Ch6 z-Transforms

6.1 Derive the z-transforms and the ROCs of the following sequences given in Table 6.1.

a) $n\alpha^n \mu[n]$

b) $r^n \cos(\omega_0 n)\mu[n]$

6.2 Determine the z-transform and the corresponding ROC of the following sequences:

a) $x_1[n] = \alpha^n \mu[n+4]$

b) $x_2[n] = -\alpha^n \mu[-n-3]$

c) $x_3[n] = nr^n \cos(\omega_0 n)\mu[n]$

d) $x_4[n] = \alpha^n \mu[-n]$

6.3 Consider the following sequences:

i. $x_1[n] = (0.2)^n \mu[n+1]$,

ii. $x_2[n] = (0.6)^n \mu[n-2]$,

iii. $x_3[n] = (0.5)^n \mu[n-6]$

iv. $x_4[n] = (-0.5)^n \mu[-n-3]$

a) Determine the ROCs of the z-transform of each of the given sequences;
b) From the ROCs determined in Part a), determine the ROCs of the following sequences:,

v. $y_1[n] = x_1[n] + x_2[n]$

vi. $y_2[n] = x_1[n] + x_4[n]$

vii. $y_3[n] = x_2[n] + x_3[n]$

viii. $y_4[n] = x_2[n] + x_4[n]$

6.4 Determine the z-transform of the two-sided sequence $v[n] = \alpha^{|n|}$, $|\alpha| < 1$.What is its ROC?

6.5 Determine the z-transform and the corresponding ROC of the following sequences. Assume $|\beta| > |\alpha| > 0$.Show their pole-zero plots and indicate clearly the ROC in these plots.

a) $x_1[n] = \alpha^n \mu[n+1] + \beta^n \mu[n+2]$,

b) $x_2[n] = \alpha^n \mu[n-1] + \beta^n \mu[-n-1]$

c) $x_3[n] = \alpha^n \mu[n+2] + \beta^n \mu[-n-1]$

- (Optional) Determine the z-transform and the corresponding ROC of the following sequences:

  d) $x_1[n] = n^2 \alpha^n \mu[n]$

  e) $x_2[n] = \dfrac{(n+1)(n+2)}{2} \alpha^n \mu[n]$

6.6 Let $X(z)$ denote the z-transform of a sequence $x[n]$ with an ROC $\mathcal{R}_x$. Determine the z-transform and the ROC of each of the following functions of $x[n]$:

a) $y_1[n] = nx[n]$,

b) $y_2[n] = n^2 x[n]$

c) $y_3[n] = (n+1)^2 x[n]$

6.7 Determine the all possible ROCs of the following z-transforms and the their corresponding inverse z-transform

a) $X_1(z) = \dfrac{7 + 3.6z^{-1}}{1 + 0.9z^{-1} + 0.18z^{-2}}$

b) $X_2(z) = \dfrac{4 - 1.6z^{-1} - 0.4z^{-2}}{(1 + 0.6z^{-1})(1 - 0.4z^{-1})}$

- (Optional) Let the z-transforms of the sequences $x[n]$ and $y[n]$ be denoted by $X(z)$ and $Y(z)$, respectively, with $\mathcal{R}_x$ and $\mathcal{R}_y$ denoting their respective ROCs. Determine the expression for the z-transform of the cross-correlation sequence $r_{xy}[\ell]$ in terms of $X(z)$ and $Y(z)$, and its ROC in terms of $\mathcal{R}_x$ and $\mathcal{R}_y$. Using this result, determine the z-transform $R_{xx}(z)$ of the autocorrelation sequence $r_{xx}[\ell]$ of the causal sequence $x[n] = \alpha^n \mu[n]$, $0 < |\alpha| < 1$, and then determine the expression for the autocorrelation sequence $r_{xx}[\ell]$ by applying the inverse z-transform to $R_{xx}(z)$.

