Chapter 4

Digital Signal Processing of Continuous-Time Signals

Chapter 4

Two major topics of this chapter:

- Time-Domain Sampling
  - **Sampling of Continuous-Time Signals**
    - **Sampling**
    - **Effect of Sampling in the Frequency Domain**
    - **Recovery of the Analog Signal**
    - **Implementation of the Sampling Process**
  - **Sampling of Bandpass Signals**
- Analog Filter Design
  - **Analog Lowpass Filter Specifications**
  - **Butterworth Approximation**
  - **Design of Other Types of Analog Filters**

Chapter 4A

Time-Domain Sampling

Part A: Time-Domain Sampling

Necessity

- Most signals in the real world are continuous in time, such as speech, music, and images.
- For processing these continuous-time signals by digital systems, we need the analog-to-digital and digital-to-analog interface circuits to convert the continuous-time signals into discrete-time digital form, and vice versa.

Part A: Time-Domain Sampling

Necessity

- It is necessary to develop the relations between the continuous-time signal and its discrete-time equivalent in the time-domain and also in the frequency-domain.
- The latter relations are important in determining conditions under which the discrete-time processing of continuous-time signals can be done free of error under ideal situations.

1 Introduction

- Digital processing of a continuous-time signal involves the following basic steps:

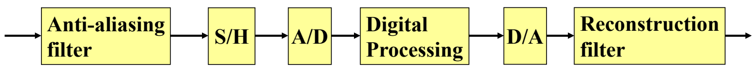
  (1) Conversion of the continuous-time signal into a discrete-time signal,

  (2) Processing of the discrete-time signal,

  (3) Conversion of the processed discrete-time signal back into a continuous-time signal

1 Introduction

- Complete block-diagram is shown below



1 Introduction

- Conversion of a continuous-time signal into digital form is carried out by an analog-to-digital (A/D) converter
- The reverse operation of converting a digital signal into a continuous-time signal is performed by a digital-to-analog (D/A) converter

1 Introduction

- Since the A/D conversion takes a finite amount of time, a sample-and-hold (S/H) circuit is used to ensure that the analog signal at the input of the A/D converter remains constant in amplitude until the conversion is complete to minimize the error in its representation
  - S/H circuit often consists of a *capacitor* to store the analogue voltage, and an electronic *switch* or *gate* to alternately connect and disconnect the capacitor from the analogue input.

1 Introduction

- The continuous-time signal to be processed usually has a larger bandwidth than the bandwidth of the available discrete-time processors.
- To prevent aliasing, an analog anti-aliasing filter is employed before the S/H circuit.

1 Introduction

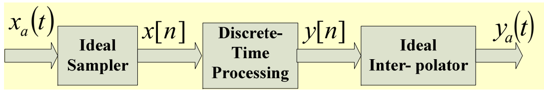
- To smooth the output signal of the D/A converter, which has a staircase-like waveform, an analog reconstruction filter is used.

1 Introduction

- Both the anti-aliasing filter and the reconstruction filter are **analog lowpass filters**, we will go throw the theory behind the design of such filters in following lecture
- Also, the most widely used IIR digital filter design method is based on the conversion of an **analog lowpass prototype**

1 Introduction

- The simplified block-diagram is shown below



1 Introduction

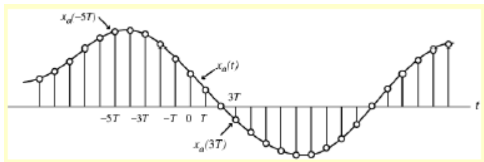
- Ideal sampler: the S/H circuit in cascade with an infinite precision A/D converter has been replaced with the ideal continuous-time to discrete-time (CT-DT) converter which develops a discrete-time equivalent $x[n]$ of the continuous-time signal $x_a(t)$

1 Introduction

- Ideal interpolator: the infinite-precision D/A converter in cascade with the ideal reconstruction filter has been replaced with the ideal discrete-time to continuous-time (DT-CT) converter, which develops a continuous-time equivalent $y_a(t)$ of the processed discrete-time signal $y[n]$.

2 Sampling of Continuous-Time Signals

- Often, a discrete-time sequence $g[n]$ is developed by uniformly sampling a continuous-time signal $g_a(t)$ as indicated below



$$g[n] = g_a(t)\big|_{t=nT} = g_a(nT) \qquad n = \cdots, -2, -1, 0, 1, 2, \cdots$$

2.1 Sampling Process

- Let $g_a(t)$ be a continuous-time signal that is sampled uniformly at $t = nT$, generating the sequence $g[n]$ where $g[n]= g_a(nT)$ with T being the sampling period $F_T = 1/T$

$$g[n] = g_a(t)|_{t=nT} = g_a(nT) \qquad n = \cdots, -2, -1, 0, 1, 2, \cdots$$

- The reciprocal of T is called the sampling frequency F_T, i.e., $\Omega_T = 2\pi F_T$ denoting the sampling angular frequency.

