Chapter 5

Finite-Length
Discrete Transforms

Part B

Discrete Fourier Transform
Properties

DFT Properties

- DFT Symmetry Relations
- DFT Theorems
- Fourier-Domain Filtering
- Computation of the DFT of Real Sequences
- Linear Convolution Using the DFT

1. DFT Symmetry Relations

- ***Symmetry Relations***

$$X[k] = X_{\text{re}}[k] + jX_{\text{im}}[k]$$

$$X_{\text{re}}[k] = \frac{1}{2}(X[k] + X^*[k]) \qquad X_{\text{im}}[k] = \frac{1}{2j}(X[k] - X^*[k])$$

$$x[n] = x_{\text{re}}[n] + jx_{\text{im}}[n]$$

$$x_{\text{re}}[n] = \frac{1}{2}(x[n] + x^*[n]) \qquad x_{\text{im}}[n] = \frac{1}{2j}(x[n] - x^*[n])$$

1. DFT Symmetry Relations

- From the definition

$$x[n] \xleftrightarrow{\text{DFT}} X[k]$$

$$x[\langle -n\rangle_N] \xleftrightarrow{\text{DFT}} X[\langle -k\rangle_N]$$

$$x^*[n] \xleftrightarrow{\text{DFT}} X^*[\langle -k\rangle_N]$$

$$x^*[\langle -n\rangle_N] \xleftrightarrow{\text{DFT}} X^*[k]$$

1. DFT Symmetry Relations

Symmetry properties of the DFT of a complex sequence

Length-N Sequence	N-point DFT
$x[n] = x_{re}[n] + jx_{im}[n]$	$X[k] = X_{re}[k] + jX_{im}[k]$
$x^*[n]$	$X^*[\langle -k\rangle_N]$
$x^*[\langle -n\rangle_N]$	$X^*[k]$
$x_{re}[n]$	$X_{pcs}[k] = \frac{1}{2}\{X[\langle k\rangle_N] + X^*[\langle -k\rangle_N]\}$
$jx_{im}[n]$	$X_{pca}[k] = \frac{1}{2}\{X[\langle k\rangle_N] - X^*[\langle -k\rangle_N]\}$
$x_{pcs}[n]$	$X_{re}[k]$
$x_{pca}[n]$	$jX_{im}[k]$

1. DFT Symmetry Relations

Symmetry properties of the DFT of a real sequence

Length-N Sequence	N-point DFT
$x[n]$	$X[k] = X_{re}[k] + jX_{im}[k]$
$x_{ev}[n]$	$X_{re}[k]$
$x_{od}[n]$	$jX_{im}[k]$
Symmetry relations	$X[k] = X^*[\langle -k\rangle_N]$ $X_{re}[k] = X_{re}[\langle -k\rangle_N]$ $X_{im}[k] = -X_{im}[\langle -k\rangle_N]$ $\lvert X[k]\rvert = \lvert X[\langle -k\rangle_N]\rvert$ $\arg X[k] = -\arg X[\langle -k\rangle_N]$

2. DFT Theorems

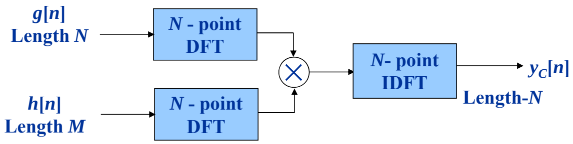
General properties of the DFT

Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
Circular time-shifting	$g[\langle n - n_0\rangle_N]$	$W_N^{kn_0} G[k]$
Circular frequency-shifting	$W_N^{-k_0 n} g[n]$	$G[\langle k - k_0\rangle_N]$
Duality	$G[n]$	$N[g\langle -k\rangle_N]$
N-point circular Convolution	$\sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N}\sum_{m=0}^{N-1} G[m]H[\langle k-m\rangle_N]$
Parseval's relation		$\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \frac{1}{N}\sum_{k=0}^{N-1} \lvert X[k]\rvert^2$

2. DFT Theorems

- Circular Convolution using the DFT

N-point circular Convolution $\sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N]$ $G[k]H[k]$



3. Fourier-Domain Filtering

- A simple approach to design the filter is to set the Fourier transform $H(e^{j\omega})$ to zero in the band containing the components of the signal $x[n]$ that need to be suppressed, and to set $H(e^{j\omega})$ equal to one in the band where the components of the signal $x[n]$ are to be preserved.
- Keep $H(e^{j\omega})$ with zero-phase.

