Chapter 5

Finite Length Discrete Transforms

Part A

The Discrete Fourier Transform (DFT)

Finite-Length Discrete Transforms

- It is convenient to map a finite-length sequence from the time domain into a finite-length sequence of the same length in a different domain, and vice-versa.
- Such transformations are usually collectively called **finite-length transforms**.

Discrete Fourier Transform

- *Orthogonal Transforms*
- The Definition of DFT
- Relation between DTFT and DFT and their inverses
- Operations on Finite-Length Sequences
  - **Circular Time-Reversal**
  - **Circular Shifting**
  - **Circular Convolution**
- Classifications of Finite-Length Sequences

1. Orthogonal Transforms

Definition: with **basis sequences** $\psi[k,n]$

$$\frac{1}{N}\sum_{n=0}^{N-1}\psi[k,n]\psi^*[l,n]=\begin{cases}1, & l=k\\ 0, & l\neq k\end{cases}$$

For length-N sequence $x[n]$, $0\leq n\leq N-1$, with $X[k]$ denoting the coefficients of its N-point orthogonal transform :

$$X[k]=\sum_{n=0}^{N-1}x[n]\psi^*[k,n] \qquad 0\leq k\leq N-1$$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X(k)\psi[k,n] \qquad 0\leq n\leq N-1$$

1. Orthogonal Transforms

- **Proof:**
- Important consequence--**Parseval's relation**

$$\sum_{n=0}^{N-1}|x[n]|^2=\frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2$$

- **Transforms with good energy compaction properties:**
  - **most of the signal energy is concentrated in a subset of the transform coefficients**
  - **remaining coefficients with very low energy to be set to zero values**

2.1 Definition

Definition

- DFT $X[k]$ is obtained by **uniformly** sampling the DTFT $X(e^{j\omega})$ over one **principal value interval** $0\leq\omega\leq 2\pi$ at $\omega_k= 2\pi k/N$, $0\leq k\leq N-1$ in the frequency domain.

Sampling the DTFT $X(e^{j\omega})$ of $x[n]$, $0\leq n\leq N-1$

$$X[k]=X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}$$

2.1 Definition

$$X[k]=X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}$$

$$=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi k}{N}n}, \quad 0\leq k\leq N-1$$

- Length-N sequence $X[k]$: discrete Fourier transform (DFT) of the sequence $x[n]$ in the frequency domain

2.1 Definition

- Using the notation $W_N = e^{-j2\pi/N}$ the DFT is usually expressed as:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad 0 \le k \le N-1$$

- Inverse discrete Fourier transform (IDFT)

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn} \qquad 0 \le k \le N-1$$

Proof

2.1 Definition

- $W_N = e^{-j2\pi/N}$: twiddle factor
  - $|W_N| = 1$
  - One of the N N-th roots of unity $W_N^0 = W_N^N = 1$
  - $W_N^{N/2} = -1$

$$W_N^k = W_N^{k+N} \qquad W_N^{k+N/2} = -W_N^k \qquad \sum_{k=0}^{N-1} W_N^k = 0$$

$$\sum_{k=0}^{N-1} W_N^{-(k-l)n} = \begin{cases} N, & \text{for } k-l = rN,\ r \text{ is an interger} \\ 0 & \text{otherwise} \end{cases}$$

2.1 Definition

Example 1

- Consider the length-N sequence

$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & 1 \le n \le N-1 \end{cases}$$

Its N-point DFT is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = x[0] W_N^0 = 1,$$

$$0 \le k \le N-1$$

2.1 Definition

Example 2

- Consider the length-N sequence defined for

$$x[n] = \cos(2\pi rn/N) \qquad 0 \le n \le N-1$$

where r is an integer in the range $0 \le r \le N-1$

- Using the Euler's function we can write

$$x[n] = \frac{1}{2}(e^{j2\pi rn/N} + e^{-j2\pi rn/N})$$

$$= \frac{1}{2}(W_N^{-rn} + W_N^{rn})$$

2.1 Definition

- The N-point DFT of $g[n]$ is thus given by

$$X[k]=\frac{1}{2}\left[\sum_{n=0}^{N-1}W_N^{-(r-k)n}+\sum_{n=0}^{N-1}W_N^{(r+k)n}\right]$$

$$=\begin{cases} N/2, & \text{for } k=r, \\ N/2, & \text{for } k=N-r, \\ 0, & \text{otherwise.}\end{cases}$$

2.1 Definition

Example 3

- Rectangular Pulse $R_N[n]$, width N
  N-point DFT is given by

$$X[k]=\sum_{n=0}^{N-1}x[n]W_N^{kn}=\sum_{n=0}^{N-1}W_N^{kn}=\frac{1-W_N^{kN}}{1-W_N^{k}}$$
$$=\frac{W_N^{kN/2}}{W_N^{k/2}}\frac{W_N^{-kN/2}-W_N^{kN/2}}{W_N^{-k/2}-W_N^{k/2}}$$
$$=\frac{\sin(k\pi)}{\sin(k\pi/N)}e^{-j\frac{N-1}{N}k\pi}$$

