Chapter 6

z-Transform

Part C

The Transfer Function

Part C: The Transfer Function

- Definition
- Transfer Function Expression
- Frequency Response from Transfer Function
- Geometric Interpretation of Frequency Response Computation
- Stability Condition in Terms of Poles Locations

2. 1 Definition

- Z-transform $H(z)$ of the impulse response $h[n]$ of the filter is called the *transfer function* or the *system function*
- Generalization of the frequency response $H(e^{j\omega})$
- Derived through the method similar to that of the frequency response

2.1 Definition

- LTI discrete-time system

$$y[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]$$

- Taking the DTFT

$$Y(e^{j\omega})=\sum_{n=-\infty}^{\infty}y[n]e^{-j\omega n}$$
$$=\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty}h[k]x[n-k]\right)e^{-j\omega n}$$

1. Definition

- Taking the z-Transform

$$Y(z)=\sum_{n=-\infty}^{\infty}y[n]z^{-n}$$
$$=\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty}h[k]x[n-k]\right)z^{-n}$$
$$=\sum_{k=-\infty}^{\infty}h[k]\left(\sum_{n=-\infty}^{\infty}x[n-k]z^{-n}\right)$$
$$=\sum_{k=-\infty}^{\infty}h[k]\left(\sum_{l=-\infty}^{\infty}x[l]z^{-(l+k)}\right)$$

2.1 Definition

- Taking the z-Transform

X(z)

$$Y(z)=\sum_{k=-\infty}^{\infty}h[k]\left(\sum_{l=-\infty}^{\infty}x[l]z^{-l}\right)z^{-k}$$
$$=\sum_{k=-\infty}^{\infty}h[k]z^{-k}X(z)$$

H(z)

- Here $Y(z)=H(z)X(z)$

- Thus $\boxed{H(z)=Y(z)/X(z)}\xrightarrow{\text{IZT}}h[n]$

2.2 Transfer function Expression

- **LTI FIR Digital Filter**
  - **Impulse response**

$$h[n],\quad N_1\le n\le N_2$$

  - **Transfer function**

$$H(z)=\sum_{n=N_1}^{N_2}h[n]z^{-n}$$

※ **All the poles of causal FIR digital filter are at $z=0$.**

2.1 Definition

Example :

- Consider the M-point **moving average** FIR filter with an impulse response

$$h(n)=\begin{cases}1/M, & 0\le n\le M-1\\ 0, & \text{otherwise}\end{cases}$$

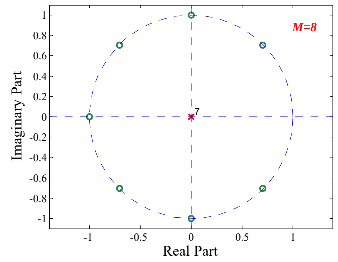
- Its transfer function is then given by

$$H(z)=\frac{1}{M}\sum_{n=0}^{M-1}z^{-n}=\frac{1-z^{-M}}{M(1-z^{-1})}=\frac{z^{M}-1}{M\cdot z^{M-1}(z-1)}$$

2.1 Definition

- The transfer function has M zeros on the unit circle at $z=e^{j2\pi k/M},\ k=0,\ldots,M-1$
- There are poles at $z=0$ and a single pole at $z=1$

The pole at $z = 1$ exactly cancels the zero at $z = 1$. The ROC is the entire z-plane except $z = 0$



2.2 Transfer function Expression

- **LTI IIR Digital Filter**
  - **LTI discrete-time system characterized by a difference equation**

$$\sum_{k=0}^{N}d_k y[n-k]=\sum_{k=0}^{M}p_k x[n-k]$$

  - **Transfer function : taking the z-transform and division**

$$H(z)=\frac{\sum_{k=0}^{M}p_k z^{-k}}{\sum_{k=0}^{N}d_k z^{-k}}$$

2.2 Transfer function Expression

- **LTI IIR Digital Filter**
  - **Or, equivalently as**

$$H(z)=z^{(N-M)}\frac{\sum_{k=0}^{M}p_k z^{M-k}}{\sum_{k=0}^{N}d_k z^{M-k}}$$

  - **Alternate form**

$$H(z)=\frac{p_0}{d_0}\frac{\prod_{k=0}^{M}(1-\xi_k z^{-1})}{\prod_{k=0}^{N}(1-\lambda_k z^{-1})}=z^{N-M}\frac{\prod_{k=0}^{M}(z-\xi_k)}{\prod_{k=0}^{N}(z-\lambda_k)}$$

2.2 Transfer function Expression

Example :