3. Fourier-Domain Filtering

- Fourier-domain filtering using DFT

Example

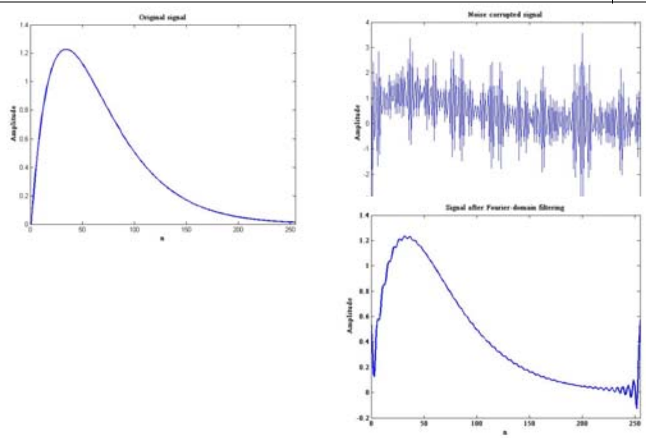
Consider the narrow-band lowpass signal

$$x[n] = 0.1ne^{-0.03n}, \quad 0 \le n \le 255,$$

The signal $x[n]$ is corrupted with a high-frequency random noise. Try to remove it.

- Take the 256-point DFT of $x_N[n]$: $X_N[k]$
- Set all samples in the range $50 \le k \le 206$ to zero values

3. Fourier-Domain Filtering



4. Computation of the DFT of Real Sequences

- *N*-Point DFTs of Two *Real Sequences* Using a Single *N*-Point DFT
- 2*N*-Point DFTs of a *Real Sequence* Using a Single *N*-Point DFT

- Length-*N* **real sequences** and *N*-point DFTs

$$h[n] \longleftrightarrow H[k] \qquad g[n] \longleftrightarrow G[k]$$

  - $2N^2$ multiplication, $2N(N-1)$ additions.
- Define a length-*N* **complex** sequence

$$x[n] = h[n] + jg[n] \longleftrightarrow X[k]$$

  - Hence, $g[n]=\mathrm{Re}\{x[n]\}$ and $h[n]=\mathrm{Im}\{x[n]\}$
  - $X[k]$denote the *N*-point DFT of $x[n]$

- Then, we arrive at

$$H[k] = X_{\mathrm{cs}}[k] = \frac{1}{2}\{X[k] + X^*[\langle -k \rangle_N]\}$$

$$G[k] = X_{\mathrm{ca}}[k] = \frac{1}{2j}\{X[k] - X^*[\langle -k \rangle_N]\}$$

- Note that

$$X^*[\langle -k \rangle_N] = X^*[\langle N-k \rangle_N]$$

Example

- We compute the 4-point DFTs of the two real sequences $g[n]$ and $h[n]$ given below

$$\{g[n]\}=\{\underset{\uparrow}{1} \;\; 2 \;\; 0 \;\; 1\}, \; \{h[n]\}=\{\underset{\uparrow}{2} \;\; 2 \;\; 1 \;\; 1\}$$

- Then $\{x[n]\} = \{g[n]\} + j\{h[n]\}$ is given by

$$\{x[n]\}=\{\underset{\uparrow}{1+j2} \;\; 2+j2 \;\; j \;\; 1+j\}$$

- 4-point DFT of $x[n]$

$$\{X[k]\}=\{4+j6\ \ 2\ \ -2\ \ j2\}$$

- Conjugate sequence

$$\{X^*[k]\}=\{4-j6\ \ 2\ \ -2\ -j2\}$$

- Circular conjugate sequence

$$\{X^*[\langle N-k\rangle_N]\}=\{4-j6\ \ -2j\ \ -2\ \ 2\}$$

- Therefore

$$\{G[k]\}=\{4\ \ 1-j\ \ -2\ 1+j\}$$

$$\{H[k]\}=\{6\ \ 1-j\ \ 0\ \ 1+j\}$$

- **Length-2*N* real sequence** $v[n]$ with an **2*N*-point** DFT $V[k]$

$$v[n] \longleftrightarrow V[k]$$

- Define two **length-*N* real sequences**:

