# Chapter 6

z-Transform

# Part B

Inverse z-Transform
&
ZT Properties

## Inverse z-Transform

- Inverse z-Transform
- z-Transform Properties
- Computation of the Convolution Sum of Finite-Length Sequences



## 1. Inverse z-Transform

**1.1 General Expression**
*** Inverse z-Transform by Lookup Table**
**1.2 Inverse z-Transform by Partial-Fraction Expansion**
**1.3 Partial-Fraction Using MATLAB**
**1.4 Inverse z-Transform via Long Division**
**1.5 Inverse z-Transform Using MATLAB**

## 1.1 General Expression

- Recall that, for $z = re^{j\omega}$, the z-transform $G(z)$ given by

$$G(z)\big|_{z=re^{j\omega}} = G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]r^{-n}e^{-j\omega n}$$

is merely the DTFT of the modified sequence $g[n]r^{-n}$

- Accordingly, the inverse DTFT is thus given by

$$g[n]r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(re^{j\omega})e^{j\omega n}d\omega$$



## 1.1 General Expression

- By making a change of variable $z = re^{j\omega}$ , the previous equation can be converted into a contour integral given by

$$\boxed{g[n] = \frac{1}{2\pi j}\oint_c G(z)z^{n-1}dz}$$

where $c$ is a counterclockwise contour of integration defined by $|z| = r$

- But the integral remains unchanged when $c$ is replaced with any contour $c'$ encircling the point $z=0$ in the ROC of $G(z)$



## 1.1 General Expression

- The contour integral can be evaluated using the Cauchy's residue theorem resulting in

$$g[n] = \sum\left[\text{residues of } G(z)z^{n-1} \text{ at the poles inside } c\right]$$

$$= -\sum\left[\begin{array}{l}\text{residues of } G(z)z^{n-1} \text{ at the poles outside } c \\ \text{only if there are any higher-order poles inside } c\end{array}\right]$$

  - The above equation needs to be evaluated at all values of $n$.
  - Difficult to arrive at a closed-form answer in most cases.



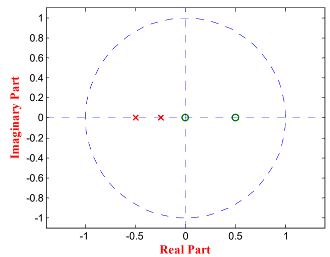
## 1.1 General Expression

**Example:**

$$G(z) = \frac{1-0.5z^{-1}}{1+0.75z^{-1}+0.125z^{-2}}$$

**Zeros:** $z=0 \quad z=0.5$

**Poles:** $z=-0.5 \quad z=-0.25$




**Three ROCs:**

$$|z| < 0.25$$

$$0.25 < |z| < 0.5$$

$$|z| > 0.5$$

## 1.1 General Expression

$$G(z)z^{n-1} = \frac{(z-0.5)z^n}{(z+0.5)(z+0.25)}$$

**Case 1:** $|z| < 0.25$

If $n \geq 0$, there is no poles inside $c$. Thus, $g[n]=0$ when $n \geq 0$

If $n < 0$, there is an $|n|$-order pole at $z=0$ which is inside $c$. In this case, we can compute the summation of the residues outside $c$ instead of that inside

$$g[n] = -\left\{\text{Res}\{z=-0.5\} + \text{Res}\{z=-0.25\}\right\}$$



## 1.1 General Expression

$$g[n] = -(z+0.5)\frac{(z-0.5)z^n}{z^2+0.75z+0.125}\bigg|_{z=-0.5}$$
$$-(z+0.25)\frac{(z-0.5)z^n}{z^2+0.75z+0.125}\bigg|_{z=-0.25}$$
$$= -4(-0.5)^n + 3(-0.25)^n \qquad n \leq -1$$



## 1.1 General Expression

**Case 2:** $0.25 < |z| < 0.5$

If $n \geq 0$, there is only one pole at $z = -0.25$ inside $c$

$$g[n] = (z+0.25)\frac{(z-0.5)z^n}{z^2+0.75z+0.125}\bigg|_{z=-0.25}$$
$$= -3(-0.25)^n \qquad n \geq 0$$



