Chapter 6

z-Transform

Contents

Part A

z-Transform

Part A: z-Transform

- **Definition: z-Transform**
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- **Region of Convergence (ROC) of a Rational z-Transform**

1. z-Transform

- **DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems.**

- **DTFT of a sequence may not exist because of the convergence condition.**

1. z-Transform

- **z-Transform(ZT)**
  - **Generalization** of the **DTFT** in **complex frequency domain**.
    - ※ **ZT** for discrete-time systems
    - ※ **Laplace-transform(LT)** for continuous-time systems
  - **Existing** for many sequences whose **DTFT does not exist**.
  - Permitting **simple algebraic manipulations**.
  - Providing a great deal of **insight** into **system design** and **behavior**.

1. 1 Definition of z-Transform

- Recall: DTFT of a sequence $g(n)$

$$G\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}$$

$G(e^{j\omega})$ can be viewed as a Fourier series and $g[n]$ is the coefficients of this series.

- Building block in DTFT: $e^{j\omega}$
  - One dimensional (single-variable) function

1. 1 Definition of z-Transform

- Define a new two dimensional variable

$$z= re^{j\omega}$$

  - z is called the complex frequency
    - ✓ r : attenuation
    - ✓ ω: real frequency
  - *simple algebraic manipulations*.

1. 1 Definition of z-Transform

- For a given sequence $g[n]$, its z-transform $G(z)$ is defined as

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

where z **= Re(z) + *j*Im(z)** is a **complex variable**.

$$g[n] \overset{z}{\longleftrightarrow} G(z) = \mathbb{Z}(g[n])$$

1. 1 Definition of z-Transform

- Let $z = re^{j\omega}$, then the z-transform

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n}$$

- DTFT of the **modified sequence** $\{g(n)r^{-n}\}$

$$G(z) = \sum_{n=-\infty}^{\infty} \left(g[n] r^{-n}\right) e^{-j\omega n} = \mathscr{F}\left(g[n] r^{-n}\right)$$

- For $r = 1$ (i.e., $|z| = 1$, ***unit circle***.), reduces to its DTFT,

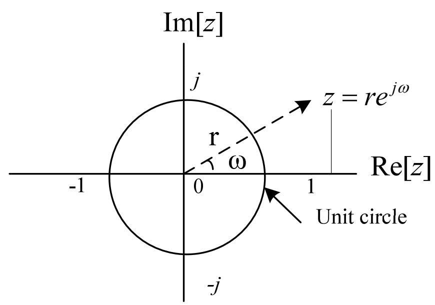
$$G(z)\Big|_{z=e^{j\omega}} = G\left(e^{j\omega}\right)$$

provided the DTFT exists!

1. 1 Definition of z-Transform

- Geometrical interpretation - considering the location of the point z in the complex z-plane.
- For fixed r and ω, the point $z = re^{j\omega}$ in the complex z-plane is at the tip of a vector of length r originating at the point $z = 0$ and subtending an angle ω with respect to the real axis.

1. 1 Definition of z-Transform



- Contour $|z| = 1$: a circle in the z-plane of **unity radius** and is called the ***unit circle***.

1. 1 Definition of z-Transform

- For r=1 (i.e., $|z|$=1), the z-transform $G(z)$ of $g[n]$ reduces to its Fourier transform $G(e^{j\omega})$, providing the latter exists.
- Evaluate $G(z)$ on the unit circle
  - Counterclockwise:

$$z: \ 1 \to j \ \to \ -1 \ \to \ -j \ \to \ 1$$
$$\omega: \ 0 \to \ \pi/2 \ \to \pi \to \ 3\pi/2 \ \to 0$$

  - Clockwise:

$$z: \ 1 \to \ -j \ \to \ -1 \ \to \ j \ \to \ 1$$
$$\omega: -2\pi \to -\pi/2 \to -\pi \to -3\pi/2 \ \to 0$$

1. 1 Definition of z-Transform

- It follows: by traversing the unit circle either clockwise or counterclockwise, we can evaluate the Fourier transform $G(e^{j\omega})$ at all values of the frequency $-\infty < \omega < \infty$ with Fourier transform exhibiting a periodic response with a period 2π .