$$t_n = nT = n / F_T = 2\pi n / \Omega_T$$

2.1 Sampling Processing

- Consider the continuous-time signal

$$g_a(t) = A\cos(2\pi f_o t + \phi) = A\cos(\Omega_o t + \phi)$$

- The corresponding discrete-time signal is

$$g[n] = A\cos(\Omega_o nT + \phi)$$
$$= A\cos(\frac{2\pi\Omega_o}{\Omega_T} n + \phi) = A\cos(\omega_o n + \phi)$$

  - $\omega_o = \frac{2\pi\Omega_o}{\Omega_T} = \Omega_o T$: **normalized digital angular frequency**

2.1 Sampling Processing

- If the unit of sampling period T is in seconds
  - The unit of normalized digital angular frequency ω_o is **radians**.
  - The unit of normalized analog angular frequency Ω_o is **radians/second**.
  - The unit of sampling frequency f_T is **hertz (Hz)**.
  - The unit of analog frequency f_o is **hertz (Hz)**.

$$\omega_0 = \frac{2\pi\Omega_0}{\Omega_T} = \Omega_0 T$$

2.1 Sampling Processing

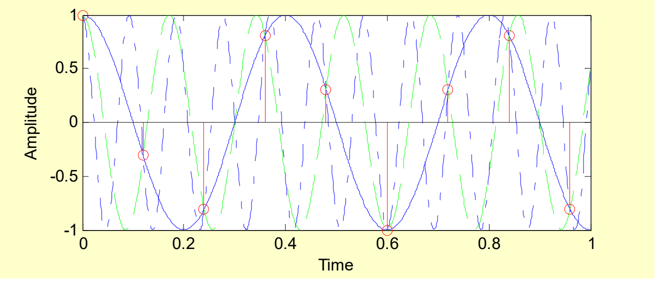
Recall

- Consider three continuous-time signals of frequencies 3 Hz, 7 Hz, and 13 Hz, are sampled at a sampling rate of 10 Hz, i.e. with $T = 0.1$ sec, generating the three sequences

$$g_1(t) = \cos(6\pi t) \Rightarrow g_1[n] = \cos(0.6\pi n)$$
$$g_2(t) = \cos(14\pi t) \Rightarrow g_2[n] = \cos(1.4\pi n)$$
$$g_3(t) = \cos(26\pi t) \Rightarrow g_3[n] = \cos(2.6\pi n)$$

Recall

- Plots of these sequences and their parent time functions



- It is obvious that **identical discrete time** signals may result from the sampling of more than one ***distinct continuous-time function***
- In fact, there exists an ***infinite number of continuous-time signals***, which when sampled lead to the **same discrete-time signal**

Example :

- This fact can also be verified by observing that

$$g_2[n] = \cos(1.4\pi n) = \cos[(2\pi - 1.4\pi)n] = \cos(0.6\pi n) = g_1[n]$$

$$g_3[n] = \cos(2.6\pi n) = \cos[(2\pi + 0.6\pi)n] = \cos(0.6\pi n) = g_1[n]$$

  - It is difficult to associate a unique continuous-time function with each of these sequences.
  - a continuous time signal of higher frequency acquiring the identity of a sinusoidal sequence of lower frequency after sampling is called **aliasing**.

- However, under certain **conditions**, it is possible to relate a unique continuous-time signal to a given discrete-time signals
- If these conditions hold, then it is possible to **recover** the original continuous-time signal from its sampled values

2.2 Effect of Sampling in the Frequency Domain

- Now, the frequency-domain representation of $g_a(t)$ is given by its continuous-time Fourier transform (CTFT):

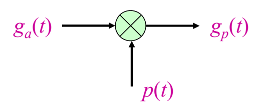
$$G_a(j\Omega) = \int_{-\infty}^{\infty} g_a(t) e^{-j\Omega t} dt$$

- The frequency-domain representation of $g[n]$ is given by its discrete-time Fourier transform (DTFT):

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n}$$

2.2 Effect of Sampling in the Frequency Domain

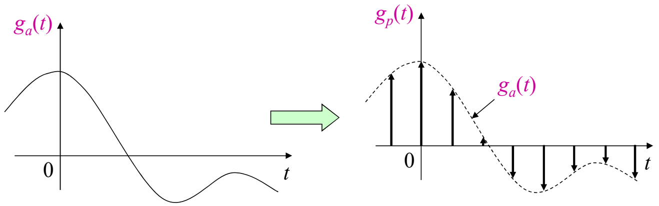
- To establish the relation between $G_a(j\Omega)$ and $G(e^{j\omega})$, we treat the sampling operation mathematically as a multiplication of $g_a(t)$ by a periodic impulse train $p(t)$:



$$g_p(t) = g_a(t) p(t)$$

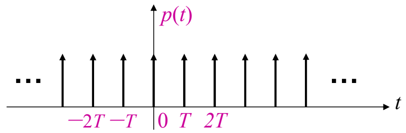
2.2 Effect of Sampling in the Frequency Domain