$$g[n]=v[2n],\quad h[n]=v[2n+1]\quad 0\le n\le N-1$$

- Let $G[k]$ and $H[k]$ denote their respective N point DFTs

$$h[n]\longleftrightarrow H[k] \qquad g[n]\longleftrightarrow G[k]$$

$$\begin{aligned}
V[k] &= \sum_{n=0}^{2N-1} v[n]W_{2N}^{nk} \\
&= \sum_{n=0}^{N-1} v[2n]W_{2N}^{2nk} + \sum_{n=0}^{N-1} v[2n+1]W_{2N}^{(2n+1)k} \\
&= \sum_{n=0}^{N-1} g[n]W_N^{nk} + \sum_{n=0}^{N-1} h[n]W_N^{nk}W_{2N}^{k} \\
&= \sum_{n=0}^{N-1} g[n]W_N^{nk} + W_{2N}^{k}\sum_{n=0}^{N-1} h[n]W_N^{nk} \qquad \boxed{0\le k\le 2N-1}
\end{aligned}$$

$$V[k]=\sum_{n=0}^{N-1} g[n]W_N^{nk} + W_{2N}^{k}\sum_{n=0}^{N-1} h[n]W_N^{nk}, \quad 0 \le k \le 2N-1$$

i.e.

$$V[k]=G[\langle k\rangle_N]+W_{2N}^{k}H[\langle k\rangle_N], \quad k=0,1,2,\ldots,2N-1$$

where the DFTs of $G[k]$ and $H[k]$ can be computed by means of the method discussed in 4.1

- **Example**

  Determine the 8-point DFT $V[k]$ of the length-8 real sequence

$$\{v[n]\}=\{\underset{\uparrow}{1}\ \ 2\ \ 2\ \ 2\ \ 0\ \ 1\ \ 1\ \ 1\},$$

  - We form two length-4 real sequences as follows

$\{g[n]\}=\{v[2n]\}=\{\underset{\uparrow}{1}\ 2\ 0\ 1\}$ $\{h[n]\}=\{v[2n+1]\}=\{\underset{\uparrow}{2}\ 2\ 1\ 1\}$

- **Example**
  - Now we have

$$V[k]=G[\langle k\rangle_4]+W_8^{k}H[\langle k\rangle_4], \quad 0\le k\le 7$$

  - Substituting the values of the 4-point DFTs $G[k]$ and $H[k]$

$$G[k]=\{4\ \ 1-j\ \ -2\ \ 1+j\}$$

$$H[k]=\{6\ \ 1-j\ \ 0\ \ 1+j\}$$

$$V[0]=G[0]+H[0]=4+6=10$$
$$V[1]=G[1]+W_8^1H[1]=(1-j)+e^{-j\pi/4}(1-j)=1-j2.4142$$
$$V[2]=G[2]+W_8^2H[2]=-2+e^{-j\pi/2}\cdot 0=-2$$
$$V[3]=G[3]+W_8^3H[3]=(1+j)+e^{-j3\pi/4}\cdot(1+j)=1-j0.4142$$
$$V[4]=G[0]+W_8^4H[0]=4-6=2$$
$$V[5]=G[1]+W_8^5H[1]=(1-j)-e^{-j\pi/4}(1-j)=1+j2.4142$$
$$V[6]=G[2]+W_8^6H[2]=-2-e^{-j\pi/2}\cdot 0=-2$$
$$V[7]=G[3]+W_8^7H[3]=(1+j)-e^{-j3\pi/4}\cdot(1+j)=1-j0.4142$$

5. Linear Convolution Using the DFT

- *Linear convolution* is a key operation in many signal processing applications
- Implementation of linear convolution using the DFT-----which can be efficiently implemented using FFT algorithms.