- (Optional) Let $H(z) = A(z^2) - z^{-1}B(z^2)$ be the z-transform of a causal sequence with its ROC including the unit circle. Define $Y(z) = A(z^2) + z^{-1}B(z^2)$, with $y[n]$ denoting its inverse z-transform. Is $y[n]$ causal? What is the ROC of $Y(z)$? If the magnitude of the DTFT $X(e^{j\omega})$ of $x[n]$ is as shown in Fig. 1, sketch the magnitude $Y(e^{j\omega})$ of the DTFT of $y[n]$.

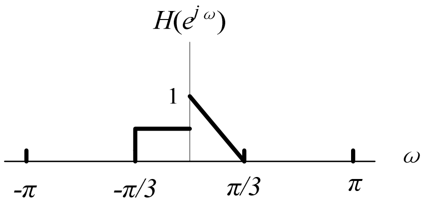


Fig. 1

6.8 Let the z-transform of a sequence $x[n]$ by $X(z)$ with $\mathcal{R}_x$ denoting its ROC. Express the z-transforms of the real and imaginary parts of $x[n]$ in terms of $X(z)$. Show also their respective ROCs.

6.9 Determine the z-transform and the ROCs of the sequences

a) $y_1[n] = \mu[n+2] - \mu[n-3]$,

b) $y_2[n] = 2n\alpha^n \mu[n], \ |\alpha| < 1$.

Show that the ROC includes the unit circle for each z-transform. Evaluate the z-transform evaluated on the unit circle for each sequence and show that it is precisely the DTFT of the respective sequence.

- (Optional) The z-transform $X(z)$ of the length-9 sequence

$$x[n] = \{3, \ 1, \ -5, \ -11, \ 0, \ -5, \ 3, \ 3, \ 8\}, \ -5 \le n \le 3$$

is sampled at 6 points $\omega_k = 2\pi k / 6, \ 0 \le k \le 5$, on the unite circle yielding the frequency samples

$$\tilde{X}[k] = X(z)|_{z=e^{j2\pi k/6}}, \ 0 \le k \le 5$$

Determine, without evaluating $\tilde{X}[k]$, the periodic sequence $\tilde{x}[n]$ whose discrete-Fourier series coefficients are given by $\tilde{X}[k]$. What is the period of $\tilde{x}[n]$.

6.10 Let $X(z)$ denote the z-transform of the length-10 sequence $x[n]$

$$\{x[n]\} = \{6,\ 8,\ -7,\ 8,\ 2,\ -8,\ -4,\ 1,\ -9,\ 5\}$$

Let $X_0[k]$ represent the samples of $X(z)$ evaluated on the unit circle at 8 equally spaced points given by $z = e^{j(2\pi k/8)}$, $0 \le k \le 7$, i.e.,

$$X_0[k] = X(z)\big|_{z=e^{j2\pi k/8}},\quad 0 \le k \le 7$$

Determine the 8-point IDFT $x_0[n]$ of $X_0[k]$ without computing the IDFT.

6.11 Consider the causal sequence $x[n] = (-0.5)^n \mu[n]$, with a z-transform given by $X(z)$

a) Determine the inverse z-transform of $X(z^3)$ without computing $X(z)$,

b) Determine the inverse z-transform of $(1+z^{-2})X(z^3)$ without computing $X(z)$,

6.12 Let

$$X(z) = \frac{3 - 7.8z^{-1}}{(1-0.7z^{-1})(1+1.6z^{-1})}$$

be the z-transform of a sequence $x[n]$. What are the possible ROCs of $X(z)$? Does the DTFT $X(e^{j\omega})$ of $x[n]$ exist? Justify your answer.

- (Optional) Determine the inverse z-transform $x[n]$ of the following z-transform:

$$X(z) = \frac{1}{1-z^{-3}},\quad |z| > 1,$$

by expanding in a power series and computing the inverse z-transform of the individual terms in the power series. Compare the result with those obtained using a partial fraction approach.