- $g_a(t)$ is a continuous-time signal consisting of a train of uniformly spaced impulses with the impulse at $t = nT$ weighted by the sampled value $g_a(nT)$ of $g_a(t)$ at that instant



2.2 Effect of Sampling in the Frequency Domain

- $p(t)$ consists of a train of ideal impulses with a period T as shown below



- The multiplication operation yields an impulse train:

$$g_p(t) = g_a(t) p(t) = \sum_{n=-\infty}^{\infty} g_a(nT) \delta(t - nT)$$

2.2 Effect of Sampling in the Frequency Domain

- Two different forms of : $G_p(j\Omega)$
  - **One form** is given by the weighted sum of the CTFTs :

$$G_p(j\Omega) = \mathcal{F}\left[\sum_{n=-\infty}^{\infty} g_a(nT)\delta(t-nT)\right] = \sum_{n=-\infty}^{\infty} g_a(nT)\mathcal{F}\left[\delta(t-nT)\right]$$

time-shifting property

$$G_p(j\Omega) = \sum_{n=-\infty}^{\infty} g_a(nT)e^{-j\Omega nT}$$

2.2 Effect of Sampling in the Frequency Domain

- Two different forms of : $G_p(j\Omega)$
  - **second form**: we note that $p(t)$ can be expressed as a Fourier series:

$$p(t) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} e^{j\frac{2\pi}{T}kt} = \frac{1}{T}\sum_{k=-\infty}^{+\infty} e^{j\Omega_T kt}$$

where

$$\Omega_T = \frac{2\pi}{T}$$

The impulse train $g_p(t)$ therefore can be expressed as

$$g_p(t) = \left(\frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\Omega_T t}\right)g_a(t)$$

2.2 Effect of Sampling in the Frequency Domain

  - From the **frequency-shifting property**, the frequency translated portions of $G_a(j\Omega)$ is given by:

$$G_a(j(\Omega - k\Omega_T))$$

- Hence, an **alternative form** of the CTFT of is given by

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j(\Omega - k\Omega_T))$$

  - periodic function of Ω consisting of a sum of **shifted and scaled replicas** of $G_a(j\Omega)$, shifted by integer multiples of Ω_T and scaled by $\frac{1}{T}$.

2.2 Effect of Sampling in the Frequency Domain

- The term on the RHS of the previous equation for $k = 0$ is the baseband portion of $G_p(j\Omega)$, and each of the remaining terms are the frequency translated portions of $G_p(j\Omega)$

2.2 Effect of Sampling in the Frequency Domain

- Thus if $\Omega_T > 2\Omega_0$, the corresponding normalized digital angular frequency ω_o of the discrete-time signal obtained by sampling the parent continuous-time sinusoidal signal will be in the range $-\pi < \omega < \pi$.

No aliasing

2.2 Effect of Sampling in the Frequency Domain

- On the other hand, if $\Omega_T > 2\Omega_0$, the normalized digital angular frequency will fold over into a lower digital frequency $\omega_0 = \langle 2\pi\Omega_0/\Omega_T \rangle_{2\pi}$ in the range $-\pi < \omega < \pi$ because of aliasing.
  - an overlap of the spectra
- To prevent aliasing, the sampling frequency Ω_T should be greater than 2 times the frequency Ω_0 of the sinusoidal signal being sampled.

Sampling Theorem

2.2 Effect of Sampling in the Frequency Domain

- The frequency range $-\Omega_T/2 \le \Omega \le \Omega_T/2$ is called the baseband or Nyquist band
- Let $g_a(t)$ be a band-limited signal with

$$G_a(j\Omega) = 0 \quad \text{for } |\Omega| > \Omega_m,$$

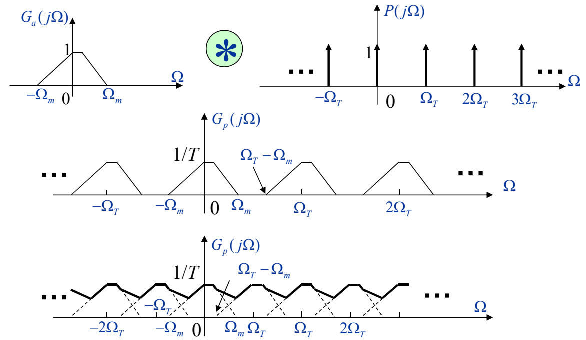
then $g_a(t)$ is uniquely determined by its samples $g_a(nT)$, $-\infty \le n \le \infty$, if

$$\Omega_T \ge 2\Omega_m$$

where $\Omega_T = \frac{2\pi}{T}$

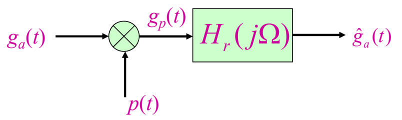
2.2 Effect of Sampling in the Frequency Domain