5. Linear Convolution Using the DFT

- **Linear Convolution** of Two Finite-Length Sequences
- **Linear Convolution** of a Finite-Length Sequence with an Infinite-Length Sequence

5.1 Linear Convolution of Two Finite-Length Sequences

- Let $g[n]$ and $h[n]$ be two finite-length sequences of length N and M, respectively
  - Denote $L=N+M-1$
  - Define two length-L sequences

$$g_e[n]=\begin{cases} g[n], & 0\le n\le N-1 \\ 0, & N\le n\le L-1 \end{cases}$$

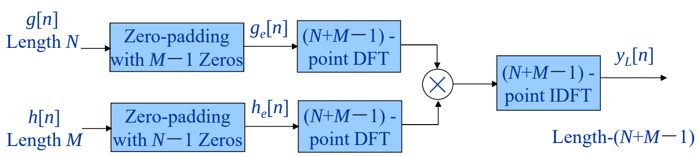
$$h_e[n]=\begin{cases} h[n], & 0\le n\le M-1 \\ 0, & M\le n\le L-1 \end{cases}$$

5.1 Linear Convolution of Two Finite-Length Sequences

- Then

$$y_L[n]=g[n] \circledast h[n]=g[n] \; Ⓝ \; h[n]$$

- The corresponding implementation scheme is illustrated below



*5.2 Linear Convolution of a Finite-Length Sequence with an Infinite-Length Sequence

- Consider the DFT-based implementation of

$$y[n] = \sum_{l=0}^{M-1} h[l]x[n-l] = h[n] \circledast x[n]$$

where ***h*[*n*]** is a finite-length sequence of **length -*M*** and ***x*[*n*]** is an **infinite length** (or a finite length sequence whose length is much greater than M)

*5.2.1 Overlap-Add Method

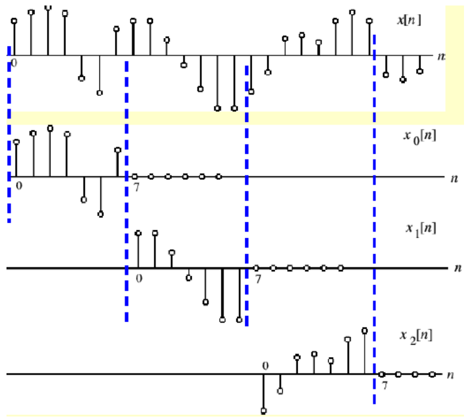
- We first segment $x[n]$, assumed to be a causal sequence here without (any) loss of generality, into a set of contiguous finite-length subsequences of length N each:

$$x[n] = \sum_{m=0}^{\infty} x_m[n-mN]$$

where

$$x_m[n] = \begin{cases} x[n+mN], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

*5.2.1 Overlap-Add Method



*5.2.1 Overlap-Add Method

- Thus we can write

$$y[n] = h[n] \circledast x[n] = \sum_{m=0}^{\infty} y_m[n-mN]$$

where

$$y_m[n] = h[n] \circledast x_m[n]$$

- Since $h[n]$ is of length M and $x_m[n]$ is of length N, the linear convolution $h[n] \circledast x_m[n]$ is of length $N+M-1$

- As a result, the desired linear convolution $y[n]= h[n] * x[n]$ has been broken up into a sum of infinite number of short-length linear convolutions of length $N+M-1$ each:

$$y_m[n]= h[n] * x_m[n]$$

- Each of these short convolutions can be implemented using the DFT-based method discussed earlier, where the DFTs (and the IDFT) are computed on the basis of $(N+M-1)$ points

- There is one more subtlety to take care of before we can implement

$$y[n] = \sum_{m=0}^{\infty} y_m[n-mN]$$

using the DFT-based approach

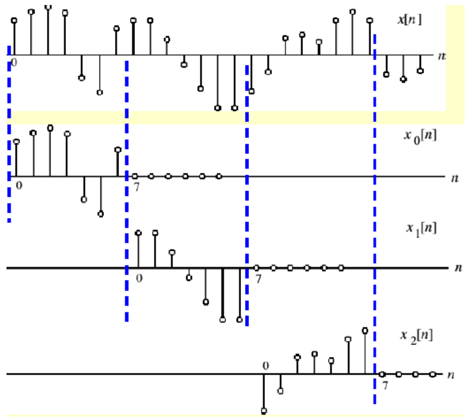
- Now the first short convolution in the above sum, $y_0[n]= h[n] * x_0[n]$ is of length $N+M-1$ and is defined for $0 \le n \le N+M-2$