1. 2 Region of Convergence (ROC)

- Conditions on the convergence of the infinite series

$$\sum_{n=-\infty}^{\infty} g[n]z^{-n}$$

For a given sequence, the set R of values of z for which its z-transform converges is called the *Region of convergence* (ROC).

※ ZT may exist for many sequences for which the DTFT does not exist

1. 2 Region of Convergence (ROC)

- From the DTFT, it follows that the series

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]r^{-n}e^{-j\omega n}$$

converges if $g[n]r^{-n}$ is absolutely summable, i.e., if

$$\sum_{n=-\infty}^{\infty} \left| g[n]r^{-n} \right| < \infty$$

1. 2 Region of Convergence (ROC)

- ROC R is an annular region of the z-plane:

$$R_{g_-} < |z| < R_{g_+}$$

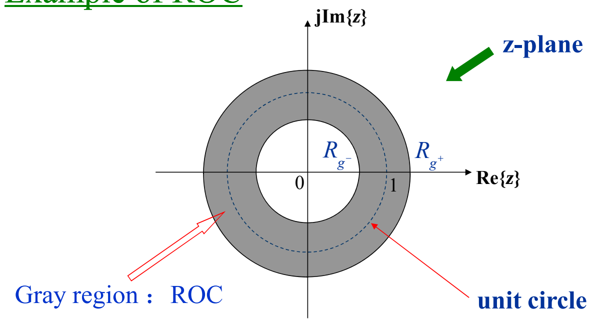
where

$$0 \le R_{g_-} < R_{g_+} \le \infty$$

  - Note: The z-transform is a form of a **Laurent series** and is an **analytic function at every point** in the ROC.
  - R_{g_-} maybe 0; R_{g_+} maybe ∞ .

1. 2 Region of Convergence (ROC)

Example of ROC



If $R_{g_+} < R_{g_-}$, ROC is a null space : ZT does not exist.

1. 2 Region of Convergence (ROC)

Example 1:

Calculate the ZT of $x[n] = a^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

Note that the above equation holds only for $|az^{-1}| < 1$,

i.e. $|z| > |a|$

Region of convergence

1. 2 Region of Convergence (ROC)

Example 2:

Calculate the ZT of $x[n] = -a^n u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n} = \sum_{n=-\infty}^{-1} \left(az^{-1}\right)^n$$

$$= \sum_{n=1}^{\infty} \left(az^{-1}\right)^{-n} = \sum_{n=1}^{\infty} \left(a^{-1}z\right)^n = \frac{z}{z-a}$$

Note that the above equation holds only for $|a^{-1}z| < 1$,

i.e. $|z| < |a|$

Region of convergence

From the above two examples, we find that

- Very different time functions can have the same z-transform expressions.
  - **ROC plays an important role in computing the z-transform or inverse z-transform**.
- **Unique sequence** can be associated with a z-transform is by **specifying its ROC**

Example 3:

Calculate the ZT $\mu(z)$ of $\mu[n]$

$$X(z) = \frac{1}{1-\alpha z^{-1}} \quad \text{for} \left|\alpha z^{-1}\right| < 1$$

by setting $\alpha = 1$

$$\mu(z) = \frac{1}{1-z^{-1}} \quad \text{for} \left|z^{-1}\right| < 1$$

$1 < |z| \le \infty$ ← Region of convergence

Example 3:

Note: The unit step sequence $\mu[n]$ is not absolutely summable, and hence its DTFT does not converge uniformly.

Example 4: Calculate the ZT of finite-length sequence

$$g[n] = \begin{cases} \alpha^n, & M \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$G(z) = \sum_{n=M}^{N-1} \alpha^n z^{-n} = z^{-M} \sum_{n=0}^{N-M-1} (\alpha z^{-1})^n$$

$$= z^{-M}\left(\frac{1-\alpha^{N-M} z^{-(N-M)}}{1-\alpha z^{-1}}\right) = \frac{z^{-M} - \alpha^{N-M} z^{-N}}{1-\alpha z^{-1}}$$

Since the sum involves a finite number of terms, the sum is finite everywhere in the z-plane except possibly z =0 and/or z = ∞ , provided $|\alpha|$ is finite.