6.13 Determine the inverse z-transforms of the following z-transforms:

a) $X_1(z) = \ln(1-\alpha z^{-1}),\ |z| > |\alpha|,$

b) $X_2(z) = \ln(\frac{\alpha - z^{-1}}{\alpha}), |z| > 1/|\alpha|$

c) $X_3(z) = \sin(z^{-1}),\ z \ne 0,$

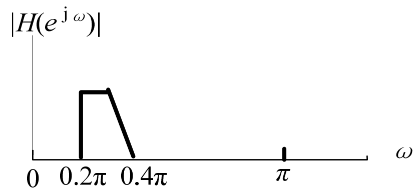
- (Optional)The z-transform of a right-sided sequence $h[n]$ is given by

$$H(z) = \frac{z+1.7}{(z+0.3)(z-0.5)}$$

Find its inverse z-transform $h[n]$ via the partial-fraction approach. Using the MATLAB verify the partial fraction expansion.

- (Optional) Prove $Y_C(z) = \langle Y_L(z) \rangle_{(z^{-N}-1)}$.

6.14 The magnitude response of a digital filter with a real-coefficient transfer function $H(z)$ is shown in Fig.2. Plot the magnitude response function of the filter $H(z^5)$.



- (Optional) Consider a sequence $x[n]$ with a z-transform $X(z)$. Define a new z-transform $\hat{X}(z)$ given by the complex natural logarithm of $X(z)$; that is $\hat{X}(z) = \ln X(z)$. The inverse z-transform of $\hat{X}(z)$ to be denoted by $\hat{x}[n]$ is called the *complex cepstrum* of $x[n]$ [Tri79]. Assume that the ROCs of both $X(z)$ and $\hat{X}(z)$ include the unit circle.

a) Relate the DTFT $X(e^{j\omega})$ of $x[n]$ to the DTFT $\hat{X}(e^{j\omega})$ of its complex cepstrum;

b) Show that the complex cepstrum of a real sequence is a real-valued sequence;

c) Let $\hat{x}_{ev}[n]$ and $\hat{x}_{od}[n]$ denote, respectively the even and odd parts of a real-valued complex cepstrum. Express $\hat{x}_{ev}[n]$ and $\hat{x}_{od}[n]$ in terms of $X(e^{j\omega})$, the DTFT of $x[n]$.

6.15 Determine the transfer function of the following causal LTI discrete-time system described by the difference equation:

$$\begin{aligned} y[n] &= 5x[n] + 9.5x[n-1] + 1.4x[n-2] - 24x[n-3] \\ &+0.1y[n-1] - 0.14y[n-2] - 0.49y[n-3] \end{aligned}$$

a) Express the transfer function in a factored form and sketch its pole-zero plot;

b) Is the corresponding system BIBO stable?

6.16 The transfer function of a causal LTI discrete-time system is given by:

$$H(z) = \frac{1-3.3z^{-1}+0.36z^{-2}}{1+0.3z^{-1}-0.18z^{-2}}$$

a) Determine the impulse response $h[n]$ of the above system

b) Determine the output $y[n]$ of the above system for all values of n for an input

$$x[n] = 2.1(0.4)^n \mu[n] + 0.3(-0.3)^n \mu[n].$$

6.17 Using z-transform methods, determine the explicit expression for the impulse response $h[n]$ of a causal LTI discrete-time system that develop an output $y[n] = 2(-0.2)^n \mu[n]$ for an input $x[n] = 3(0.5)^n \mu[n]$.

6.18 A causal LTI discrete-time system is described by the difference equation

$$y[n] = 0.4y[n-1] + 0.05y[n-2] + 3x[n]$$

where $x[n]$ and $y[n]$ are, respectively, the input and the output sequences of the system.

a) Determine the transfer function $H(z)$ of the system.

b) Determine the impulse response $h[n]$ of the system.

c) Determine the step response $s[n]$ of the system.

