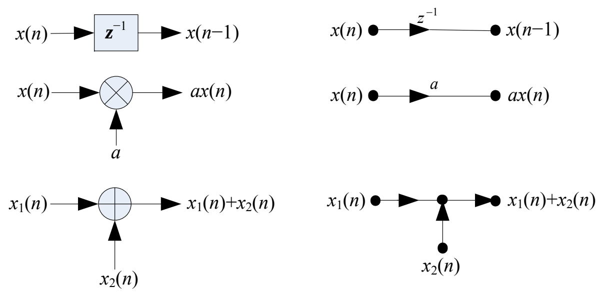
1.1 Basic Building Blocks

- The computational algorithm of an LTI digital filter can be conveniently represented in block diagram form using the basic building blocks shown below



1.1 Basic Building Blocks

- The corresponding signal flow charts are shown on the right-hand side



1.1 Basic Building Blocks

- Advantages of block diagram/signal flow chart representation
  - Easy to write down the computational algorithm by inspection.
  - Easy to analyze the block diagram to determine the explicit relation between the output and input.
  - Easy to manipulate a block diagram to derive other “equivalent” block diagrams yielding different computational algorithms.

1.1 Basic Building Blocks

- Advantages of block diagram/signal flow chart representation (const.)
  - Easy to determine the hardware requirements.
  - Easier to develop block diagram representations from the transfer function directly.

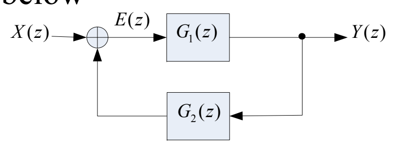
1.2 Analysis of Block Diagrams

- **Steps of Analyzing Block Diagrams**
  - Carried out by writing down the expressions for the output signals of each adder as a sum of its input signals, and developing a set of equations relating the filter input and output signals in terms of all internal signals
  - Eliminating the unwanted internal variables then results in the expression for the output signal as a function of the input signal and the filter parameters that are the multiplier coefficients

1.2 Analysis of Block Diagrams

Example:

- Consider the single-loop feedback structure shown below



The output $E(z)$ of the adder is

$$E(z) = X(z) + G_2(z)Y(z)$$

But from the figure $Y(z) = G_1(z)E(z)$

1.2 Analysis of Block Diagrams

- Eliminating $E(z)$ from the previous two equations we arrive at

$$\left[1 - G_1(z)G_2(z)\right]Y(z) = G_1(z)X(z)$$

which leads to

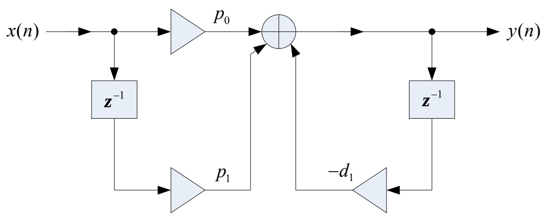
$$H(z) = \frac{Y(z)}{X(z)} = \frac{G_1(z)}{1 - G_1(z)G_2(z)}$$

1.3 Canonic and Noncanonic Structures

- A digital filter structure is said to be canonic if the number of delays in the block diagram representation is equal to the order of the transfer function
- Otherwise, it is a noncanonic structure
- The structure shown in the next slide is noncanonic as it employs two delays to realize a first-order difference equation

1.3 Canonic and Noncanonic Structures

$$y[n] = -d_1 y[n-1] + p_0 x[n] + p_1 x[n-1]$$



2. Equivalent Structures

- Two digital filter structures are defined to be equivalent if they have the same transfer function
- There are a number of methods for the generation of equivalent structures
- However, a fairly simple way to generate an equivalent structure from a given realization is via the transpose operation

2. Equivalent Structures

- **Transpose Operation**

(1) Reverse all paths

(2) Replace pick-off nodes by adders, and vice versa

(3) Interchange the input and output nodes

※ All other methods for developing equivalent structures are based on a specific algorithm for each structure

2. Equivalent Structures

- There are literally an infinite number of equivalent structures realizing the same transfer function
- It is thus impossible to develop all equivalent realizations
- In this course we restrict our attention to a discussion of some commonly used structures