## 1.1 General Expression

If $n < 0$, there are one first-order pole at $z=-0.25$ and one $|n|$th-order pole at $z=0$ inside $c$, respectively. Thus, we can compute the summation of the residues outside $c$ instead of that inside

$$g[n] = -(z+0.5)\frac{(z-0.5)z^n}{z^2+0.75z+0.125}\bigg|_{z=-0.5}$$
$$= -4(-0.5)^n \qquad n \leq -1$$

Hence, we can rewrite $g[n]$ as follows

$$g[n] = -3(-0.25)^n u[n] - 4(-0.5)^n u[-n-1]$$

## 1.1 General Expression

**Case 3:** $|z| > 0.5$

If $n \geq 0$, there are two first-order poles at $z = -0.25$ and $z = -0.5$ inside $c$

$$g[n] = (z+0.5)\frac{(z-0.5)z^n}{z^2+0.75z+0.125}\bigg|_{z=-0.5}$$

$$+ (z+0.25)\frac{(z-0.5)z^n}{z^2+0.75z+0.125}\bigg|_{z=-0.25}$$

$$= 4(-0.5)^n - 3(-0.25)^n \qquad n \geq 0$$



## 1.1 General Expression

If $n<0$, there are two first-order poles and one $|n|$th-order pole at $z=-0.25$, $z=-0.25$ and $z=0$ inside $c$, respectively. Thus, we can compute the summation of the residues outside $c$ instead of that inside. Because there is no poles outside $c$. Thus, $g[n]=0$ in this case

**Summary:**

$$g[n] = \begin{cases} -4(-0.5)^n u[-n-1] + 3(-0.25)^n u[-n-1], & |z| < 0.25 \\ -3(-0.25)^n u[n] - 4(-0.5)^n u[-n-1], & 0.25 < |z| < 0.5 \\ 4(-0.5)^n u[n] - 3(-0.25)^n u[n], & |z| > 0.5 \end{cases}$$



## 1.2 Inverse z-Transform by Partial-Fraction Expansion

- A rational z-transform $G(z)$ with a causal inverse transform $g[n]$ has an ROC that is exterior to a circle
- Here it is more convenient to express $G(z)$ in a partial-fraction expansion form and then determine $g[n]$ by summing the inverse transform of the individual simpler terms in the expansion



## 1.2 Inverse z-Transform by Partial-Fraction Expansion

- A rational $G(z)$ can be expressed as

$$G(z) = \frac{P(z)}{D(z)} = \sum_{i=0}^{M} p_i z^{-i} \Bigg/ \sum_{i=0}^{N} d_i z^{-i}$$

- If then $G(z)$ can be re-expressed as

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)}$$

**Proper Fraction (真分数)**

where the degree of $P_1(z)$ is less than $N$

## 1.2 Inverse z-Transform by Partial-Fraction Expansion

- Rational $G(z)$

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)}$$

- Then

$$\sum_{\ell=0}^{M-N} z^{-\ell} \longleftrightarrow \sum_{\ell=0}^{M-N} \delta[n-\ell]$$

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## 1.2 Inverse z-Transform by Partial-Fraction Expansion

**Example 1: Determine the inverse z-transform**

$$G(z) = \frac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

By long division we arrive at

$$1.5\delta[n-1]$$

$$G(z) = -3.5 + 1.5z^{-1} + \frac{5.5 + 2.1z^{-1}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

$$-3.5\delta[n]$$

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## 1.2 Inverse z-Transform by Partial-Fraction Expansion

- Rational $G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)}$

- For simple poles, let

$$H(z) = \frac{P_1(z)}{D(z)} = \sum_{\ell=0}^{N} \left( \frac{\rho_\ell}{1 - \lambda_\ell z^{-1}} \right) \longleftrightarrow \sum_{\ell=0}^{N} \rho_\ell (\lambda_\ell)^n \mu[n]$$

poles: $\{\lambda_\ell\}, \quad 1 \le \ell \le N$

Constants: $\rho_\ell = (1 - \lambda_\ell z^{-1}) H(z)\big|_{z=\lambda_\ell}$

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## 1.2 Inverse z-Transform by Partial-Fraction Expansion