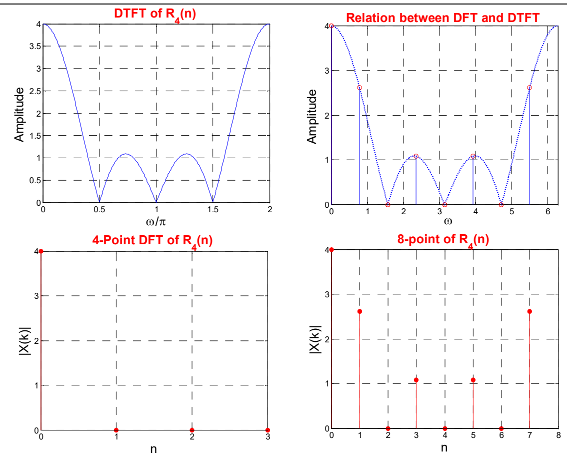
2.1 Definition

- $2N$-point DFT is given by

$$X[k]=\sum_{n=0}^{2N-1}x[n]W_{2N}^{kn}=\sum_{n=0}^{N-1}W_{2N}^{kn}$$
$$=\frac{1-W_{2N}^{kN}}{1-W_{2N}^{k}}=e^{-j\frac{N-1}{2N}k\pi}\frac{\sin(k\pi/2)}{\sin(k\pi/2N)}$$

- **Length of DFT plays a very important role in DFT**

2.1 Definition



2.1 Definition

- Mapping Relations between time-domain and frequency-domain transforms

(Time-domain)		(Frequency-domain)
Continuous	$\Leftrightarrow$	Aperiodical
Discrete	$\Leftrightarrow$	Periodical
Periodical	$\Leftrightarrow$	Discrete
Aperiodical	$\Leftrightarrow$	Continuous

2.1 Definition

- ***Type* 1:** Continuous-Time Fourier Transform (CTFT)

Continuous Aperiodical $x_a(t) \Leftrightarrow X_a(j\Omega)$ Aperiodical Continuous

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$$

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t} d\Omega$$

2.1 Definition

- ***Type* 2:** Continuous-Time Fourier Series (CTFS)

Continuous Periodical $x_a(t) \Leftrightarrow X_a(jk\Omega_0)$ Aperiodical Discrete

$$X_a(jk\Omega_0) = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x_a(t) e^{-jk\Omega_0 t} dt$$

$$x_a(t) = \sum_{k=-\infty}^{\infty} X_a(jk\Omega_0) e^{jk\Omega_0 t}$$

2.1 Definition

- ***Type* 3:** Discrete-Time Fourier Transform (DTFT)

Discrete Aperiodical $x[n] \Leftrightarrow X(e^{j\omega})$ Periodical Continuous

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

2.1 Definition

- ***Type* 4:** Discrete Fourier Transform (DFT)

Discrete Periodical $\quad x[n] \Longleftrightarrow X[k] \quad$ Periodical Discrete

$$X[k]=\sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad 0 \le k \le N-1$$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}, \quad 0 \le n \le N-1$$

2.1 Definition

- The computation of the DFT and the IDFT requires, respectively, approximately N^2 complex multiplications and $N(N-1)$ complex additions.
- However, elegant methods have been developed to reduce the computational complexity to about $N(\log_2 N)$ operations.
- These techniques are usually called fast Fourier transform (FFT) algorithms .

2.2 Matrix Relations

- Since MATLAB stands for *MAtrix LABoratory*, we represent DFT definition in terms of matrix form

$$X[k]=\sum_{n=0}^{N-1} x[n]W_N^{kn}, \quad 0 \le k \le N-1$$

can be expressed in matrix form as

$$\mathbf{X}=\mathbf{D}_N\mathbf{x}$$

2.2 Matrix Relations

- Where

$$\mathbf{X}=\begin{bmatrix} X[0] & X[1] & \cdots & X[N-1] \end{bmatrix}^T$$

$$\mathbf{x}=\begin{bmatrix} x[0] & x[1] & \cdots & x[N-1] \end{bmatrix}^T$$

And $\mathbf{D}_N$ is the $N \times N$ DFT matrix given by

$$\mathbf{D}_N=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}_{N\times N}$$

2.2 Matrix Relations

- Likewise, the IDFT relations can be expressed in

$$x[n] = \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad 0 \le n \le N-1$$

can be expressed in matrix form as

$$\mathbf{x} = \mathbf{D}_N^{-1} \mathbf{X}$$

Where $\mathbf{D}_N^{-1}$ is the $N \times N$ IDFT matrix

2.2 Matrix Relations

- where

$$\mathbf{D}_N^{-1} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)(N-1)} \end{bmatrix}_{N \times N}$$