2. Equivalent Structures

- Under infinite precision arithmetic any given realization of a digital filter behaves identically to any other equivalent structure
- However, in practice, due to the finite wordlength limitations, a specific realization behaves totally differently from its other equivalent realizations

2. Equivalent Structures

- Hence, it is important to choose a structure that has the least quantization effects when implemented using finite precision arithmetic
- One way to arrive at such a structure is to determine a large number of equivalent structures, analyze the finite wordlength effects in each case, and select the one showing the least effects

2. Equivalent Structures

- In certain cases, it is possible to develop a structure that by construction has the least quantization effects
- Here, we review some simple realizations that in many applications are quite adequate

3. FIR Digital Filter Structures

- **Direct Form**
- **Cascade Form**
- **Polyphase Realization**
- **Linear-phase Structure**
- **Tapped Delay Line**

3. FIR Digital Filter Structures

- A causal FIR filter of order N is characterized by a transfer function $H(z)$ given by

$$H(z) = \sum_{k=0}^{N} h[k] z^{-k}$$

which is a polynomial in z^{-1}

- In the time-domain the input-output relation of the above FIR filter is given by

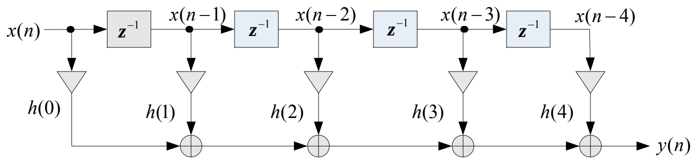
$$y[n] = \sum_{k=0}^{N} h[k] x[n-k]$$

3.1 Direct Form FIR Digital Filter Structures

- An FIR filter of order N is characterized by $N+1$ coefficients and, in general, require $N+1$ multipliers and N two-input adders
- Structures in which the multiplier coefficients are precisely the coefficients of the transfer function are called direct form structures

3.1 Direct Form FIR Digital Filter Structures

- A direct form realization of an FIR filter can be readily developed from the convolution sum description as indicated below for $N = 4$



3.1 Direct Form FIR Digital Filter Structures

- An analysis of this structure yields

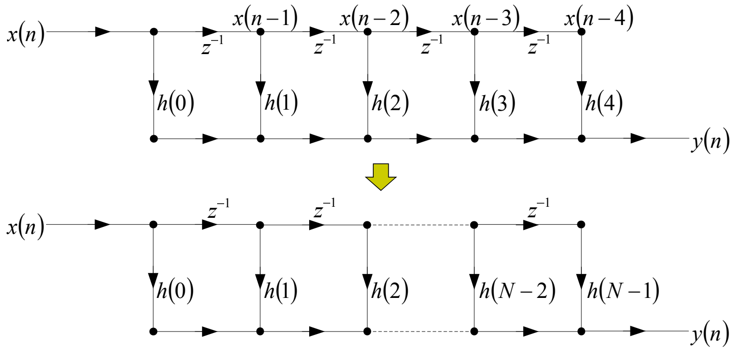
$$y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + h[3]x[n-3] + h[4]x[n-4]$$

which is precisely of the form of the convolution sum description

- The direct form structure shown on the previous slide is also known as a tapped delay line or a transversal (横截型) filter.

3.1 Direct Form FIR Digital Filter Structures

General Form



3.2 Cascade Form FIR Digital Filter Structures

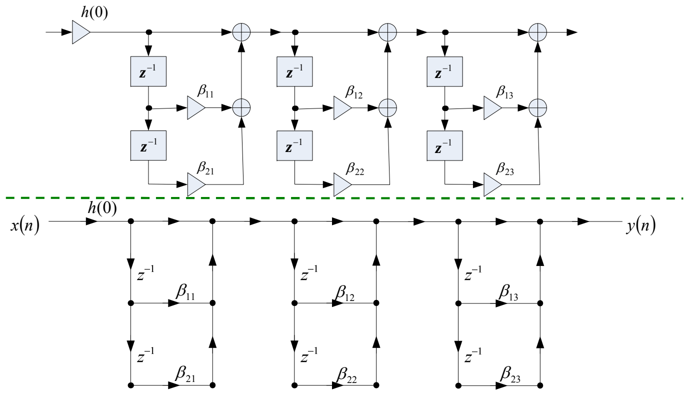
- A higher-order FIR transfer function can also be realized as a cascade of second order FIR sections and possibly a first-order section
- To this end we express $H(z)$ as