- (Optional) Fig. 3a) and b) show, respectively, the DPCM (*differential pulse-code modulation*) coder and decoder often employed for the compression of the digital signal. The linear predictor $P(z)$ in the encoder develops a prediction $\hat{x}[n]$ of the input signal $x[n]$, and the difference signal $d[n] = x[n] - \hat{x}[n]$ is quantized by the quantizer Q developing the quantized output $u[n]$, which is represented with fewer bits than that of $x[n]$. The output of the encoder is transmitted over a channel to the decoder. In the absence of any errors due to transmission and quantization, the input $v[n]$ to the decoder is equal to $u[n]$, and the decoder generates the output $y[n]$, which is equal to the input $x[n]$. Determine the transfer function $H(z) = U(z)/X(z)$ of the encoder in the absence of any quantization, and the transfer function $G(z) = Y(z)/V(z)$ of the decoder for the case of each of the following predictors, and show that $G(z)$ is the inverse of $H(z)$ in each case.

a) $P(z) = h_1 z^{-1}$

b) $P(z) = h_1 z^{-1} + h_2 z^{-2}$

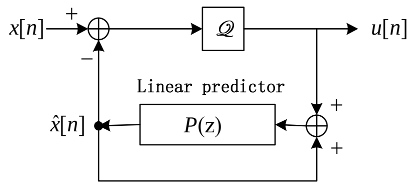


Fig.3 a)

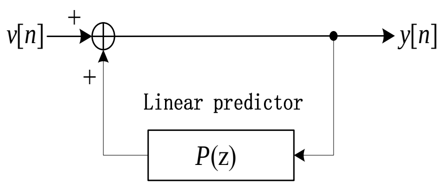


Fig.3 b)

6.19 Determine the closed-form expression for the frequency response $H(e^{j\omega})$ of an LTI discrete-time system characterized by the an impulse response

$$h[n] = \delta[n] - \alpha\delta[n-R], \quad |\alpha| < 1,$$

a) Determine the maximum and the minimum values of its magnitude response.
b) How many peaks and dips of the magnitude response occur in the range $0 \le \omega < 2\pi$? What are the location of the peaks and dips?
c) Sketch the magnitude and the phase response for *R*=5.

- (Optional) An IIR LTI discrete-time system is described by the difference equation

$$y[n] + a_1 y[n-1] + a_3 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

where $y[n]$ and $x[n]$ denote , respectively, the output and the input sequences.

a) Determine the expression for its frequency response;

b) For what values of the constants b_0, b_1 ,and b_2, will the magnitude response be a constant for all value of ω.

6.20 The frequency response $H(e^{j\omega})$ of a length-4 FIR filter with a real and antisymmetric impulse response has the following specific values: $H(e^{j3\pi/2}) = -5 - j5$ and $H(e^{j\pi}) = 2$. Determine $H(z)$.

6.21 A causal stable LTI discrete-time system is characterized by an impulse response

$$h_1[n] = 1.9\delta[n] + 0.5(-0.2)^n \mu[n] - 0.6(0.7)^n \mu[n]$$

Determine the impulse response $h_2[n]$ of its inverse system, which is causal and stable.

(Pay attention to the definition of the **reverse system**!!)

6.22 Consider the FIR LTI discrete-time system characterized by the difference equation

$$y[n] = x[n] + \alpha x[n-M]$$

a) Determine the impulse response $h[n]$.
b) Determine the impulse response $g[n]$ of its causal reverse system.
c) Check the stability of the inverse system.
(Pay attention to the definition of the **reverse system**!!)

- (Optional)

d) Let $G(z)$ be a causal stable rational transfer function with an ROC given by $|z| \geq \beta$. For what values of the constant α will the transfer function $H(z) = G(z/\alpha)$ remain stable? What is the ROC of $H(z)$.
e) Consider the causal stable transfer function

$$G(z) = \frac{1 - 0.5z^{-1} + 2z^{-2}}{(1 + 0.9z^{-1})(1 + 0.4z^{-1})}$$

Develop a transfer function $H(z)$ by scaling the complex variable z by a constant α, i.e., $H(z) = G(z/\alpha)$. Determine the range of values of α for which $H(z)$ remains stable.

6.23 The transfer function of an LTI discrete-time system is given by

$$H(z) = \frac{3 + 5.9z^{-1}}{(1 - 3.5z^{-1})(1 + 0.6z^{-1})}$$

a) How many ROCs are associated with $H(z)$?

b) Does the frequency response $H(e^{j\omega})$ of the system exist? Justify your answer.

c) Can the system by stable? If it is stable, can it by causal?
d) Determine the forms of its impulse response $h[n]$.
(Hint: Find all the ROCs and discuss accordingly.)