- A **causal** LTI IIR digital filter is described by a constant coefficient difference equation, given by

$$y[n]=x[n-1]-1.2x[n-2]+x[n-3] \\ \qquad +1.3y[n-1]-1.04y[n-2]+0.222y[n-3]$$

- Its transfer function is therefore given by

$$H(z)=\frac{Y(z)}{X(z)}=\frac{z^{-1}-1.2z^{-2}+z^{-3}}{1-1.3z^{-1}+1.04z^{-2}-0.222z^{-3}}$$

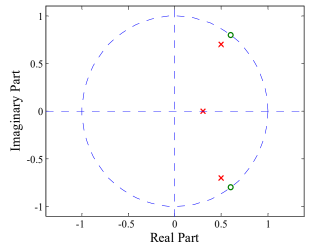
2.2 Transfer function Expression

Example :

- Alternative form $H(z)=\dfrac{z^2-1.2z+1}{z^3-1.3z^2+1.04z^2-0.222}$

$$H(z)=\frac{(z-0.6+j0.8)(z-0.6-j0.8)}{(z-0.3)(z-0.5+j0.7)(z-0.5+j0.7)}$$

- Zero-pole plot
  - ROC: $|z|>\sqrt{0.74}$



2.3 Frequency Response from Transfer Function

- If ROC of the transfer function $H(z)$ includes the unit circle, the frequency response $H(e^{j\omega})$ of the LTI digital filter can be obtained simply as follows:

$$H(e^{j\omega})=H(z)\big|_{z=e^{j\omega}}$$

- Transfer function $H(z)$ can be determined from its Fourier transform $H(e^{j\omega})$ by analytic continuation

$$H(z)=H(e^{j\omega})\big|_{\omega=\frac{1}{j}\ln z}$$

2.3 Frequency Response from Transfer Function

- For a stable rational transfer function in the form

$$H(z)=\frac{p_0}{d_0}\cdot z^{(N-M)}\cdot\frac{\prod_{k=1}^{M}(z-\xi_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

- The factored form of the frequency response is given by

$$H(e^{j\omega})=\frac{p_0}{d_0}\cdot e^{j\omega(N-M)}\cdot\frac{\prod_{k=1}^{M}\left(e^{j\omega}-\xi_k\right)}{\prod_{k=1}^{N}\left(e^{j\omega}-\lambda_k\right)}$$

2.3 Frequency Response from Transfer Function

- zero factor $\left(e^{j\omega}-\xi_k\right)$
- pole factor $\left(e^{j\omega}-\lambda_k\right)$
- The magnitude function is given by

$$\left|H(e^{j\omega})\right| = \left|\frac{p_0}{d_0}\right|\left|e^{j\omega(N-M)}\right|\frac{\prod_{k=1}^{M}\left|e^{j\omega}-\xi_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega}-\lambda_k\right|}$$

$$= \left|\frac{p_0}{d_0}\right|\cdot\frac{\prod_{k=1}^{M}\left|e^{j\omega}-\xi_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega}-\lambda_k\right|}$$

2.3 Frequency Response from Transfer Function

- The phase response for a rational transfer function is of the form

$$\arg H(e^{j\omega}) = \arg(p_0/d_0)+\omega(N-M)$$

$$+\sum_{k=1}^{M}\arg(e^{j\omega}-\xi_k)-\sum_{k=1}^{N}\arg(e^{j\omega}-\lambda_k)$$

2.3 Frequency Response from Transfer Function

- For a real coefficient transfer function, the magnitude-squared function

$$\left|H(e^{j\omega})\right|^2 = \left|\frac{p_0}{d_0}\right|^2\frac{\prod_{k=1}^{M}(e^{j\omega}-\xi_k)(e^{-j\omega}-\xi_k^{\,*})}{\prod_{k=1}^{N}(e^{j\omega}-\lambda_k)(e^{-j\omega}-\lambda_k^{\,*})}$$

$$= H(e^{j\omega})H^*(e^{j\omega})$$

$$= H(e^{j\omega})H(e^{-j\omega}) = H(z)H(z^{-1})\Big|_{z=e^{j\omega}}$$

2.3 Frequency Response from Transfer Function

- Phase response of an LTI discrete-time system with a real-coefficient transfer function

$$\theta(\omega) = \tan^{-1}\left[\frac{H_{\mathrm{im}}(z)}{H_{\mathrm{re}}(z)}\right]_{z=e^{j\omega}} = \frac{1}{2j}\ln\left[\frac{H(z)}{H(z^{-1})}\right]_{z=e^{j\omega}}$$