$$H(z) = h[0]\prod_{k=1}^{K}\left(1 + \beta_{1k} z^{-1} + \beta_{2k} z^{-2}\right)$$

where $K = N/2$ if N is even, and $K = (N+1)/2$ if N is odd, with $\beta_{2K} = 0$

3.2 Cascade Form FIR Digital Filter Structures

- A cascade realization for $N = 6$ is shown below



3.3 Polyphase Realization

- The polyphase decomposition of $H(z)$ leads to a parallel form structure
- To illustrate this approach, consider a causal FIR transfer function $H(z)$ with $N = 8$:

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6} + h[7]z^{-7} + h[8]z^{-8}$$

3.3 Polyphase Realization

- $H(z)$ can be expressed as a sum of two terms, with one term containing the even-indexed coefficients and the other containing the odd-indexed coefficients:

$$H(z) = \left(h[0] + h[2]z^{-2} + h[4]z^{-4} + h[6]z^{-6} + h[8]z^{-8}\right) + \left(h[1]z^{-1} + h[3]z^{-3} + h[5]z^{-5} + h[7]z^{-7}\right)$$

$$= \left(h[0] + h[2]z^{-2} + h[4]z^{-4} + h(6)z^{-6} + h[8]z^{-8}\right) + z^{-1}\left(h[1] + h[3]z^{-2} + h[5]z^{-4} + h[7]z^{-6}\right)$$

3.3 Polyphase Realization

- By using the notation

$$E_0(z) = h[0] + h[2]z^{-1} + h[4]z^{-2} + h[6]z^{-3} + h[8]z^{-4}$$
$$E_1(z) = h[1] + h[3]z^{-1} + h[5]z^{-2} + h[7]z^{-3}$$

we can express $H(z)$ as

$$H(z) = E_0(z^2) + z^{-1}E_1(z^2)$$

3.3 Polyphase Realization

- In a similar manner, by grouping the terms in the original expression for $H(z)$, we can re-express it in the form

$$H(z) = E_0(z^3) + z^{-1}E_1(z^3) + z^{-2}E_2(z^3)$$

where we have

$$E_0(z) = h[0] + h[3]z^{-1} + h[6]z^{-2}$$
$$E_1(z) = h[1] + h[4]z^{-1} + h[7]z^{-2}$$
$$E_2(z) = h[2] + h[5]z^{-1} + h[8]z^{-2}$$

3.3 Polyphase Realization

- The decomposition of $H(z)$ in the form

$$H(z) = E_0(z^2) + z^{-1}E_1(z^2)$$

or

$$H(z) = E_0(z^3) + z^{-1}E_1(z^3) + z^{-2}E_2(z^3)$$

is more commonly known as the polyphase decomposition

3.3 Polyphase Realization

- In the general case, an L-branch polyphase decomposition of an FIR transfer function of order N is of the form

$$H(z) = \sum_{m=0}^{L-1} z^{-m} E_m(z^L)$$

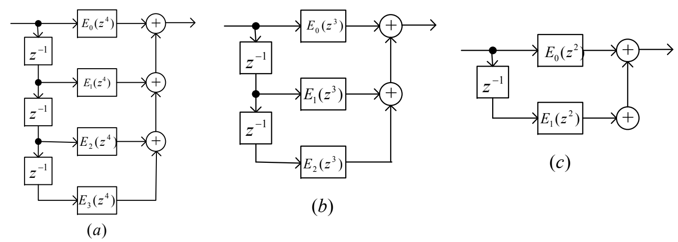
where

$$E_m(z) = \sum_{n=0}^{\lfloor (N+1)/L \rfloor} h[Ln+m]z^{-m}$$

with $h[n] = 0$ for $n > N$

3.3 Polyphase Realization

- Figures below show the 4-branch, 3-branch, and 2-branch polyphase realization of a transfer function $H(z)$