- Note: $D_N^{-1} = \frac{1}{N} D_N^*$

2.2 Matrix Relations

- Obviously, the relation between the two coefficient matrices can be expressed as follows

$$\mathbf{D}_N^{-1} = \frac{1}{N} \mathbf{D}_N^*$$

- Therefore, the algorithms designed for DFT are applicable to IDFT

2.3 DFT Computation Using MATLAB

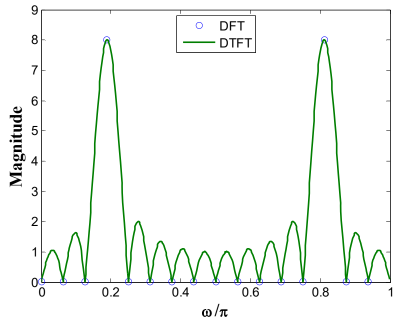
- **Built-in** Functions to compute the DFT and the IDFT are `fft` and `ifft`

```
fft(x) ifft(X)
fft(x,M) ifft(x,M)
```

- These functions make use of FFT algorithms which are computationally highly efficient compared to the direct computation

2.3 DFT Computation Using MATLAB

- Sequence $\cos(6\pi n/16) \quad 0 \le n \le 15$



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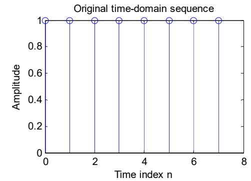
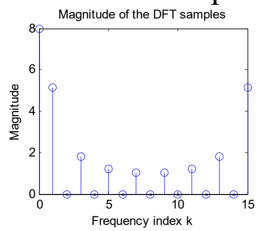
2.3 DFT Computation Using MATLAB

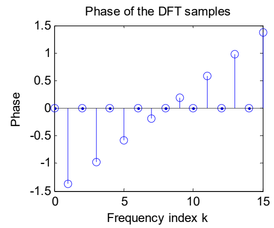
- N-point sequence $\mu[n]$

$$u[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$



Determine the M-point DFT.





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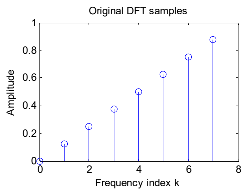
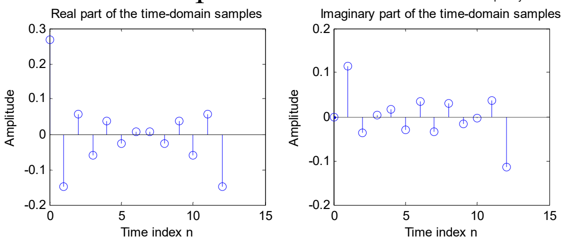
2.3 DFT Computation Using MATLAB

- N-point sequence $\mu[n]$

$$V[k] = \begin{cases} k/K, & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$



Determine the M-point DFT.



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3. Relations between DTFT and DFT and their inverses

- **Relations: (for finite $x[n]$ of length N)**

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$$

$X[k]$ is obtained by uniformly sampling on the ω-axis between

$$X[k] = X(e^{j\omega})\Big|_{\omega=2\pi k/N} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \quad 0 \le k \le N-1$$

$$X[k] \underset{\text{sampling}}{\overset{?}{\rightleftarrows}} X(e^{j\omega})$$

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3.1 Numerical Computation of the DTFT Using the DFT

- A practical approach to the numerical computation of the DTFT of a finite-length sequence.
- Let $X(e^{j\omega})$ be the DTFT of a length-N sequence $x[n]$. We wish to evaluate $X(e^{j\omega})$ at a dense grid of frequencies , where ***M* >> *N***:

$$\omega_k = 2\pi k / M, \quad 0 \le k \le M-1$$

3.1 Numerical Computation of the DTFT Using the DFT

$$X(e^{j\omega_k}) = \sum_{n=0}^{M-1} x[n] e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/M}$$

- Define a new sequence

$$X_e[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & N \le n \le M-1 \end{cases}$$

- Then

$$X_e(e^{j\omega_k}) = \sum_{n=0}^{M-1} x_e[n] e^{-j2\pi kn/M}$$

3.1 Numerical Computation of the DTFT Using the DFT

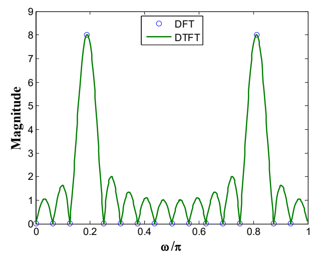
- Thus $X_e(e^{j\omega_k})$ is essentially an M-point DFT $X_e[k]$ of the length-M sequence $x_e[n]$
- The DFT $X_e[k]$ can be computed very efficiently using the FFT algorithm if M is an integer power of 2.