- $N > M \geq 0$, the ROC is the entire z-plane excluding the origin z =0.
- $M < 0$ and $N > 0$, the ROC is the entire z-plane excluding z =0 and z =∞ .
- $M < N < 0$, the ROC is the entire z-plane excluding z = ∞ .

Example 5: The finite energy sequence

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

has a DTFT given by

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega < |\omega| \leq \pi \end{cases}$$

which converges in the mean-square sense.

- However, $h_{LP}[n]$ does not have a z-transform as it is not absolutely summable for any value of r.

- The DTFT $H(e^{j\omega})$ of a sequence $h[n]$ converges uniformly if and only if the ROC of the z-transform $H(z)$ of $h[n]$ includes the unit circle.
- The existence of the DTFT does not always imply the existence of the z-transform.

Some commonly used z-transform pairs.

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n \mu[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n \cos\omega_0 n)\mu[n]$	$\frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > r$
$(r^n \sin\omega_0 n)\mu[n]$	$\frac{1-(r\sin\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > r$

2 Rational z-Transform

- LTI discrete-time systems, all involved ZT are rational functions of z^{-1}
- Ratios of two polynomials in z^{-1}:

(Form I)

$$G(z)=\frac{P(z)}{D(z)}=\frac{p_0+p_1z^{-1}+\cdots+p_{M-1}z^{-(M-1)}+p_Mz^{-M}}{d_0+d_1z^{-1}+\cdots d_{N-1}z^{-(N-1)}+d_{N-1}z^{-N}}$$

  - Degree of numerator polynomial $P(z)$: M
  - Degree of denominator polynomial $D(z)$: N

2 Rational z-Transform

- Alternate representation of a rational z-transform is as a ratio of two polynomials in z:

$$G(z)=z^{(N-M)}\frac{p_0z^M+p_1z^{M-1}+\cdots+p_{M-1}z+p_M}{d_0z^N+d_1z^{N-1}+\cdots d_{N-1}z+d_{N-1}}$$

(Form II)

2 Rational z-Transform

- A rational z-transform can be alternatively written in factored form as

$$G(z)=\frac{p_0}{d_0}\frac{\prod_{l=1}^{M}(1-\xi_l z^{-1})}{\prod_{l=1}^{N}(1-\lambda_l z^{-1})}$$ **(Form III)**

$$G(z)=z^{(N-M)}\frac{p_0}{d_0}\frac{\prod_{l=1}^{M}(z-\xi_l)}{\prod_{l=1}^{N}(z-\lambda_l)}$$ **(Form IV)**

2 Rational z-Transform

- Roots $z=\xi_l$ of the numerator polynomial, $G(\xi_l)=0$, known as the zeros of $G(z)$
- Roots $z=\lambda_l$ of the denominator polynomial, $G(\lambda_l)=\infty$, known as the poles of $G(z)$

2 Rational z-Transform

Consider: $$G(z)=\frac{p_0}{d_0}\frac{\prod_{l=1}^{M}(1-\xi_l z^{-1})}{\prod_{l=1}^{N}(1-\lambda_l z^{-1})}$$

- Rational z-transform can be represented completely by :

  the locations of its poles $\{\lambda_l\}$ 、 zeros $\{\xi_l\}$

  and the gain constant $\dfrac{p_0}{d_0}$.