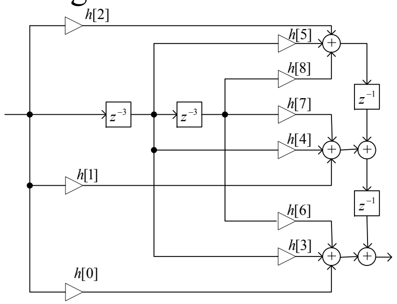


3.3 Polyphase Realization

- The subfilters $E_m(z^L)$ in the polyphase realization of an FIR transfer function are also FIR filters and can be realized using any methods described so far
- However, to obtain a canonic realization of the overall structure, the delays in all subfilters must be shared

3.3 Polyphase Realization

- Figure below shows a canonic realization of a length-9 FIR transfer function obtained using delay sharing



3.4 Linear-Phase FIR Digital Filter Structures

- Linear-phase FIR filter of length N is characterized by the symmetric impulse response

$$h[n] = h[N-n]$$

- An antisymmetric impulse response condition

$$h[n] = -h[N-n]$$

results in a constant group delay and "linear-phase" property

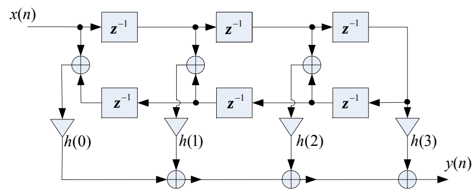
- Symmetry of the impulse response coefficients can be used to reduce the number of multiplications

3.4 Linear-Phase FIR Digital Filter Structures

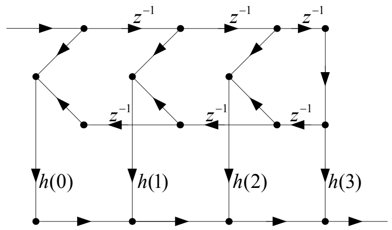
- Length $N+1$ is odd (N=6)

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[2]z^{-4} + h[1]z^{-5} + h[0]z^{-6}$$

$$= h[0](1+z^{-6}) + h[1](z^{-1}+z^{-5}) + h[2](z^{-2}+z^{-4}) + h[3]z^{-3}$$



3.4 Linear-Phase FIR Digital Filter Structures

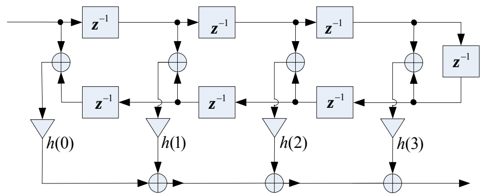


- The Type 1 linear-phase structure for a length-7 FIR filter requires 4 multipliers, whereas a direct form realization requires 6 multipliers

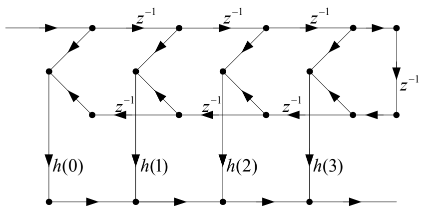
3.4 Linear-Phase FIR Digital Filter Structures

- Length $N+1$ is even ($N=7$)

$$H(z)=h[0]+h[1]z^{-1}+h[2]z^{-2}+h[3]z^{-3}$$
$$+h[3]z^{-4}+h[2]z^{-5}+h[1]z^{-7}+h[0]z^{-7}$$
$$=h[0](1+z^{-7})+h[1](z^{-1}+z^{-6})+h[2](z^{-2}+z^{-5})+h[3](z^{-3}+z^{-4})$$



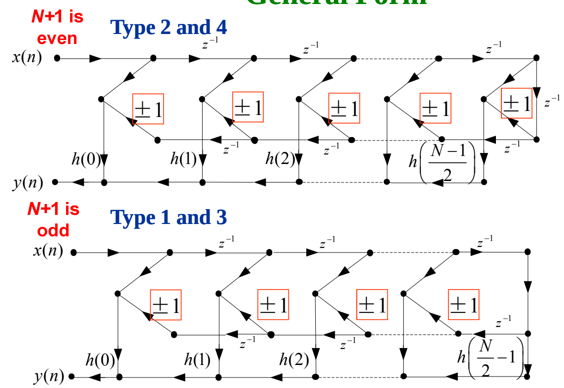
3.4 Linear-Phase FIR Digital Filter Structures