3.1 Numerical Computation of the DTFT Using the DFT

Example

- Compute the DFT and the DTFT of the sequence, as shown below

$$\cos(6\pi n/16) \quad 0 \le n \le 15$$



3.1 Numerical Computation of the DTFT using DFT

- *The function* **`freqz`** *employs this approach to evaluate the frequency response at a prescribed set of frequencies of a DTFT expressed as a rational function in*

3.2 DTFT from DFT by interpolation

$$X[k] \xrightarrow{?} X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} = \frac{1}{N}\sum_{n=0}^{N-1}\Bigg[\underbrace{\sum_{k=0}^{N-1} X[k]W_N^{-kn}}_{\text{IDFT}\,=\,x[n]}\Bigg]e^{-j\omega n}$$

$$= \frac{1}{N}\underbrace{\sum_{k=0}^{N-1} X[k]\sum_{n=0}^{N-1}}_{\text{Exchange of the order of summations}} e^{-j[\omega-(2\pi k/N)]n}$$

IDFT

$x[n]$

Exchange of the order of summations

3.2 DTFT from DFT by interpolation

- Let $S = \sum_{n=0}^{N-1} e^{-j[\omega-(2\pi k/N)]n}$ and $r = e^{-j[\omega-(2\pi k/N)]}$
- Thus

$$S = \sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r} = \frac{1-e^{-j(\omega N-2\pi k)}}{1-e^{-j[\omega-(2\pi k/N)]}}$$

$$= \frac{\sin\left(\dfrac{\omega N-2\pi k}{2}\right)}{\sin\left(\dfrac{\omega N-2\pi k}{2N}\right)} \cdot e^{-j[\omega-(2\pi k/N)][(N-1)/2]}$$

3.2 DTFT from DFT by interpolation

$$X(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\frac{\sin\left(\dfrac{\omega N-2\pi k}{2}\right)}{\sin\left(\dfrac{\omega N-2\pi k}{2N}\right)} \cdot e^{-j[\omega-(2\pi k/N)][(N-1)/2]}$$

$$X(e^{j\omega}) = \sum_{k=0}^{N-1} X[k]\Phi(\omega - \frac{2\pi k}{N})$$ interpolation formula

$$\Phi(\omega) = \frac{\sin\left(\dfrac{\omega N}{2}\right)}{N\sin\left(\dfrac{\omega}{2}\right)} \cdot e^{-j\omega[(N-1)/2]}$$

3.2 DTFT from DFT by interpolation

- DTFT can be **possibly** determined by the following *interpolation formula*

$$X(e^{j\omega}) = \sum_{k=0}^{N-1} X[k]\Phi(\omega - \frac{2\pi k}{N})$$

$$X(k) \xrightleftharpoons[\text{sampling}]{\text{interpolation}} X(e^{j\omega})$$

$$X(e^{j\omega})\Big|_{\omega=2\pi\ell/N} = X[\ell]$$

3.3 Sampling the DTFT

- Sequence $x[n]$, $0 \le k \le N-1$ with a DTFT $X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega\ell}$$

  - Uniformly sample $X(e^{j\omega})$ at N equally spaced points $\omega_k = 2\pi\, k/N,\quad 0 \le k \le N-1$ developing the N frequency samples $\{X(e^{j\omega_k})\}$
  - Let $Y[k] = X(e^{j\omega_k}),\ \ 0 \le k \le N-1$

$$Y[k] = X(e^{j\omega_k})\,|_{\omega_k = 2\pi k/N} = \sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell},\quad 0 \le k \le N-1$$

  - IDFT of $Y[k]$ $\quad y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k] W_N^{-kn},\quad 0 \le n \le N-1$

3.3 Sampling the DTFT

- i.e. $\quad y[n] = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell} W_N^{-kn}$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell]\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-\ell)}\right]$$

- Making use of the identity

$$\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-\ell)} = \begin{cases} 1, & \text{for } \ell = n + mN \\ 0, & \text{otherwise} \end{cases}$$

3.3 Sampling the DTFT

We arrive at the desired relation

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN],\quad 0 \le n \le N-1$$

- Thus $y[n]$ is obtained from $x[n]$ by adding an infinite number of ***shifted replicas*** of $x[n]$, with each replica shifted by an integer multiple of N sampling instants, and observing the sum only for the interval $0 \le n \le N-1$

3.3 Sampling the DTFT

- For finite length-M sequences $x(n)$

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \quad 0 \le n \le N-1$$

assume that the samples outside the specified range are zeros.

  - If $M \le N$, then $y[n]=x[n]$ for $0 \le n \le N-1$
  - If $M > N$, there is a time-domain aliasing of samples of $x[n]$ in generating $y[n]$, and $x[n]$ cannot be recovered from $y[n]$

Sampling Theorem in Frequency-Domain

3.3 Sampling the DTFT

- Example Let $x[n]=\{\underset{\uparrow}{0}\ 1\ 2\ 3\ 4\ 5\}$

Sampling 4 point at its DTFT.

Can we recover $x[n]$ from the sampling?