- Illustration of the frequency-domain effects of time-domain sampling



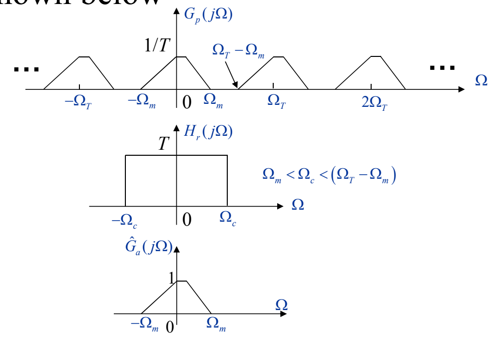
2.2 Effect of Sampling in the Frequency Domain

- If $\Omega_T \geq 2\Omega_m$, $g_a(t)$ can be recovered exactly from $g_p(t)$ by passing it through an ideal lowpass filter $H_r(j\Omega)$ with a gain T and a cutoff frequency Ω_c greater than Ω_m and less than $\Omega_T - \Omega_m$ as shown below



2.2 Effect of Sampling in the Frequency Domain

- The spectra of the filter and pertinent signals are shown below



2.2 Effect of Sampling in the Frequency Domain

- On the other hand, if $\Omega_T < 2\Omega_m$, due to the overlap of the shifted replicas of $G_a(j\Omega)$, the spectrum $G_a(j\Omega)$ cannot be separated by filtering to recover because of the distortion caused by a part of the replicas $G_a(j\Omega)$ immediately outside the baseband folded back or aliased into the baseband.

2.2 Effect of Sampling in the Frequency Domain

- Sampling Theorem

  Let $g_a(t)$ be a band-limited signal with CTFT

$$G_a(j\Omega) = 0 \quad for \quad |\Omega| > \Omega_m$$

  Then $g_a(t)$ is uniquely determined by its samples $g[n] = g_a(nT), \; -\infty < n < \infty$, if

$$\Omega_T \geq 2\Omega_m$$

  where $\boldsymbol{\Omega}_T = \frac{2\pi}{T}$.

2.2 Effect of Sampling in the Frequency Domain

$\Omega_T \ge 2\Omega_m$	**Nyquist condition**
$\Omega_T / 2$	**Folding frequency**
Ω_m	**Nyquist frequency**
$2\Omega_m$	**Nyquist rate**
When $\Omega_T > 2\Omega_m$	**Oversampling**
When $\Omega_T < 2\Omega_m$	**Undersampling**
When $\Omega_T = 2\Omega_m$	**Critical sampling**

Note: A pure sinusoid may not be recoverable from its critically sampled version

2.2 Effect of Sampling in the Frequency Domain

Example

- In high-quality analog music signal processing, a bandwidth of 20 kHz has been determined to preserve the fidelity (保真度)
- Hence, in compact disc (CD) music systems, a sampling rate of 44.1 kHz, which is slightly higher than twice the signal bandwidth, is used

2.2 Effect of Sampling in the Frequency Domain

Example

- Consider the three continuous time sinusoidal signals at a rate of $T = 0.1$ sec :

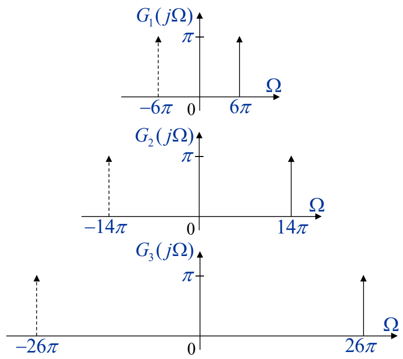
$$g_1(t) = \cos(6\pi t),\ g_2(t) = \cos(14\pi t),\ g_3(t) = \cos(26\pi t)$$

- Their corresponding CTFTs are:

$$G_1(j\Omega) = \pi\left[\delta(\Omega - 6\pi) + \delta(\Omega + 6\pi)\right]$$
$$G_2(j\Omega) = \pi\left[\delta(\Omega - 14\pi) + \delta(\Omega + 14\pi)\right]$$
$$G_3(j\Omega) = \pi\left[\delta(\Omega - 26\pi) + \delta(\Omega + 26\pi)\right]$$

2.2 Effect of Sampling in the Frequency Domain

- These three transforms are plotted below



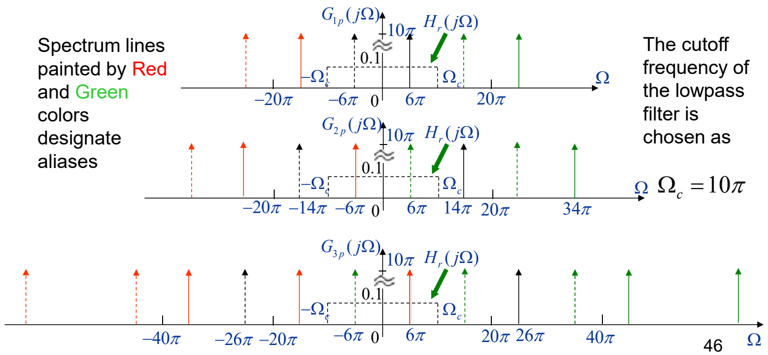
2.2 Effect of Sampling in the Frequency Domain