- The Type 2 linear-phase structure for a length-8 FIR filter requires 4 multipliers, whereas a direct form realization requires 7 multipliers

3.4 Linear-Phase FIR Digital Filter Structures

General Form



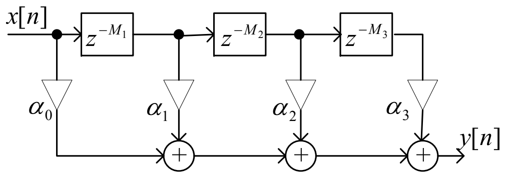
(N+1) /2 multipliers

Direct Form needs N multipliers

N/2 multipliers

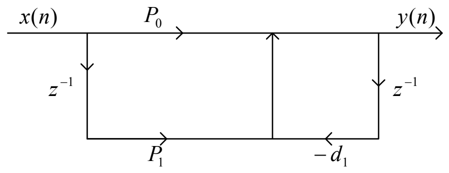
3.5 Tapped Delay Line

- The structure consists of a chain of $M_1 + M_2 + M_3$ unit delays with taps at the input, at the end of first M1 delays, at the end of next M2 delays, and at the output, respectively.



3.5 Tapped Delay Line

- The direct form FIR structure of the figure can be seen to be a special case of a tapped delay line, where there is a tap after each unit delay.



4. IIR Digital Filter Structures

- **Direct Form**
- **Cascade Form**
- **Parallel Form**

4.1 Direct Form IIR Digital Filter Structures

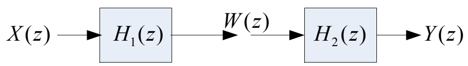
- The causal IIR digital filters we are concerned with in this course are characterized by a real rational transfer function of z^{-1} or, equivalently by a constant coefficient difference equation.
- From the difference equation representation, it can be seen that the realization of the causal IIR digital filters requires some form of feedback.

4.1 Direct Form IIR Digital Filter Structures

- Direct forms -- Coefficients are directly the transfer function coefficients
- Consider for simplicity a 3rd-order IIR filter with a transfer function (assuming $d_0 = 1$)

$$H(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}$$

- We can implement $H(z)$ as a cascade of two filter sections as shown below

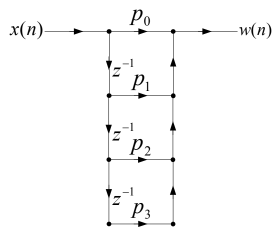


4.1 Direct Form IIR Digital Filter Structures

- where $H_1(z) = P(z) = p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}$

$$H_2(z) = 1 / D(z)$$

- The filter section $H_1(z)$ can be seen to be an FIR filter and can be realized as shown below

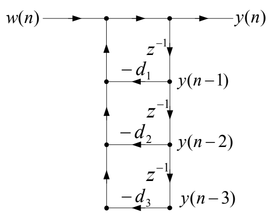


4.1 Direct Form IIR Digital Filter Structures

- The time-domain representation of $H_2(z)$ is given by

$$y[n] = w[n] - d_1 y[n-1] - d_2 y[n-2] - d_3 y[n-3]$$

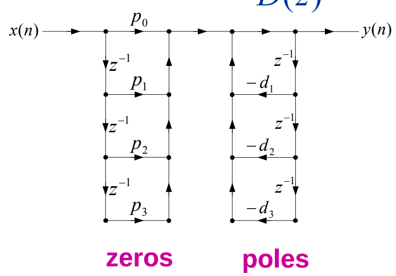
- Realization of $H_2(z)$ follows from the above equation and is shown below



4.1 Direct Form IIR Digital Filter Structures

- Considering the basic cascade realization results in *Direct form* I :