2.3 Frequency Response from Transfer Function

- Group delay of an LTI discrete-time system with a real-coefficient transfer function

$$\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega} = -\mathrm{Re}\left(z \frac{d\left[\ln H(z)\right]}{dz} \right)\Bigg|_{z=e^{j\omega}}$$

2.4 Geometric Interpretation of Frequency Response Computation

- For a stable rational transfer function in the form

$$H(e^{j\omega}) = \frac{p_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}\left(e^{j\omega} - \xi_k\right)}{\prod_{k=1}^{N}\left(e^{j\omega} - \lambda_k\right)}$$

  - Typical factor $e^{j\omega} - \rho e^{j\phi}$
    - Pole factor: $e^{j\omega} - \xi_k$
    - Zero factor: $e^{j\omega} - \lambda_k$

2.4 Geometric Interpretation of Frequency Response Computation

$$\left|H(e^{j\omega})\right| = \left|\frac{p_0}{d_0}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega} - \xi_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega} - \lambda_k\right|}$$

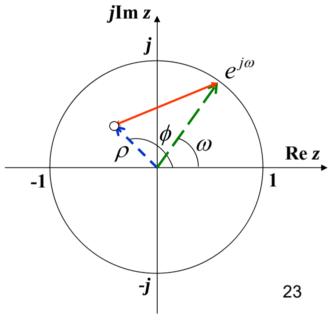
The product of magnitudes of *M* zero vectors

The product of magnitudes of *N* pole vectors

constant term

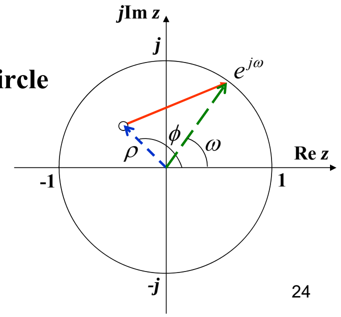
zero or pole vector $(e^{j\omega} - \rho e^{j\phi})$

represents a vector starting at the point $z = \rho e^{j\phi}$ and ending on the unit circle at $z = e^{j\omega}$



2.4 Geometric Interpretation of Frequency Response Computation

- **As ω varies from 0 to 2 π**
  - **Tip of the vector moves counterclockwise from the point z = 1 tracing the unit circle and back to the point z = 1.**



- Likewise, from

$$\arg H(e^{j\omega}) = \arg(p_0/d_0) + \omega(N-M) + \sum_{k=1}^{M} \arg(e^{j\omega} - \xi_k) - \sum_{k=1}^{N} \arg(e^{j\omega} - \lambda_k)$$

  - Phase response at a specific value of ω is obtained by
    - adding the phase of the term p_0/d_0 and the linear-phase term $\omega(N-M)$ to the sum of the angles of the zero vectors minus the angles of the pole vectors.

- Thus, an *approximate plot* of the magnitude and phase responses of the transfer function of an LTI digital filter can be developed by *examining the pole and zero locations*
- Now, a zero (pole) vector has the smallest (largest) magnitude when $\omega = \varphi$

- To highly *attenuate* signal components in a specified frequency range, we need to place zeros very close to or on the unit circle in this range.
- Likewise, to highly *emphasize* signal components in a specified frequency range, we need to place poles very close to the unit circle in this range.

2.5 Stability Condition in Terms of Poles Locations

- A causal LTI digital filter is BIBO stable if and only if its impulse response $h[n]$ is absolutely summable, i.e.,

$$\boldsymbol{S} = \sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

- We now develop a stability condition in terms of the pole locations of the transfer function $H(z)$

2.5 Stability Condition in Terms of Poles Locations

- ROC of the z-transform $H(z)$
  - $|z|=r$ for which $h[n]r^{-n}$ is absolutely summable
- Thus, if the ROC includes the unit circle $|z|=1$, then the digital filter is stable, and vice versa
- Meanwhile, for LTI system causality we require that $h[n]=0$, for $n<0$. This implies that the ROC of $H(z)$ must be outside some circle of radius R_{x-}

2.5 Stability Condition in Terms of Poles Locations

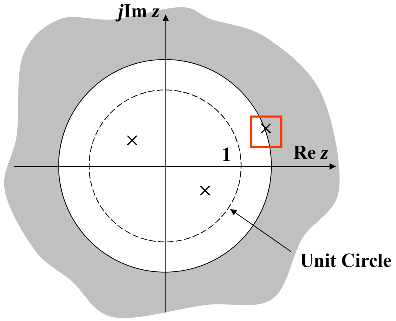
Theorem

A causality LTI system is stable **if and only if** the system function $H(z)$ has **all its poles inside the unit circle**. So, the ROC will include the unit circle and entire z-plane including the point $z=\infty$