3.3 Sampling the DTFT

- By sampling its DTFT $X(e^{j\omega})$ at $\omega_k=2\pi\, k/4$, $0 \le k \le 3$, and then applying a 4-point IDFT to these samples, we arrive at the sequence $y[n]$ given by

$$y[n]= x[n] + x[n+4] + x[n-4]\ ,\ 0 \le k \le 3$$

i.e. $y[n]= \{\underset{\uparrow}{4}\ \ 6\ \ 2\ \ 3\}$

⇒ $\{x[n]\}$ cannot be recovered from $\{y[n]\}$

4. Operations on Finite-length Sequences

- Let $x[n]$ be a sequence of length N defined for $0 \le n \le N-1$, the time-reversal and time-shift of the sequence is no longer defined in $0 \le n \le N-1$.
- We thus need to define another type of operations that will keep the reversed and shifted sequences in the range $0 \le n \le N-1$.
- Similarly, another type of convolution needs to be defined that ensure the convoluted sequence is in the range $0 \le n \le N-1$.

4.1 Circular Time-Reversal Operation

- The time-reversal operation on a finite-length sequence that develops a sequence also defined for the same range of the time index *n*, is obtained by using the modulo operation.
- Let 0, 1,…, *N*-1 be a set of *N* positive integers, and let *m* be any integer. The integer *r* obtained by evaluating *m* modulo *N* is called the **residue** and is an integer with a value between 0 and *N*-1.

$$r = \langle m \rangle_N = m \text{ modulo } N \qquad r = m + \ell N$$

4.1 Circular Time-Reversal Operation

- Thus, the time-reversal version $\{y[n]\}$ of the length-*N* sequence $\{x[n]\}$ defined for $0 \le n \le N-1$ is given by

$$\{y[n]\} = x[\langle -n \rangle_N], \quad 0 \le n \le N-1$$
$$= x[\langle -n + \ell N \rangle_N] R_N[n]$$
$$= \begin{cases} x[n], & n = 0, \\ x[N-n], & \text{otherwise.} \end{cases}$$

4.2 Circular Time-Shifting Operation

- The time-shifting operation on a finite-length sequence that results in another sequence of the same length and defined for the same range of value of *n*, is referred to as the circular time-shifting operation.
- Such a shifting operation is achieved by using the **modulo operation**.

4.2 Circular Time-Shifting Operation

- The circular time-shifting operation of a length-*N* sequence $x[n]$ by an arbitrary amount n_0 sample period is defined by the equation

$$x_c[n] = x\left[\langle n - n_0 \rangle_N\right]$$

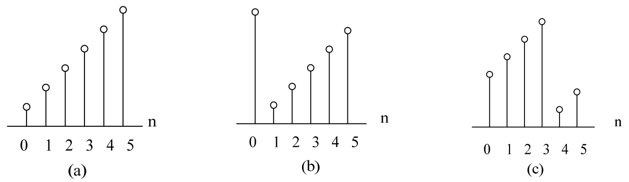
where $x_c[n]$ is also a length-*N* sequence.

- If $n_0 > 0$ (right circular shift)

$$x_c[n] = \begin{cases} x[n-n_0], & \text{for } n_0 \le n \le N-1, \\ x[N-n_0+n], & \text{for } 0 \le n < n_0. \end{cases}$$

4.2 Circular Time-Shifting Operation

- Given a length-6 sequence $x[n]$, its circularly shifted versions are shown



$x[n]$ $\quad x\left[\langle n-1\rangle_6\right]=x\left[\langle n+5\rangle_6\right]$ $\quad x\left[\langle n-4\rangle_6\right]=x\left[\langle n+2\rangle_6\right]$

4.2 Circular Time-Shifting Operation

- As can be seen from the figures, a right circular shift by n_0 is equivalent to a left circular shift by N-n_0 sample periods.

- A circular shift by an integer number n_0 greater than N is equivalent to a circular shift by $\langle n_0\rangle_N$.

4.2 Circular Time-Shifting Operation

- In the **frequency domain**, the circular shifting operation by k_0 samples on the length-N DFT sequence $X[k]$ is defined by

$$X_c[k]=X\left[\langle k-k_0\rangle_N\right]$$

where $X_c[k]$ is also a length-N DFT.

4.2 Circular Time-Shifting Operation

Steps to get a circular shift of an M-point sequence $x[n]$

- **Periodize**

$$y[n]=x[\langle n\rangle_N]$$

- **Time-shifting**

$$y_1[n]=y[n-n_0]=x\left[\langle n-n_0\rangle_N\right]$$

- **Principal value**

$$x_C[n]=y_1[n]\cdot R_N[n]$$

4.2 Circular Time-Shifting Operation

- DFT of the circular shift sequence

$$y[n] = x[\langle n+m\rangle_N]R_N[\langle n+m\rangle_N]$$

$$Y[k] = \mathrm{DFT}\left[y[n]\right]$$

$$= \sum_{n=0}^{N-1} x[\langle n+m\rangle_N]R_N[n]W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x[\langle n+m\rangle_N]W_N^{kn}$$