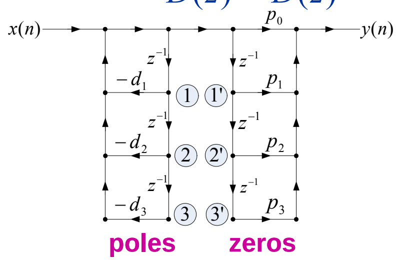
$$H(z) = P(z) \cdot \frac{1}{D(z)}$$



4.1 Direct Form IIR Digital Filter Structures

- Changing the order of blocks in cascade results in *Direct form* II :

$$H(z) = P(z) \cdot \frac{1}{D(z)} = \frac{1}{D(z)} \cdot P(z)$$

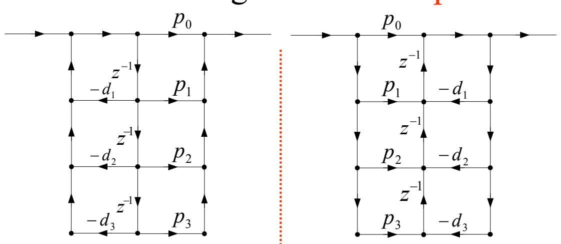


4.1 Direct Form IIR Digital Filter Structures

- Observe in the direct form structure shown below, the signal variable at nodes 1 and 1' are the same, and hence the two top delays can be shared
- Following the same argument, the bottom two delays can be shared
- Sharing of all delays reduces the total number of delays to 3 resulting in a canonic realization along with its transpose structure.

4.1 Direct Form IIR Digital Filter Structures

- Sharing of all delays reduces the total number of delays to 3 resulting in a canonic realization shown below along with its transpose structure.



  - Direct form realizations of an N-th order IIR transfer function should be evident.

4.2 Cascade Realizations

- By expressing the numerator and the denominator polynomials of the transfer function as a product of polynomials of lower degree, a digital filter can be realized as a cascade of low-order filter sections
- Consider, for example, $H(z)=P(z)/D(z)$ expressed as

$$H(z) = H_1(z)H_2(z)\cdots H_k(z) = \frac{P_1(z)P_2(z)\cdots P_k(z)}{D_1(z)D_2(z)\cdots D_k(z)}$$

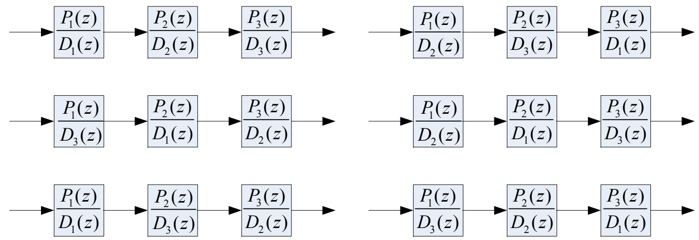
4.2 Cascade Realizations

- Consider, for example, $H(z)=P(z)/D(z)$ expressed as

$$H(z)=H_1(z)H_2(z)\cdots H_k(z)$$
$$=\frac{P_1(z)P_2(z)\cdots P_k(z)}{D_1(z)D_2(z)\cdots D_k(z)}$$

4.2 Cascade Realizations

- Examples of cascade realizations obtained by different pole-zero pairings are shown below



4.2 Cascade Realizations

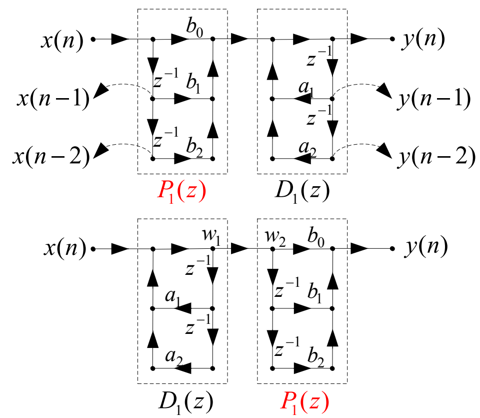
- There are altogether a total of 36 $(P_3^2 \cdot P_3^2)$ different cascade realizations of

$$H(z)=\frac{P(z)}{D(z)}=\frac{P_1(z)P_2(z)P_3(z)}{D_1(z)D_2(z)D_3(z)}$$

  based on pole-zero-pairings and ordering

- Due to finite wordlength effects, each such cascade realization behaves differently from Others