Let

$$H(z) = \frac{5.5 + 2.1z^{-1}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

$$= \frac{2.75 + 0.25j}{1 - (-0.4 + 0.2j)z^{-1}} + \frac{2.75 - 0.25j}{1 - (-0.4 - 0.2j)z^{-1}}$$

$$(2.75 + 0.25j)(-0.4 + 0.2j)^n u[n]$$

$$(2.75 - 0.25j)(-0.4 - 0.2j)^n u[n]$$

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## 1.2 Inverse z-Transform by Partial-Fraction Expansion

- Rational $G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)}$
- For multiple poles, let

$$H(z) = \frac{P_1(z)}{D(z)} = \sum_{\ell=0}^{N-L}\left(\frac{\rho_\ell}{1-\lambda_\ell z^{-1}}\right) + \sum_{i=1}^{L}\left(\frac{\gamma_i}{(1-\nu z^{-1})^i}\right)$$

constants:

$$\gamma_i = \frac{1}{(L-i)!(-\nu)^{L-i}} \frac{d^{L-i}}{d(z^{-1})^{L-i}}[(1-\nu z^{-1})^L G(z)]\Big|_{z=\nu} \quad 1 \le i \le L$$

multiple poles: $\nu$



## 1.2 Inverse z-Transform by Partial-Fraction Expansion

**Solutions:**

**Step 1--** Converting $G(z)$ into the form of proper fractions by long division

**Step 2–** Summing up the inverse transform of the individual simpler terms in the expansion

* Assumming that $g[n]$ is causal



## 1.2 Inverse z-Transform by Partial-Fraction Expansion

**Example 2: Determine the inverse z-transform**

$$H(z) = \frac{z(z+2)}{(z-0.2)(z+0.6)} = \frac{(1+2z^{-1})}{(1-0.2z^{-1})(1+0.6z^{-1})}$$

A partial-fraction expansion of $H(z)$ is

$$H(z) = \frac{\rho_1}{1-0.2z^{-1}} + \frac{\rho_2}{1+0.6z^{-1}}$$

$$\rho_1 = (1-0.2z^{-1})H(z)\Big|_{z=0.2} = \frac{1+2z^{-1}}{1+0.6z^{-1}}\Big|_{z=0.2} = 2.75$$



## 1.2 Inverse z-Transform by Partial-Fraction Expansion

**Example 2:**

Now $$H(z) = \frac{\rho_1}{1-0.2z^{-1}} + \frac{\rho_2}{1+0.6z^{-1}}$$

$$\rho_1 = (1-0.2z^{-1})H(z)\Big|_{z=0.2} = \frac{1+2z^{-1}}{1+0.6z^{-1}}\Big|_{z=0.2} = 2.75$$

$$\rho_2 = (1+0.6z^{-1})H(z)\Big|_{z=0.6} = \frac{1+2z^{-1}}{1-0.2z^{-1}}\Big|_{z=0.2} = -1.75$$

## 1.2 Inverse z-Transform by Partial-Fraction Expansion

**Example 2:**

Hence $$H(z) = \frac{2.75}{1-0.2z^{-1}} + \frac{1.75}{1+0.6z^{-1}}$$

Inverse transform is therefore given by

$$h[n] = 2.75(0.2)^n \mu[n] - 1.75(-1.6)^n \mu[n]$$



## 1.2 Inverse z-Transform by Partial-Fraction Expansion

**Example 3: Determine the inverse z-transform**

$$G(z) = \frac{z^3}{(z-\frac{1}{2})(z+\frac{1}{3})^2}, \quad |z| > \frac{1}{2}$$

A partial-fraction expansion of $H(z)$ is

$$G(z) = \frac{0.36}{1-\frac{1}{2}z^{-1}} + \frac{0.24}{1+\frac{1}{3}z^{-1}} + \frac{0.4}{(1+\frac{1}{3}z^{-1})^2}$$