- These continuous-time signals sampled at a rate of $T = 0.1$ sec, i.e., with a sampling frequency $\Omega_T = 20\pi$ rad/sec
- The sampling process generates the continuous-time impulse trains, $g_{1p}(t)$, $g_{2p}(t)$, and $g_{3p}(t)$
- Their corresponding CTFTs are given by

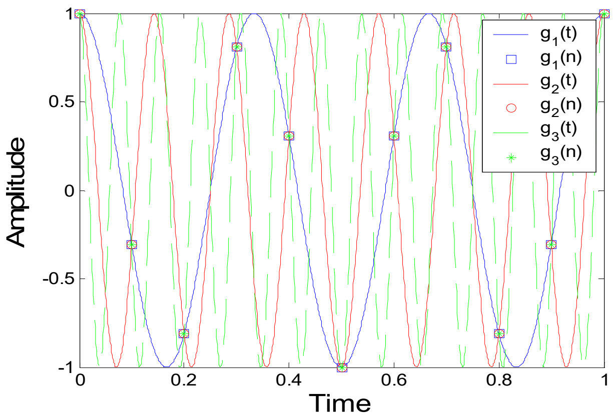
$$G_{lp}(j\Omega) = 10 \sum_{k=-\infty}^{\infty} G_l(j(\Omega - k\Omega_T)), \quad 1 \le l \le 3$$

2.2 Effect of Sampling in the Frequency Domain

- Plots of the 3 CTFTs are shown below



2.2 Effect of Sampling in the Frequency Domain



2.2 Effect of Sampling in the Frequency Domain

- We now derive the relation between the DTFT of $g[n]$ and the CTFT of $g_p(t)$
- To this end we compare

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega}$$

with

$$G_p(j\Omega) = \sum_{n=-\infty}^{\infty} g_a(nT) e^{-j\Omega nT}$$

and make use of

$$g[n] = g_a(nT), \quad -\infty < n < \infty$$

2.2 Effect of Sampling in the Frequency Domain

- Observation: We have

$$G(e^{j\omega}) = G_p(j\Omega)\Big|_{\Omega=\omega/T}$$

or, equivalently,

$$G_p(j\Omega) = G(e^{j\omega})\Big|_{\omega=\Omega T}$$

- From the above observation and

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j(\Omega - k\Omega_T))$$

2.2 Effect of Sampling in the Frequency Domain

- We arrive at the desired result given by

$$G(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\Omega - jk\Omega_T\right)\Big|_{\Omega=\omega/T}$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\frac{\omega}{T} - jk\Omega_T\right)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right)$$

2.2 Effect of Sampling in the Frequency Domain

- The relation derived on the previous slide can be alternately expressed as

$$G(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\Omega - jk\Omega_T\right)$$

from $G(e^{j\omega}) = G_p(j\Omega)\Big|_{\Omega=\omega/T}$

or from $G_p(j\Omega) = G(e^{j\omega})\Big|_{\omega=\Omega T}$

2.2 Effect of Sampling in the Frequency Domain

- It follows that $G(e^{j\omega})$ is obtained from $G_p(j\Omega)$ by applying the mapping $\Omega = \omega / T$
- Now, the CTFT $G_p(j\Omega)$ is a **periodic function** of Ω with a **period** $\Omega_T = 2\pi / T$
- Because of the mapping, the DTFT $G(e^{j\omega})$ is a **periodic function of ω** with a **period** 2π

- We now derive the expression for the output $\hat{g}_a(t)$ of the ideal lowpass reconstruction filter $H_r(j\Omega)$ as a function of the samples $g[n]$
- The impulse response $h_r(t)$ of the lowpass reconstruction filter is obtained by taking the inverse DTFT of :

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \le \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

- Thus, the impulse response is given by

$$h_r(\mathrm{t}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega)e^{j\Omega t}d\Omega = \frac{T}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega t}d\Omega$$

$$= \frac{\sin(\Omega_c t)}{\Omega_T t/2}, \quad -\infty < t < \infty$$

- The input to the lowpass filter is the impulse train $g_p(t)$

$$g_p(t) = \sum_{n=-\infty}^{\infty} g[n]\delta(t - nT)$$

- Therefore, the output $\hat{g}_a(t)$ of the ideal lowpass filter is given by:

$$\hat{g}_a(t) = g_p(t) * h_r(t) = \sum_{n=-\infty}^{\infty} g(n)h_r(t - nT)$$

- Substituting $h_r(t)$ in the above and assuming $\Omega_c = \Omega_T/2 = \pi/T$ for simplicity, we get

$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g[n]\frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