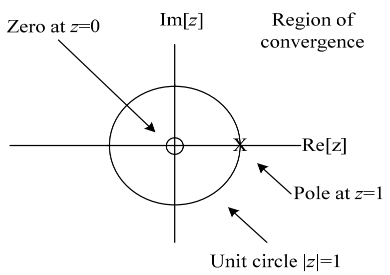
2 Rational z-Transform

Consider: $$G(z)=z^{(N-M)}\frac{p_0}{d_0}\frac{\prod_{l=1}^{M}(z-\xi_l)}{\prod_{l=1}^{N}(z-\lambda_l)}$$

- Note $G(z)$ has M finite zeros and N finite poles
  - If $N > M$ there are additional $N-M$ zeros at $z = 0$ (the origin in the z-plane)
  - If $N < M$ there are additional $M-N$ poles at $z = 0$

2 Rational z-Transform

Example: The z-transform $H(z)=\dfrac{1}{1-z^{-1}}$, for $|z|>1$ has zero at z=0 and a pole at z=1

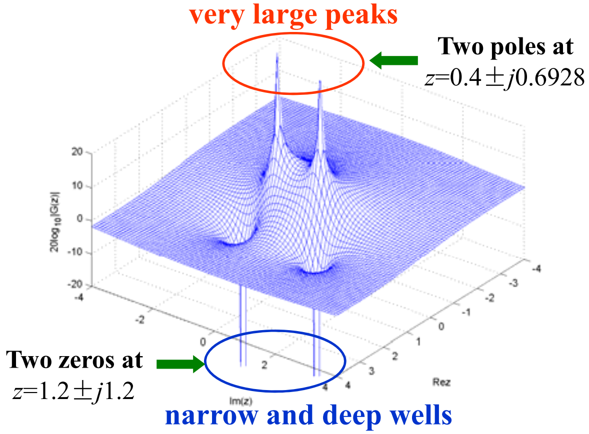


2 Rational z-Transform

- A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude $20\log_{10}|G(z)|$ as shown on next slide for

$$G(z)=\frac{1-2.4z^{-1}+2.88z^{-2}}{1-0.8z^{-1}+0.64z^{-2}}$$

2 Rational z-Transform



3 General Form of ROC of a Rational z- Transform

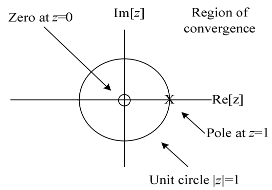
- Without the knowledge of the ROC, there is no unique relationship between a sequence and its z-transform. Hence, the z-transform must always be specified with its ROC.
- If the ROC of a z-transform includes the unit circle, the DTFT of the sequence is obtained by simply evaluating the z-transform on the unit circle.

3 General Form of ROC of a Rational z- Transform

- There is a **relationship between the ROC** of the z-transform of the impulse response of a **causal LTI discrete-time system** and its **BIBO stability**.
- The **ROC** of a rational z-transform is **bounded by the locations of its poles**. It is instructive to examine the pole-zero plot of a z-transform.

3 General Form of ROC

Example Consider again the pole-zero plot of the z-transform $\mu(z)$



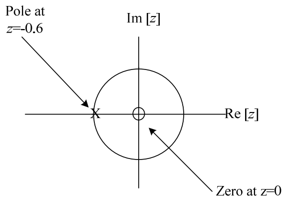
- the ROC is the region of the z-plane just outside the circle centered at the origin and going through the pole at z = 1

3 General Form of ROC

Example

The z-transform the sequence $h[n] = (-0.6)^n \mu[n]$ is given by

$$H(z) = \frac{1}{1+0.6z^{-1}}, \quad |z| > 0.6$$



- Here the ROC is just outside the circle going through the point $z = -0.6$

3 General Form of ROC of a Rational z- Transform

- In general, there are **four types of ROCs** for z-transforms, and they depend on the **type of the corresponding time functions**.
  - Finite-length sequence
  - Right-sided sequence
  - Left-sided sequence
  - Two-sided (infinite duration) sequence

3.1 General Form of ROC

– Finite-length Sequence

A **finite-length sequence** $g[n]$ is defined for $M \le n \le N$ with $|g[n]| < \infty$.