## 1.2 Inverse z-Transform by Partial-Fraction Expansion

**Example 3:**

$$\frac{1}{1-\frac{1}{2}z^{-1}} \xrightarrow{\text{IZT}} 0.36(\frac{1}{2})^n \mu[n] \qquad \frac{1}{1+\frac{1}{3}z^{-1}} \xrightarrow{\text{IZT}} 0.24(-\frac{1}{3})^n \mu[n]$$

$$\frac{0.4}{\left(1+\frac{1}{3}z^{-1}\right)^2} \xrightarrow{\text{IZT}} 0.4(n+1)(-\frac{1}{3})^n \mu[n]$$

$$g[n] = [0.36(\frac{1}{2})^n + 0.24(-\frac{1}{3})^n + 0.4(n+1)(-\frac{1}{3})^n]\mu[n]$$



## 1.2 Inverse z-Transform by Partial-Fraction Expansion

- Enlargement of ROC caused by pole-zero cancellation

  Consider two causal sequences $g[n]$ and $h[n]$ , with z-transforms $G(z)$ and $H(z)$, respectively, as given below:

$$G(z) = \frac{2+1.2z^{-1}}{1-0.2z^{-1}}, \qquad |z| > 0.2$$

$$H(z) = \frac{3}{1+0.6z^{-1}}, \qquad |z| > 0.6$$

## 1.2 Inverse z-Transform by Partial-Fraction Expansion

- The intersection of the two ROCs is |z| > 0.6. The product of the above two z-transforms is

$$G(z)H(z) = \left(\frac{2+1.2z^{-1}}{1-0.2z^{-1}}\right)\left(\frac{3}{1+0.6z^{-1}}\right) = \frac{6}{1-0.2z^{-1}}$$

**whose ROC is given by |z| > 0.2, which is larger than the region |z| > 0.6**



## 1.3 Partial-Fraction Expansion Using MATLAB

- **[*r*,*p*,*c*]= `residuez`(num,den)** develops the partial-fraction expansion of a rational z-transform with numerator and denominator coefficients given by vectors **num** and **den**
  - Vector ***r*** contains the residues
  - Vector ***p*** contains the poles
  - Vector ***c*** contains the constants $\eta_l$



## 1.3 Partial-Fraction Expansion Using MATLAB

- **[num,den]=`residuez`(*r*,*p*,*c*)** converts a z-transform expressed in a partial-fraction expansion form to its rational form



## 1.4 Inverse z-Transform via Long Division

- The *z*-transform $G(z)$ of a **causal sequence** $\{g[n]\}$ can be expanded in a **power series** in $z^{-1}$ by **long division**
- In the series expansion, the coefficient multiplying the term $z^{-n}$ is then the $n$-th sample $g[n]$
- For a rational *z*-transform expressed as a ratio of polynomials in $z^{-1}$, the power series expansion can be obtained by **long division**.

## 1.4 Inverse z-Transform via Long Division

**Example 1:**

- Consider $X(z)=\dfrac{1+2z^{-1}}{1+0.4z^{-1}-0.12z^{-2}}$

  Long division of the numerator by the denominator yields

  $$X(z)=1+1.6z^{-1}-0.52z^{-2}+0.4z^{-3}-0.224z^{-4}+\cdots$$

- Hence

  $$\{x[n]\}=\{\underset{\substack{\uparrow\\ n=0}}{1},\ 1.6,\ -0.52,\ 0.4,\ -0.2224,\ldots\}\quad n\geq 0$$

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## 1.4 Inverse z-Transform via Long Division

**Example 2:**

- Consider $X(z)=\dfrac{z^{-1}}{(1-z^{-1})^2}.$

  Long division $\dfrac{1}{(1-z^{-1})^2}=1+2z^{-1}+3z^{-2}+4z^{-3}+\ldots.$

  Then $\dfrac{z^{-1}}{(1-z^{-1})^2}=z^{-1}+2z^{-2}+3z^{-3}+4z^{-4}+\ldots.$

- Hence

  $$\{x[n]\}=\{\underset{\uparrow}{0}\quad 1\quad 2\quad 3\quad 4\quad \ldots\},\quad \text{for}\quad n\geq 0.\qquad x[n]=\begin{cases} n, & n\geq 0,\\ 0, & n<0.\end{cases}$$