- It can be shown that when $\Omega_c = \Omega_T/2$ in

$$h_r(t) = \frac{\sin \Omega_c t}{\Omega_T t/2}$$

$h_r(0)=1$ and $h_r(nT)=0$ for $n \neq 0$
- As a result, from

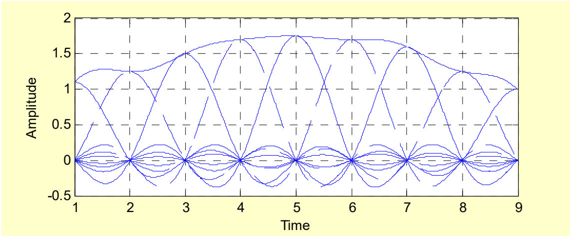
$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g(n)\frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

we observe $\hat{g}_a(rT) = g(r) = g_a(rT)$

for all integer values of r in the range $-\infty < r < \infty$

2.3 Recovery of the Analog Signal

- The ideal bandlimited interpolation process is illustrated below

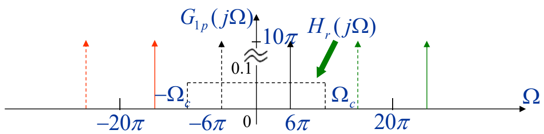


2.3 Recovery of the Analog Signal

- The relation $\hat{g}_a(rT) = g(r) = g_a(rT)$, r is integer, holds whether or not the condition of the sampling theorem is satisfied
- However, $\hat{g}_a(t) = g_a(t)$ for all values of t only if the sampling frequency Ω_T satisfies the condition of the sampling theorem

2.4 Implication of the Sampling Process

- Consider again the three continuous-time signals: $g_1(t) = \cos(6\pi t)$, $g_2(t) = \cos(14\pi t)$, and $g_3(t) = \cos(26\pi t)$
- The plot of the CTFT $G_{1p}(j\Omega)$ of the sampled version of $g_1(t)$ is shown below



2.4 Implication of the Sampling Process

- From the plot, it is apparent that we can recover any of its frequency-translated versions $\cos\left[(20k \pm 6)\pi t\right]$ outside the baseband by passing through an ideal analog bandpass filter $g_{1p}(t)$ with a passband centered at $\Omega = (20k \pm 6)\pi$
- For example, to recover the signal $\cos(34\pi t)$, it will be necessary to employ a bandpass filter with a frequency response

A small number

$$H_r(j\Omega) = \begin{cases} 0.1, & (34-\Delta)\pi \le |\Omega| \le (34+\Delta)\pi \\ 0, & \text{otherwise} \end{cases}$$

2.4 Implication of the Sampling Process

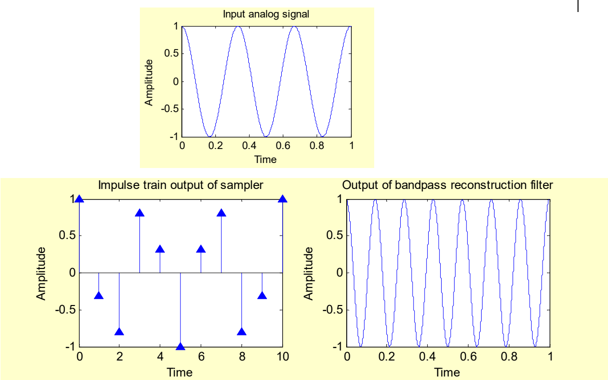
- Likewise, we can recover the aliased baseband component $\cos(6\pi t)$ from the sampled version of either $g_{2p}(t)$ or $g_{3p}(t)$ by passing it through an ideal lowpass filter with a frequency response:

$$H_r(j\Omega) = \begin{cases} 0.1, & 0 \le |\Omega| \le (6+\Delta)\pi \\ 0, & \text{otherwise} \end{cases}$$

2.4 Implication of the Sampling Process

- There is no aliasing distortion unless the original continuous-time signal also contains the component $\cos(6\pi t)$
- Similarly, from either $g_{2p}(t)$ or $g_{3p}(t)$ we can recover any one of the frequency-translated versions, including the parent continuous-time signal $\cos(14\pi t)$ or $\cos(26\pi t)$ as the case may be, by employing suitable filters

2.4 Implication of the Sampling Process



3 Sampling of Bandpass Signals

- The conditions developed earlier assumed that the continuous-time signal is band-limited in the frequency range from dc to some frequency Ω_m .
- Such a continuous-time signal is commonly referred to as a lowpass signal

3 Sampling of Bandpass Signals

- There are applications where the continuous-time signal is bandlimited to a higher frequency range a $\Omega_L \le |\Omega| \le \Omega_H$ with $\Omega_L > 0$
- Such a signal is usually referred to as the bandpass signal
- To prevent aliasing, a bandpass signal can of course be sampled at a rate greater than twice the highest frequency, i.e. by ensuring $\Omega_T \ge 2\Omega_H$

3 Sampling of Bandpass Signals

- However, due to the bandpass spectrum of the continuous-time signal, the spectrum of the discrete-time signal obtained by sampling will have spectral gaps with no signal components present in these gaps
- Moreover, if Ω_H is very large, the sampling rate also has to be very large which may not be practical in some situations