Example :

$$g[n] = \begin{cases} \alpha^n, & M \le n \le N \\ 0, & \text{otherwise} \end{cases}$$

$$G(z) = \sum_{n=M}^{N} g[n] z^{-n} = \sum_{n=0}^{N-M} g[n+M] \frac{z^{N-M-n}}{z^N}$$

3.1 General Form of ROC

- A finite-length sequence is bounded sequence with **converge everywhere in the z-plane except possible z =0 and/or z=∞**

➢ **N> M> 0, the ROC is the entire z-plane excluding the origin z =0.**

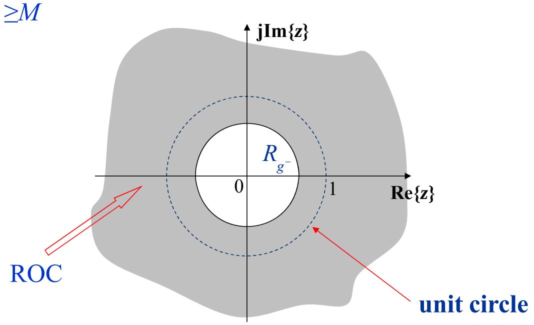
➢ **M < 0 and N > 0, the ROC is the entire z-plane excluding z =0 and z = ∞ .**

➢ **M<N < 0, the ROC is the entire z-plane excluding z = ∞ .**

3.1 General Form of ROC

– Right-sided Sequence

A **right-sided sequence** $u[n]$ with nonzero sample values only for $n \geq M$



3.1 General Form of ROC

If $M \geq 0$, $R_{g-} < |z| \leq \infty$ $\quad R_{g^+} = \infty$

If $M < 0$, $R_{g-} < |z| < \infty$ $\quad R_{g^+} < \infty$

If $M = 0$, $u[n]$ is called a **causal sequence**

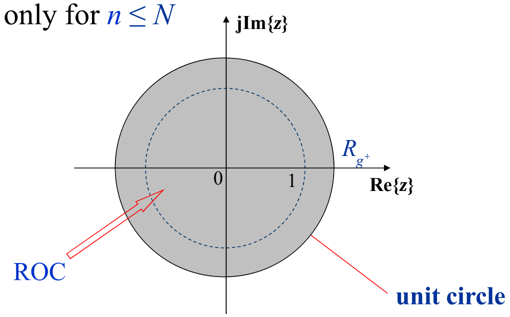
Comment

All causal sequences (or the impulse responses of LTI systems) are right-sided, while not all right-sided sequences correspond to causal systems.

3.1 General Form of ROC

– Left-sided Sequence

A **left-sided sequence** $v[n]$ with nonzero sample values only for $n \leq N$



3.1 General Form of ROC

If $N > 0$, $0 < |z| < R_{g+}$ $\quad R_{g-} > 0$

If $N \leq 0$, $0 \leq |z| < R_{g+}$ $\quad R_{g-} = 0$

If $N = 0$, $v[n]$ is called a **anticausal sequence**

3.1 General Form of ROC

– Two-sided Sequence

The z-Transform of a **two-sided sequence** $w[n]$ can be expressed as

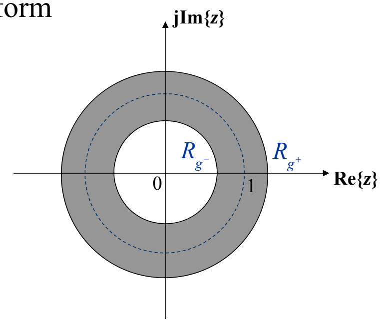
$$W(z)=\sum_{n=-\infty}^{\infty} w[n]z^{-n}=\sum_{n=0}^{\infty} w[n]z^{-n}+\sum_{n=-\infty}^{-1} w[n]z^{-n}$$

A right-sided sequence + A left-sided sequence

$|z|>R_{g-}$ $\quad$ $|z|<R_{g+}$

3.1 General Form of ROC

Obviously, the ROC of $W(z)$ is the intersection of $|z|>R_{g-}$ and $|z|<R_{g+}$. If $R_{g+}>R_{g-}$, its ROC has the following form



But, if $R_{g+}<R_{g-}$, its ROC is a null space, i.e., the transform does not exist