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## 1.5 Inverse z-Transform Using MATLAB

- The function **impz** can be used to find the inverse of a rational $z$-transform $G(z)$
- The function computes the coefficients of the power series expansion of $G(z)$
- The number of coefficients can either be user specified or determined automatically

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## 2. z-Transform Properties

**Useful properties of the z-transform.**

| Property | Sequence | z-Transform | ROC |
|---|---|---|---|
| | $g[n]$<br>$h[n]$ | $G(z)$<br>$H(z)$ | $\Re_g$<br>$\Re_h$ |
| Conjugation | $g^*[n]$ | $G^*(z^*)$ | $\Re_g$ |
| Time-reversal | $g[-n]$ | $G(1/z)$ | $1/\Re_g$ |
| Linearity | $\alpha g[n]+\beta h[n]$ | $\alpha G(z)+\beta H(z)$ | Includes $\Re_g \cap \Re_h$ |
| Time-shifting | $g[n-n_0]$ | $z^{-n_0}G(z)$ | $\Re_g$ except possibly the point z=0 or ∞ |
| Multiplication by an exponential sequence | $\alpha^n g[n]$ | $G(z/\alpha)$ | $\lvert\alpha\rvert\Re_g$ |

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## 2. z-Transform Properties

**Useful properties of the z-transform.**

| Property | Sequence | z-Transform | ROC |
|---|---|---|---|
| | $g[n]$ | $G(z)$ | $\Re_g$ |
| | $h[n]$ | $H(z)$ | $\Re_h$ |
| Differentiation of G(z) | $ng[n]$ | $-z\dfrac{dG(z)}{dz}$ | $\Re_g$, except possibly the point z=0 or ∞ |
| Convolution | $g[n]*h[n]$ | $G(z)H(z)$ | Includes $\Re_g \cap \Re_h$ |
| Modulation | $g[n]h[n]$ | $\frac{1}{2\pi j}\oint_C G(v)H(z/v)v^{-1}dv$ | Includes $\Re_g\Re_h$ |
| Parseval's relation | $\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j}\oint_C G(v)H^*(1/v^*)v^{-1}dv$ | | |



## 2. z-Transform Properties

**Example 1:**

$$y[n] = (n+1)\alpha^n u[n]$$

$y[n]$ can be rewritten as $y[n] = nx[n] + x[n]$

where $x[n] = \alpha^n u[n]$

- The z-transform of $x[n]$ is given by

$$X(z) = \frac{1}{1-\alpha z^{-1}} \qquad |z| > |\alpha|$$



## 2. z-Transform Properties

- Using the differentiation property, we arrive at the z-transform of $nx[n]$ as

$$-z\frac{dX(z)}{dz} = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} \qquad |z| > |\alpha|$$

- Using the linearity property we finally obtain

$$Y(z) = \frac{1}{(1-\alpha z^{-1})^2} \qquad |z| > |\alpha|$$



## 2. z-Transform Properties

**Example 2:**

$$x[n] = r^n \cos(\omega_0 n)\mu[n]$$

$x[n]$ can be rewritten as $x[n] = v[n] + v^*[n]$

where $$v[n] = \frac{1}{2}r^n e^{j\omega_0 n}\mu[n] = \frac{1}{2}\alpha^n \mu[n]$$

- The z-transform of $x[n]$ is given by

$$V(z) = \frac{1}{2}\cdot\frac{1}{1-\alpha z^{-1}} = \frac{1}{2}\cdot\frac{1}{1-re^{j\omega_0}z^{-1}}, \qquad |z| > |\alpha| = r$$

## 2. z-Transform Properties

Using the conjugation property we obtain the $z$-transform of $v^*[n]$ as

$$V^*(z^*) = \frac{1}{2}\cdot\frac{1}{1-\alpha^* z^{-1}} = \frac{1}{2}\cdot\frac{1}{1-re^{-j\omega_0}z^{-1}}, \qquad |z|>|\alpha| = r$$