4.2 Circular Time-Shifting Operation

$$Y[k] = \sum_{n'=m}^{N-1+m} x[\langle n'\rangle_N]W_N^{k(n'-m)}$$

$$= W_N^{-km}\sum_{n'=m}^{N-1+m} x[\langle n'\rangle_N]W_N^{kn'}$$

$$= W_N^{-km}\left(\sum_{n'=0}^{N-1}(.) - \sum_{n'=0}^{m-1}(.) + \sum_{n'=N}^{N-1+m}(.)\right)$$

$$= W_N^{-km}\sum_{n'=0}^{N-1} x[\langle n'\rangle_N]W_N^{kn'}$$

$$= W_N^{-km}\sum_{n'=0}^{N-1} x[n']W_N^{kn'} = W_N^{-km}X[k]$$

4.3 Circular Convolution

- ***Analogous*** to linear convolution, but with a ***subtle difference***
- **Comparison** of linear convolution with circular convolution
  - Consider two length-N sequences, $g[n]$ and $h[n]$ respectively. Their linear convolution results in a length-($2N$-1) sequence $y_L[n]$ given by

$$y_L[n] = \sum_{m=0}^{N-1} g[m]h[n-m], \quad 0 \le n \le 2N-2$$

4.3 Circular Convolution

	linear convolution	circular convolution
Length of convolution	$2N-1$	*to be specified*
Convolution Formulas	$y_L(n) = \sum_{m=0}^{N-1} g(m)h(n-m)$	$y_C(n) = \sum_{m=0}^{N-1} g(m)h(\langle n-m\rangle_N)$
Convolution Signs	$\circledast$ or $*$	$\textcircled{N}$
Condition of equivalence	?	

4.3 Circular Convolution

- To develop a convolution-like operation resulting in a length-N sequence $y_C[n]$, we need to utilize a circular time-reversal, and then apply a circular time-shift.
- Resulting operation, called a ***circular convolution***, is defined by

$$y_C[n] = \sum_{m=0}^{N-1} g[m]h[\langle n-m \rangle_N], \quad 0 \le n \le N-1$$

4.3 Circular Convolution

- Since the operation defined involves two length-N sequences, it is often referred to as an N-point circular convolution, denoted as

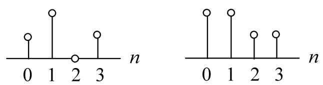
$$y_C[n] = g[n] \, Ⓝ \, h[n]$$

- The circular convolution is commutative, i.e.

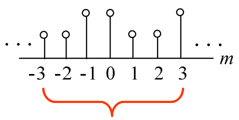
$$g[n] \, Ⓝ \, h[n] = h[n] \, Ⓝ \, g[n]$$

4.3 Circular Convolution

Example 1 *Length of Circular Convolution is* 4

$g[n]$ ➡  ⬅ $h[n]$

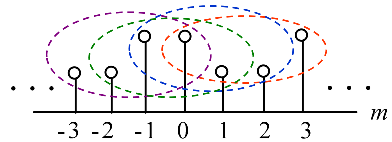
Step 1: Perform *Circular time-reversal operation* on $h[m]$ (or $g[m]$)

$h[\langle -m \rangle_4]$ ➡ 

These seven samples are enough to calculate the circular convolution because of the periodicity of DFT

4.3 Circular Convolution

Step 2: Perform *Circular time-shift operation*



Red	$h[\langle -m \rangle_4]R_4[m]$	**{2 1 1 2}**
Blue	$h[\langle 1-m \rangle_4]R_4(m)$	**{2 2 1 1}**
Green	$h[\langle 2-m \rangle_4]R_4(m)$	**{1 2 2 1}**
Purple	$h[\langle 3-m \rangle_4]R_4(m)$	**{1 1 2 2}**

4.3 Circular Convolution

Step 3: Perform *multiplication and summation of sequences* over the region of $0 \le m \le 3$ for $n=0, n=1, n=2$ and $n=3$ respectively

```
        1 2 0 1               1 2 0 1
        2 1 1 2               2 2 1 1
y(0)=  2+2+0+2= 6     y(1)=  2+4+0+1= 7

        1 2 0 1               1 2 0 1
        1 2 2 1               1 1 2 2
y(2)=  1+4+0+1= 6     y(3)=  1+2+0+2= 5
```


4.3 Circular Convolution

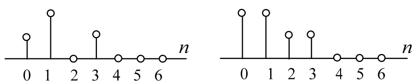
Example 2 *Length of Circular Convolution is 7*

- **In order to develop the 7-point circular convolution on the sequences in the former example, we extended these two sequences to length 7 by appending each with 3 zero-valued samples, i.e.**

$$g_e[n] = \begin{cases} g[n], & 0 \le n \le 3 \\ 0, & 4 \le n \le 6 \end{cases}$$

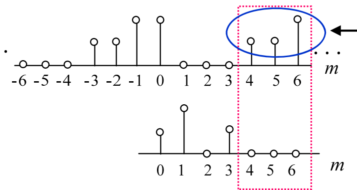
$$h_e[n] = \begin{cases} h[n], & 0 \le n \le 3 \\ 0, & 4 \le n \le 6 \end{cases}$$