4.2 Cascade Realizations



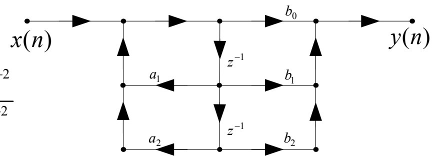
4.2 Cascade Realizations

- Usually, the polynomials are factored into a product of 1st-order and 2nd-order (sos) polynomials:

$$H(z) = p_0 \prod_k \left(\frac{1+\beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1+\alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \right)$$

for a first-order factor $\alpha_{2k} = \beta_{2k} = 0$

$$H_k(z) = \frac{1+\beta_{1j} z^{-1} + \beta_{2j} z^{-2}}{1-\alpha_{1j} z^{-1} + \alpha_{2j} z^{-2}}$$



4.2 Cascade Realizations

- Realizing complex conjugate poles and zeros with second order blocks results in real coefficients

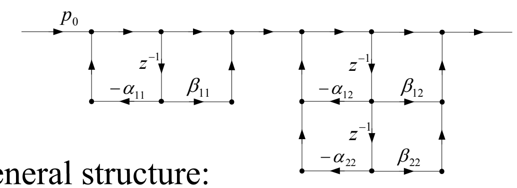
Example

- Third order transfer function

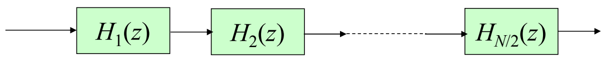
$$H(z) = \frac{P(z)}{D(z)} = p_0 \left(\frac{1+\beta_{11} z^{-1}}{1+\alpha_{11} z^{-1}} \right) \left(\frac{1+\beta_{12} z^{-1} + \beta_{22} z^{-2}}{1+\alpha_{12} z^{-1} + \alpha_{22} z^{-2}} \right)$$

4.2 Cascade Realizations

- One possible realization is shown below



- General structure:



4.2 Cascade Realizations

Example

- Direct form II and cascade form realizations of

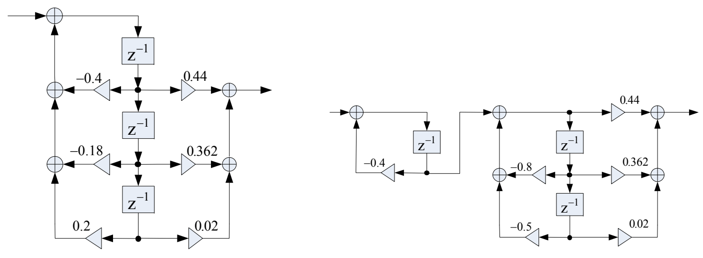
$$H(z) = \frac{0.44z^2 + 0.362z + 0.02}{(z^2 + 0.8z + 0.5)(z - 0.4)}$$

$$= \left(\frac{0.44 + 0.362z^{-1} + 0.02z^{-2}}{1 + 0.8z^{-1} + 0.5z^{-2}} \right) \left(\frac{z^{-1}}{1 - 0.4z^{-1}} \right)$$

4.2 Cascade Realizations

Example

- Direct form II and cascade form realizations



Direct Form II   Cascade Form

4.3 Parallel Realizations

- Parallel realizations are obtained by making use of the partial fraction expansion of the transfer function

Parallel form I:

$$H(z) = \gamma_0 + \sum_k \left(\frac{\gamma_{0k} + \gamma_{1k} z^{-1}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \right)$$

for a real pole

$$\alpha_{2k} = \gamma_{1k} = 0$$

4.3 Parallel Realizations

- Parallel realizations are obtained by making use of the partial fraction expansion of the transfer function

Parallel form II:

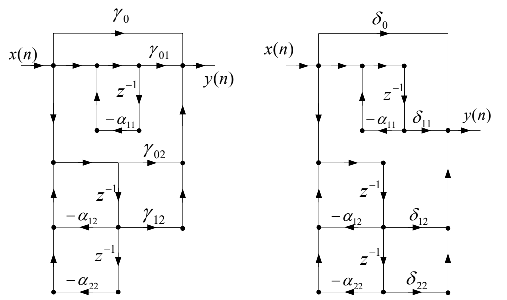
$$H(z) = \delta_0 + \sum_k \left(\frac{\delta_{1k} z^{-1} + \delta_{2k} z^{-2}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \right)$$

for a real pole

$$\alpha_{2k} = \delta_{2k} = 0$$

4.3 Parallel Realizations

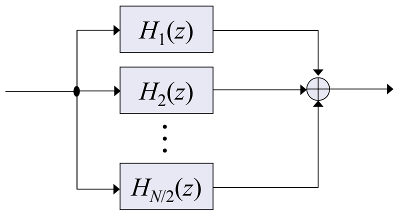
- The two basic parallel realizations of a 3rd order IIR transfer function are shown below



Parallel Form I   Parallel Form II

4.3 Parallel Realizations

- General structure:



- Easy to realize:
  No choices in section ordering and
  No choices in pole and zero pairing

4.3 Parallel Realizations

Example

- A partial-fraction expansion of

$$H(z)=\frac{0.44+0.362z^{-2}+0.002z^{-3}}{1+0.4z^{-1}+0.18z^{-2}-0.2z^{-3}}$$

in z^{-1} yields

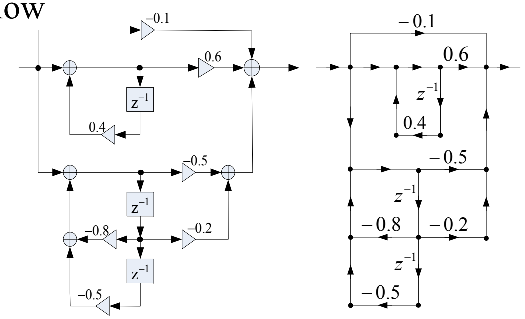
$$H(z)=-0.1+\frac{0.6}{1-0.4z^{-1}}+\frac{-0.5-0.2z^{-1}}{1+0.8z^{-1}+0.5z^{-2}}$$

- Likewise, a partial-fraction expansion of $H(z)$ in z yields

$$H(z)=\frac{0.24z^{-1}}{1-0.4z^{-1}}+\frac{0.2z^{-1}+0.25z^{-2}}{1+0.8z^{-1}+0.5z^{-2}}$$

4.3 Parallel Realizations

- Their realizations are *parallel form I* shown below

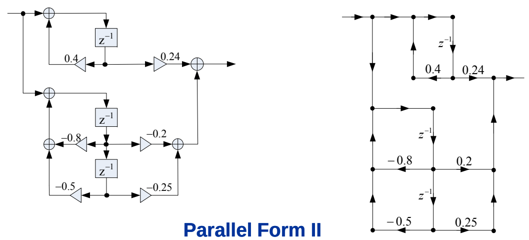


Parallel Form I

4.3 Parallel Realizations

- Likewise, a partial-fraction expansion of $H(z)$ in z yields *parallel form II*

$$H(z)=\frac{0.24z^{-1}}{1-0.4z^{-1}}+\frac{0.2z^{-1}+0.25z^{-2}}{1+0.8z^{-1}+0.5z^{-2}}$$



Parallel Form II

4.3 Parallel Realizations

- Consider

$$H(z)=\frac{1-z^{-N}}{N}\cdot\sum_{k=0}^{N-1}\frac{H[k]}{1-W_N^{-k}z^{-1}}$$

$$H(z)=\frac{1}{N}H_c(z)\cdot\sum_{k=0}^{N-1}H_k(z)$$

$$\begin{cases} H_c(z)=1-z^{-N} \\ H_k(z)=\dfrac{H[k]}{1-W_N^{-k}z^{-1}} \end{cases}$$

4.3 Parallel Realizations

- Consider $H(z)=\frac{1-z^{-N}}{N}\cdot\sum_{k=0}^{N-1}\frac{H[k]}{1-W_N^{-k}z^{-1}}$