Thus $\quad X(z) = V(z) + V^*(z^*)$

$$= \frac{1}{2}\left(\frac{1}{1-re^{j\omega_0}z^{-1}} + \frac{1}{1-re^{-j\omega_0}z^{-1}}\right),$$

$$= \frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}} \qquad |z|>r$$



## 2. z-Transform Properties

**Example 3:**

Determine the z-transform $V(z)$ of the sequence $v[n]$

$$d_0 v[n] + d_1 v[n-1] = p_0\delta[n] + p_1\delta[n-1] \qquad |d_1/d_0| < 1$$

We have

$$d_0 V(z) + d_1 z^{-1} V(z) = p_0 + p_1 z^{-1}$$

Therefore

$$V(z) = \frac{p_0 + p_1 z^{-1}}{d_0 + d_1 z^{-1}}$$



## 2. z-Transform Properties

**Example 4:**

Determine the energy of the sequence

$$x[n] = a^n u[n], \qquad 0<a<1$$

Using the **Parseval's relation**

$$\xi_x = \frac{1}{2\pi j}\oint_c X(v)X(v^{-1})v^{-1}dv$$

Therefore

$$\frac{1}{2\pi j}\oint_c X(z)X(z^{-1})z^{-1}dz = [\text{sum of residues of } X(z)X(z^{-1})z^{-1} \text{ inside } c]$$



## 2. z-Transform Properties

**Example 4:**

The ZT is given by $\quad \mathrm{X}(z) = \dfrac{1}{1-\alpha z^{-1}}, |z|>\alpha.$

Then $\quad X(z)X(z^{-1})z^{-1} = \dfrac{z^{-1}}{(1-\alpha z^{-1})(1-\alpha z)} = \dfrac{1}{(z-\alpha)(1-\alpha z)}$

It has a pole at $z=\alpha$ inside the unit circle

$$residue = \left.\frac{1}{1-\alpha z}\right|_{z=\alpha} = \frac{1}{1-\alpha^2} \qquad \xi_x = \frac{1}{1-\alpha^2}$$

## 3. Computation of the Convolution Sum of Finite-Length Sequences

- Linear Convolution using Polynomial Multiplication
- Circular Convolution using Polynomial Multiplication



## 3.1 Linear Convolution

- For length-$L$+1 and length-$M$+1 sequence $x[n]$ and $h[n]$

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + ... + x[L]z^{-L}$$

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + ... + h[M]z^{-M}$$

  - Polynomials in $z^{-1}$ of degree ***L*** and ***M***
- From the convolution theorem:

$$y_L[n] = x[n] * h[n] \longleftrightarrow Y(z) = H(z)X(z)$$



## 3.1 Linear Convolution

$$Y(z) = y[0] + y[1]z^{-1} + y[2]z^{-2} + ... + y[L+M]z^{-(L+M)}$$

- **Polynomials in $z^{-1}$ of degree *L+M***

- **Coefficients of the polynomial are precisely the samples of the sequence.**

$n$-th coefficient of $Y(z)$

$$y[n] = \sum_{k=0}^{L+M} x[k]h[n-k]$$



## 3.1 Linear Convolution

**Example 1:**

- Linear convolution of Right-Sided Sequences

$$x[n] = \{\underset{\uparrow}{-2}, 0, 1, -1, 3\} \qquad h[n] = \{\underset{\uparrow}{1}, 2, 0, 1\}$$

using the Polynomial Multiplication Method

$$X(z) = -2 + z^{-2} - z^{-3} + 3z^{-4} \quad H(z) = 1 + 2z^{-1} - z^{-3}$$

$$Y(z) = X(z)H(z) = (-2 + z^{-2} - z^{-3} + 3z^{-4})(1 + 2z^{-1} - z^{-3})$$

$$= -2 - 4z^{-1} + z^{-2} + 3z^{-3} + z^{-4} + 5z^{-5} + z^{-6} - 3z^{-7}$$

$$y[n] = \{\underset{\uparrow}{-2}, -4, 1, 3, 1, 5, 1, -3\}$$

## 3.1 Linear Convolution

**Example 2:**

- Linear convolution of Two-Sided Sequences

$$x[n]=\{3,\ \underset{\uparrow}{-2},\ 4\} \qquad h[n]=\{\underset{\uparrow}{4},\ 2,\ -1\}$$