4.3 Circular Convolution

$g_e[n]$ ➡  ⬅ $h_e[n]$

Perform *Circular time-reversal operation* on $h_e[m]$

$h_e[\langle -m \rangle_7]$ ➡
$g_e[m]$ ➡



These three samples are not involved in the circular convolution operation

4.3 Circular Convolution

- In this case, the procedure of circular convolution is equivalent to that of linear convolution over the region of principle value.
- Obviously, this conclusion always holds when the length of Circular Convolution is not less than 7

Summary

Provided that the length of Circular Convolution is not less than $N+M-1$ where N and M are the lengths of the two sequences involved, the procedure of circular convolution is equivalent to that of linear convolution

5.1 Classification Based on Conjugate Symmetry

- **Based on Conjugate Symmetry**

It has been discussed in Ch.2 of 4th edition.

- **Circular Conjugate Symmetry**

A length-N circular conjugate-symmetric sequence $x[n]$

$$x[n] = x^*[\langle -n\rangle_N] = x^*[N-n], \qquad 0 \le n \le N-1$$

A length-N circular conjugate-antisymmetric sequence

$$x[n] = -x^*[\langle -n\rangle_N] = -x^*[N-n], \qquad 0 \le n \le N-1$$

5.1 Classification Based on Conjugate Symmetry

A length-N sequence $x[n]$ can be expressed as

$$x[n] = x_{pcs}[n] + x_{pca}[n] \qquad 0 \le n \le N-1$$

where

circular (periodic) conjugate-symmetric part

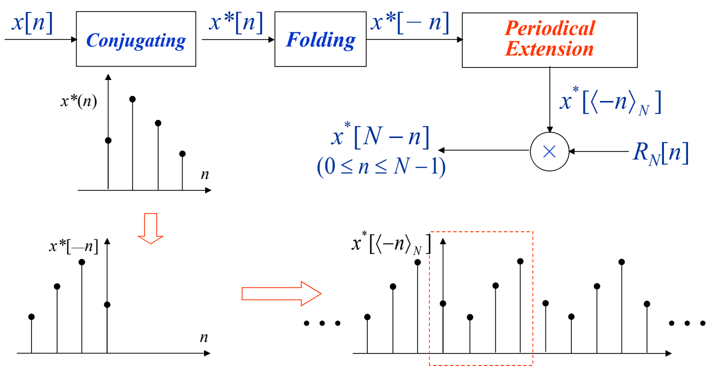
$$x_{pcs}[n] = \frac{1}{2}\left(x[n] + x^*[\langle -n\rangle_N]\right), \qquad 0 \le n \le N-1$$

circular (periodic) conjugate-antisymmetric part

$$x_{pca}[n] = \frac{1}{2}\left(x[n] - x^*[\langle -n\rangle_N]\right), \qquad 0 \le n \le N-1$$

$$X[k] = X_{pcs}[k] + X_{pca}[k], \qquad 0 \le k \le N-1$$

5.1 Classification Based on Conjugate Symmetry



5.1 Classification Based on Conjugate Symmetry

Example

- Consider the length-4 sequence defined for

$$\{u[n]\} = \{1+j4,\ -2+j3,\ 4-j2,\ -5-j6\} \qquad 0 \le n \le 3$$

Conjugate sequence

$$\{u^*[n]\} = \{1-j4,\ -2-j3,\ 4+j2,\ -5+j6\}$$

Circular conjugate sequence

$$\{u^*[\langle -n\rangle_4]\} = \{1-j4,\ -5+j6,\ 4+j2,\ -2-j3\}$$

5.1 Classification Based on Conjugate Symmetry

Conjugate-symmetric part

$$\{u_{PCS}[n]\} = \frac{1}{2}\{u[n] + u^*[\langle -n\rangle_4]\}$$
$$= \{1,\ -3.5 + j4.5,\ 4,\ -3.5 - j4.5\}$$

Circular conjugate-antisymmetric part

$$\{u_{pca}[n]\} = \frac{1}{2}\{u[n] - u^*[\langle -n\rangle_4]\}$$
$$= \{j4,\ 1.5 - j1.5,\ -j2,\ -1.5 - j1.5\}$$

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5.2 Classification Based on Geometric Symmetry