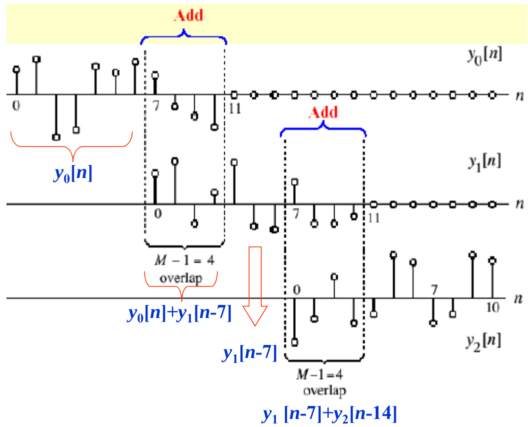
- The second short convolution $y_1[n]= h[n] * x_1[n]$ is also of length $N+M-1$ but is defined for $N \le n \le 2N+M-2$

  ➡ There is an overlap of $M-1$ samples between these two short linear convolutions

- Likewise, the third short convolution $y_2[n]= h[n] * x_2[n]$, is also of length $N+M-1$ but is defined for $2N \le n \le 3N+M-2$

- Thus there is an **overlap** of $M-1$ samples between $h[n] * x_1[n]$ and $h[n] * x_2[n]$
- In general, there will be an overlap of $M-1$ samples between the samples of the short convolutions $h[n] * x_{r-1}[n]$ and $h[n] * x_r[n]$
- This process is illustrated in the figure on the next slide for $M = 5$ and $N = 7$.





- Therefore, $y[n]$ is obtained by a linear convolution of $h[n]$ and $x[n]$ is given by

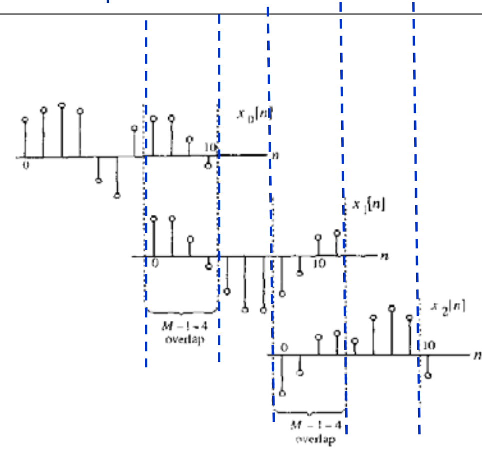
$$
\begin{array}{ll}
y[n] = y_0[n], & 0 \le n \le 6 \\
y[n] = y_0[n] + y_1[n-7] & 7 \le n \le 10 \\
y[n] = y_1[n-7] & 11 \le n \le 13 \\
y[n] = y_1[n-7] + y_2[n-14] & 14 \le n \le 17 \\
y[n] = y_2[n-14] & 18 \le n \le 20 \\
\vdots &
\end{array}
$$

- *Overlap add method*： since the results of the short linear convolutions overlap and the overlapped portions are added to get the correct final result.
- Function **`fftfilt`** can be used to implement the above method.
- Program 5_5 illustrates the use of **`fftfilt`** in the filtering of a noise-corrupted signal using a length-3 *moving average filter*

- Let $h[n]$ be a length-M sequence
- We first segment $x[n]$, into a set of contiguous finite-length subsequences of smaller length N

$$x_m[n] = x[n + m(N - M + 1)] \quad 0 \le n \le N-1$$

with $M \le N$



- Let $w_m[n]= h[n] \circledast x_m[n]$
- We reject the first $M-1$ samples of $w_m(n)$ and abut the remaining $N+M-1$ samples of $w_m(n)$ to form the linear convolution $y_L[n]=h(n) * x_m(n)$
- If $y_m[n]$ denotes the saved portion of $w_m[n]$, i.e.

$$y_m[n] = \begin{cases} 0, & 0 \le n \le M-2 \\ w_m[n], & M-1 \le n \le N-2 \end{cases}$$

- Then

$$y_L[n + m(N - M + 1)] = y_m[n], \quad M-1 \le n \le N-1$$

- Input is segmented into overlapping sections
- Parts of the results of the circular convolutions are saved and abutted to determine the linear convolution.

*5.2.1 Overlap-Save Method