- *An causal FIR digital filter with bounded impulse response is always stable (?)*

2.5 Stability Condition in Terms of Poles Locations

Proof: (Hint : reduction to absurdity)



2.5 Stability Condition in Terms of Poles Locations

- On the other hand, an causal IIR filter may be unstable if not designed properly.
- Furthermore, an originally stable IIR filter characterized by infinite precision coefficients may become unstable when coefficients get quantized due to implementation

Example 1:

Under what conditions, the following system is stable or causal ?

$$H(z)=\frac{-3z^{-1}}{2-5z^{-1}+2z^{-2}}$$

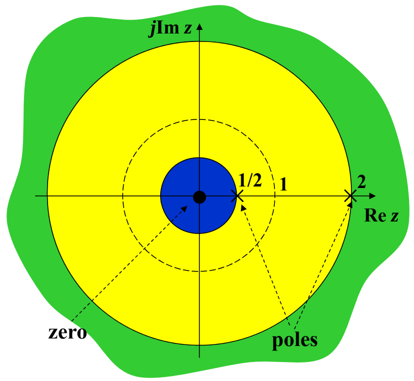
Solution:

Step 1----Determine the zeros and poles of *H(z)*

$$H(z)=\frac{-3z^{-1}}{2-5z^{-1}+2z^{-2}}=\frac{-3z}{(2z-1)(z-2)}$$

$z_{zero}=0$, $z_{pole}=0.5, 2$

Step 2----plot the zeros and poles on the z-plane



Step 3----Determine all possible ROCs according to the distribution of the zeros and poles

ROC1={|z|<0.5}
—Blue Area

ROC2={0.5<|z|<2}
—yellow Area

ROC3={|z|>2}
—Green Area

Step4----Discuss the system's stability and causality

Case 1: ROC1={|z|<0.5} Because the unit circle does not lie in this area and the ROC is inside of the circle with radius 0.5, the system is anti-causal and unstable.

Case 2: ROC2={0.5<|z|<2} Because the unit circle lies in this area and the ROC is an annulus bounded by 0.5 and 2 , the system is anti-causal and stable.

Case 3: ROC3={|z|>2} Because the unit does not lie in this area and the ROC is outside of the circle with radius 2, the system is causal and unstable.

Example 2:

Consider a causal LTI discrete-time system with an impulse response

$$h[n]=\alpha^n\mu[n]$$

- **Solution: For this system**

$$S=\sum_{n=-\infty}^{\infty}\left|\alpha^n\mu[n]\right|=\sum_{n=-\infty}^{\infty}\left|\alpha^n\right|=\frac{1}{1-|\alpha|},\quad \text{if } |\alpha|<1$$

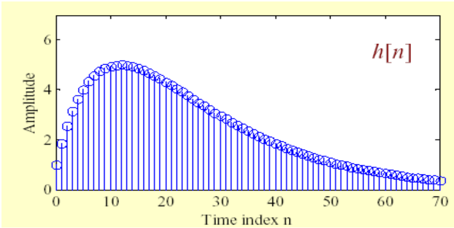
  - **Therefore** $S<\infty$ **if** $|\alpha|<1$ **for which the system is BIBO stable**
  - **If** $|\alpha|=1$**, the system in not BIBO stable.**

Example 3:

Consider a causal LTI discrete-time IIR system an

$$H(z) = \frac{1}{1 - 1.845\,z^{-1} + 0.850586\,z^{-2}}$$

The plot of impulse response is



Example 3:

- **The absolute summablility condition is satisfied.**

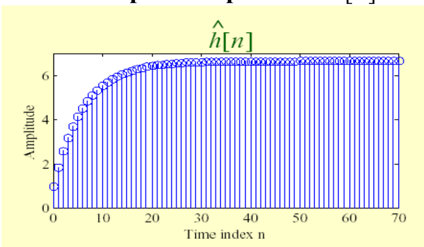
  Hence, $H(z)$ is a stable transfer function.

- Now, consider the case **when the transfer function coefficients are rounded to values with 2 digits after the decimal point:**

$$\hat{H}(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

Example 3:

- **The plot of the impulse response of $\hat{h}[n]$ is shown below**



- **Absolute summability condition is violated.**

  Thus, $\hat{H}(z)$ is an unstable transfer function

Example 4:

Consider a causal LTI IIR discrete-time system with decaying impulse response

$$h[n] = \begin{cases} \frac{1}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

- **Solution** $H(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} = \log_e\left(\frac{1}{1 - z^{-1}}\right),$
  which has infinite number of poles on the unit circle at $|z| = 1$
  and hence, it is unstable.