3.1 General Form of ROC

Example

Consider the sequences for $|\alpha|<|\beta|$ where α and β can be either complex or real

$x[n]=(\alpha^n+\beta^n)\mu[n]$ ➡ $|\beta|<|z|\le\infty$

$x[n]=-(\alpha^n+\beta^n)\mu[-n-1]$ ➡ $0\le|z|<|\alpha|$

$x[n]=\alpha^n\mu[n]-\beta^n\mu[-n-1]$ ➡ $|\alpha|<|z|<|\beta|$

$x[n]=-\alpha^n\mu[-n-1]+\beta^n\mu[n]$ ➡ $|z|<|\alpha|\cap|\beta|<|z|=\varnothing$

3.1 General Form of ROC

Finally, for a two-sided sequence, some of the poles contribute to terms in the parent sequence for $n<0$ and the other poles contribute to terms $n>0$.

- ROC is thus **bounded**
  - on the outside by the pole with the smallest magnitude that contributes for $n<0$
  - on the inside by the pole with the largest magnitude that contributes for $n>0$

3.1 General Form of ROC

Example

Consider the two-sided sequence $x[n]=a^n$, where a can be either complex or real. Its z-Transform is given by

$$X(z)=\sum_{n=0}^{\infty}a^n z^{-n}+\sum_{n=-\infty}^{-1}a^n z^{-n}$$

$$|z|>|a| \qquad |z|<|a|$$

There is no overlap between these two regions. Hence, its z-transform does not exist

3.1 General Form of ROC

Summary

- In general, if the rational z-transform has N poles with R distinct magnitudes, then it has R+1 possible ROCs
- Thus, there are R+1 distinct sequences with the same z-transform

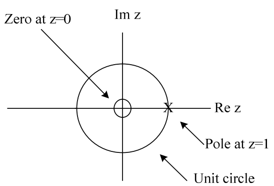
Hence, a rational z-transform with a specified ROC has a unique sequence as its inverse z-transform.

3. 2 ROC of Rational z-Transform

Recall

ZT $\mu(z)$ of $\mu[n]$ $\qquad \mu(z)=\dfrac{1}{1-z^{-1}}$ $\qquad$ for $|z^{-1}|<1$

$1<|z^{-1}|\leq\infty$ Region of convergence



3. 2 ROC of Rational z-Transform

- The ROC of a rational z-transform is bounded by the locations of its poles
- ROC:
  - unique relationship between a sequence and its z-transform.
  - ROC includes the unit circle:
    - ✓ **DTFT exists** $\leftrightarrow$ **BIBO Stable**
  - ROC outside circular:
    - ✓ **Causal**

- To understand the relationship between the poles and the ROC, it is instructive to examine the pole-zero plot of a z-transform

- The pole-zero can be easily determined using MATLAB

  [z,p,k] = `tf2zp`(num,den)

  determines the *zeros, poles,* and the *gain constant* of a rational z-transform with the numerator coefficients specified by the vector **num** and the denominator coefficients specified by the vector **den**.

- **[num,den] = `zp2tf`(z,p,k)** implements the reverse process
- The factored form of the z-transform can be obtained using **sos = `zp2sos`(z,p,k)** where **sos** stands for second-order section
- The above statement computes the coefficients of each second-order factor given as an $L \times 6$ matrix **sos**

$$\mathbf{sos} = \begin{bmatrix} b_{01} & b_{11} & b_{21} & a_{01} & a_{11} & a_{12} \\ b_{02} & b_{12} & b_{22} & a_{02} & a_{12} & a_{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{0L} & b_{1L} & b_{2L} & a_{0L} & a_{1L} & a_{2L} \end{bmatrix}$$

where

$$G(z) = \prod_{k=1}^{L} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{a_{0k} + a_{1k} z^{-1} + a_{2k} z^{-2}}$$

3.3 Determine the ROC by MATLAB

- The pole-zero plot is determined using the function **zplane**
- The z-transform can be either described in terms of its zeros and poles:

  zplane(zeros,poles)

  or, it can be described in terms of its numerator and denominator coefficients:

  zplane(num,den)