- **Based on Geometric Symmetry**

A length-N ***symmetry sequence*** $x[n]$ satisfies the condition

$$x[n] = x[N-1-n]$$

A length-N ***antisymmetry sequence*** $x[n]$ satisfies the condition

$$x[n] = -x[N-1-n]$$

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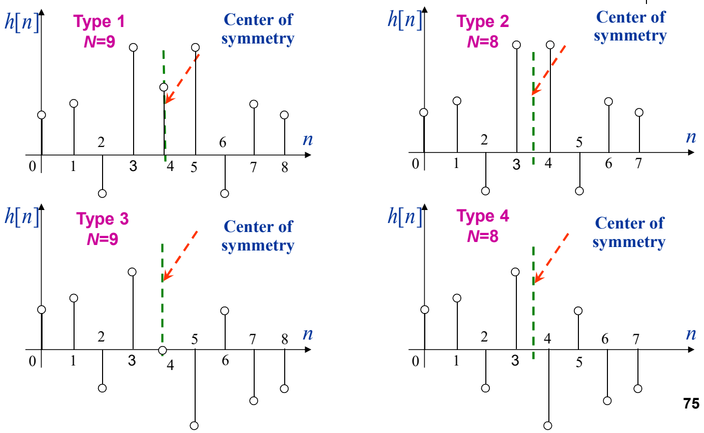
5.2 Classification Based on Geometric Symmetry



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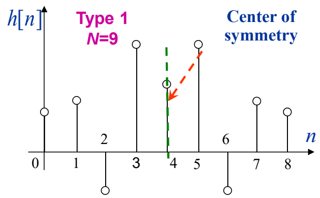
5.2 Classification Based on Geometric Symmetry



Symmetric Sequence with Odd Length

$$\theta(\omega) = -\left(\frac{N-1}{2}\right)\omega + \beta, \quad \beta\text{=0 or } \pi$$

$$X(e^{j\omega}) = e^{-j(N-1)\omega/2}\left\{x\left[\frac{N-1}{2}\right] + 2\sum_{n=1}^{\frac{N-1}{2}} x\left[\frac{N-1}{2} - n\right]\cos(\omega n)\right\}$$

$$X[k] = e^{-j(N-1)\pi k/N}\left\{x\left[\frac{N-1}{2}\right] + 2\sum_{n=1}^{(N-1)/2} x\left[\frac{N-1}{2} - n\right]\cos\left(\frac{2\pi kn}{N}\right)\right\}$$

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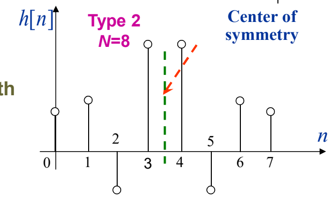
5.2 Classification Based on Geometric Symmetry

Symmetric Sequence with Even Length



$$\theta(\omega) = -\left(\frac{N-1}{2}\right)\omega + \beta, \quad \beta = 0 \text{ or } \pi$$

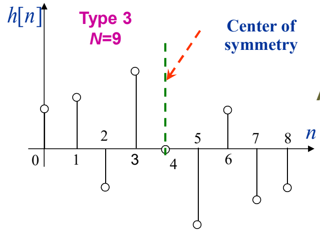
$$X\left(e^{j\omega}\right) = e^{-j(N-1)\omega/2}\left\{2\sum_{n=1}^{N/2} x\left[\frac{N}{2}-n\right]\cos\left(\omega\left(n-\frac{1}{2}\right)\right)\right\}$$

$$X[k] = e^{-j(N-1)\pi k/N}\left\{2\sum_{n=1}^{N/2} x\left[\frac{N}{2}-n\right]\cos\left(\frac{\pi k(2n-1)}{N}\right)\right\}$$

5.2 Classification Based on Geometric Symmetry

$$X\left(e^{j\omega}\right) = je^{-j(N-1)\omega/2}\left\{2\sum_{n=1}^{(N-1)/2} x\left[\frac{N-1}{2}-n\right]\sin(\omega n)\right\}$$

$$X[k] = je^{-j(N-1)\pi k/N}\left\{2\sum_{n=1}^{(N-1)/2} x\left[\frac{N-1}{2}-n\right]\sin\left(\frac{2\pi kn}{N}\right)\right\}$$



Antisymmetric Sequence with Odd Length

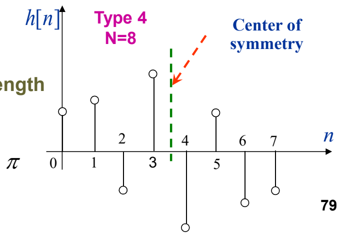
$$\theta(\omega) = -\left(\frac{N-1}{2}\right)\omega + \frac{\pi}{2} + \beta, \quad \beta = 0 \text{ or } \pi$$

5.2 Classification Based on Geometric Symmetry

$$X\left(e^{j\omega}\right) = je^{-j(N-1)\omega/2}\left\{2\sum_{n=1}^{N/2} x\left[\frac{N}{2}-n\right]\sin\left(\omega\left(n-\frac{1}{2}\right)\right)\right\}$$

$$X[k] = je^{-j(N-1)\pi k/N}\left\{2\sum_{n=1}^{N/2} x\left[\frac{N}{2}-n\right]\sin\left(\frac{\pi k(2n-1)}{N}\right)\right\}$$

Antisymmetric Sequence with Even Length



$$\theta(\omega) = -\left(\frac{N-1}{2}\right)\omega + \frac{\pi}{2} + \beta, \quad \beta = 0 \text{ or } \pi$$