3 Sampling of Bandpass Signals

- A more practical approach is to use under-sampling
- Let $\Delta\Omega = \Omega_H - \Omega_L$ define the bandwidth of the bandpass signal
- Assume first that the highest frequency Ω_H contained in the signal is an integer multiple of the bandwidth, i.e.,

$$\Omega_H = M(\Delta\Omega)$$

3 Sampling of Bandpass Signals

- We choose the sampling frequency Ω_T to satisfy the condition

$$\Omega_T = 2(\Delta\Omega) = \frac{2\Omega_H}{M}$$

which is smaller than $2\Omega_H$, the *Nyquist rate*

- Substitute the above expression for Ω_T in

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\Omega - jk\Omega_T\right)$$

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\Omega - j2k(\Delta\Omega)\right)$$

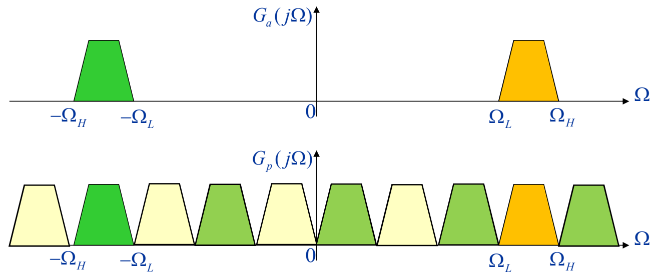
3 Sampling of Bandpass Signals

- As before, $G_p(j\Omega)$ consists of a sum of $G_a(j\Omega)$ and replicas of $G_a(j\Omega)$ shifted by integer multiples of twice the bandwidth $\Delta\Omega$ and scaled by $1/T$

- The amount of shift for each value of k ensures that there will be no overlap between all shifted replicas → *no aliasing*

3 Sampling of Bandpass Signals

- Figure below illustrates the idea behind



3 Sampling of Bandpass Signals

- As can be seen, $g_a(t)$ can be recovered from $g_p(t)$ by passing it through an ideal bandpass filter with a passband given by $\Omega_L \le |\Omega| \le \Omega_H$ and a gain of T
- Note: Any of the replicas in the lower frequency bands can be retained by passing $g_p(t)$ through bandpass filters with passbands

$$\Omega_L - k(\Delta\Omega) \le |\Omega| \le \Omega_H - k(\Delta\Omega) \quad , \quad 1 \le k \le M-1$$

providing a translation to lower frequency ranges

3 Sampling of Bandpass Signals

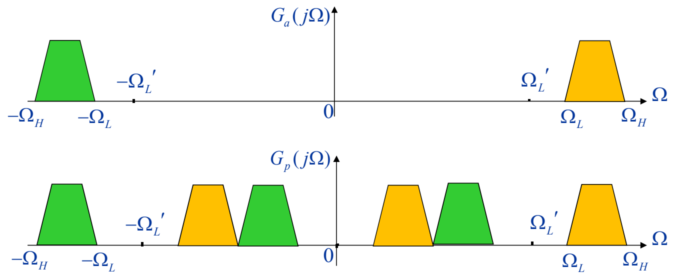
- If Ω_H is not an integer multiple of the band-width $\Delta\Omega = \Omega_H - \Omega_L$, we can

extend the band-width either to the right or to the left artificially

so that the highest frequency contained in the bandpass signal is an integer multiple of the extended bandwidth.

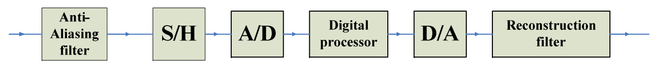
3 Sampling of Bandpass Signals

- Figure below illustrates the idea behind



*1 Anti-Aliasing Filter Design

- Analog anti-aliasing lowpass filter is the first circuit in the interface between the continuous-time and the discrete-time domains.



*1 Anti-Aliasing Filter Design

- Ideally, the anti-aliasing filter $H_a(s)$ should have a lowpass frequency response

$$H_a(j\Omega) = \begin{cases} 1, & |\Omega| < \dfrac{\Omega_T}{2} \\ 0, & |\Omega| \geq \dfrac{\Omega_T}{2} \end{cases}$$

- Such a "brick-wall" type frequency response cannot be realized using practical analog circuit components and, hence, must be approximated.

*1 Anti-Aliasing Filter Design

- A practical anti-aliasing filter should have
  - a passband magnitude response approximating unity with an acceptable tolerance,
  - a stopband magnitude response exceeding a minimum attenuation level
  - an acceptable transition band separating the passband and the stopband, with a transmission zero at infinity.
- In many applications, it is also desirable to have a linear-phase response in the passband.