using the Polynomial Multiplication Method

$$X(z)=3z-2+4z^{-1} \qquad H(z)=4+2z^{-1}-z^{-2}$$

$$Y(z)=X(z)H(z)=(3z-2+4z^{-1})(4+2z^{-1}-z^{-3})$$

$$=12z-2+9z^{-1}+10z^{-2}-4z^{-3}$$

$$y[n]=\{12,\ \underset{\uparrow}{-2},\ 9,\ 10,\ -4\}$$



## 3.2 Circular Convolution

- The circular convolution can also be related to polynomial multiplication but requires a **modulo operation** after the multiplication.



## 3.2 Circular Convolution

- For length-$N$ causal sequences $x[n]$ and $y[n]$, $0 \le n \le N-1$

$$X(z)=x[0]+x[1]z^{-1}+x[2]z^{-2}+\ldots+x[N-1]z^{-(N-1)}$$

$$H(z)=h[0]+h[1]z^{-1}+h[2]z^{-2}+\ldots+h[N-1]z^{-(N-1)}$$

  - Polynomials in $z^{-1}$ of degree ***N*-1**

- From the convolution definition:

$$Y_L(z)=y_L[0]+y_L[1]z^{-1}+y_L[2]z^{-2}+\ldots+y_L[2N-2]z^{-(2N-2)}$$

$$\boxed{Y_C(z)=\left\langle Y_L(z)\right\rangle_{(z^{-N}-1)}}$$

**Take $z^{-N}=1$**



## 3.2 Circular Convolution

**Example 1:**

- Circular convolution of causal sequences $g[n]$ and $h[n]$

(0 ≤n ≤3) using the Polynomial Multiplication Method.

$$G(z)=g[0]+g[1]z^{-1}+g[2]z^{-3}+g[3]z^{-3}$$

$$H(z)=h[0]+h[1]z^{-1}+h[2]z^{-3}+h[3]z^{-3}$$

$$Y_L(z)=X(z)H(z)$$

$$=y_L[0]+y_L[1]z^{-1}+y_L[2]z^{-3}+y_L[3]z^{-3}$$

$$+y_L[4]z^{-4}+y_L[5]z^{-5}+y_L[6]z^{-6}$$

## 3.2 Circular Convolution

**Example 1:**

- Where

$$
\begin{aligned}
y_L[0] &= g[0]h[0] \\
y_L[1] &= g[0]h[1] + g[1]h[0] \\
y_L[2] &= g[0]h[2] + g[1]h[1] + g[2]h[0] \\
y_L[3] &= g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0] \\
y_L[4] &= g[1]h[3] + g[2]h[2] + g[3]h[1] \\
y_L[5] &= g[2]h[3] + g[3]h[2] \\
y_L[6] &= g[3]h[3]
\end{aligned}
$$

- Then

$$Y_C(z) = \langle Y_L(z) \rangle_{(z^{-4}-1)}$$

Take $z^{-4}=1$



## 3.2 Circular Convolution

**Example 1:**

$$
\begin{aligned}
Y_C(z) &= y_L[0] + y_L[1]z^{-1} + y_L[2]z^{-2} + y_L[3]z^{-3} + y_L[4] + y_L[5]z^{-1} + y_L[6]z^{-2} \\
&= (y_L[0] + y_L[4]) + (y_L[1] + y_L[5])z^{-1} + (y_L[2] + y_L[6])z^{-2} + y_L[3]z^{-3} \\
&= y_C[0] + y_C[1]z^{-1} + y_C[2]z^{-2} + y_C[3]z^{-3}
\end{aligned}
$$

where

$$
\begin{aligned}
y_C[0] &= y_L[0] + y_L[4] = g[0]h[0] + g[1]h[3] + g[2]h[2] + g[3]h[1] \\
y_C[1] &= y_L[1] + y_L[5] = g[0]h[1] + g[1]h[0] + g[2]h[3] + g[3]h[2] \\
y_C[2] &= y_L[2] + y_L[6] = g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[3] \\
y_C[3] &= y_L[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]
\end{aligned}
$$