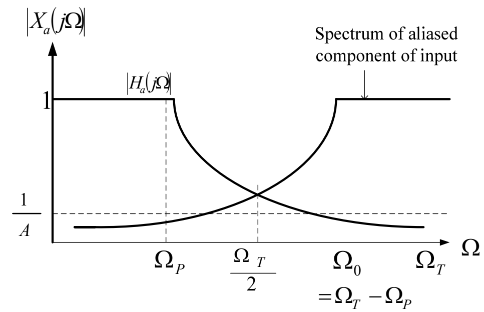
*1 Anti-Aliasing Filter Design

- The passband edge frequency Ω_p, the stopband edge frequency Ω_s, and the sampling frequency Ω_T must satisfy the relation

$$\Omega_p < \Omega_s \le \frac{\Omega_T}{2}$$

- The maximum aliasing distortion comes from the *signal components in the replicas of the input spectrum adjacent to the baseband*.

*1 Anti-Aliasing Filter Design

- 

*1 Anti-Aliasing Filter Design

- The the frequency $\Omega_0 = \Omega_T - \Omega_p$ is aliased into Ω_p, and if the acceptable amount of aliased spectrum at Ω_p is

$$\alpha_P = -20\log_{10}\left(\frac{1}{A}\right)$$

then the minimum attenuation of the anti-aliasing filter at Ω_0 must also be α_P.

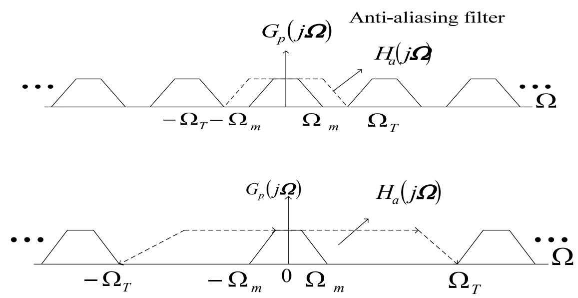
*1 Anti-Aliasing Filter Design

- In practice, the sampling frequency chosen depends on the specific application.
- In applications requiring minimal aliasing, the sampling rate is typically chosen to be 3 to 4 times the passband edge Ω_p of the anti-aliasing analog filter.
- In noncritical applications, a sampling rate of twice the passband edge Ω_p of the anti-aliasing analog filter is more than adequate.

*1 Anti-Aliasing Filter Design

- Requirements for the analog anti-aliasing filter can be relaxed by *oversampling* the analog signal and then *decimating* the high-sampling-rate digital signal to the desired low-rate digital signal.
- The decimation process can be implemented completely in the digital domain by first passing the high-rate digital signal through a digital anti-aliasing filter and then downsampling its output.

*1 Anti-Aliasing Filter Design

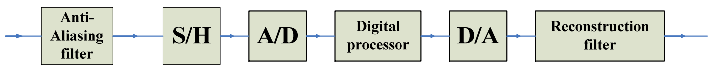
- 

*1 Anti-Aliasing Filter Design

- Note that the transition band of the analog anti-aliasing filter with a higher sampling rate is considerably more than 3 times that needed in the former situation.
- As a result, the filter specifications are met more easily with a much lower order analog filter.

*2 Reconstruction Filter Design

- The output of the D/A converter is finally passed through an analog reconstruction or smoothing filter to eliminate all the replicas of the spectrum outside the baseband.



*2 Reconstruction Filter Design

- The impulse response $h_r(t)$ of the lowpass reconstruction filter $H_r(j\Omega)$ is obtained by taking the inverse DTFT of :

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \le \Omega_T / 2 \\ 0, & |\Omega| > \Omega_T / 2 \end{cases}$$

The reconstructed analog equivalent $y_a(t)$

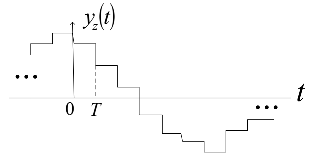
$$y_a(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}.$$

*2 Reconstruction Filter Design

- An ideal impulse-train D/A output $y_p(t)$, followed by a linear, time-invariant analog circuit (zero-order hold operation) with an impulse response $h_z(t)$ that is a rectangular pulse of width T and unity height.

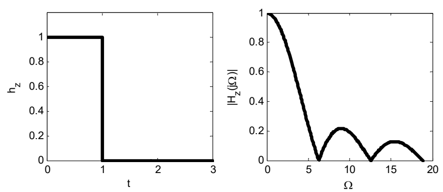
$$y_p(t) \longrightarrow \boxed{h_z(t)} \longrightarrow y_z(t)$$

$$Y_z(j\Omega) = H_z(j\Omega) Y_P(j\Omega)$$



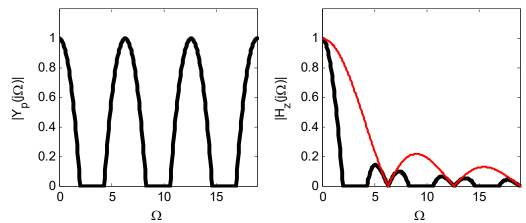
*2 Reconstruction Filter Design

- The magnitude response of the zero-order hold circuit, has a lowpass characteristic with zeros at $\pm T$, $\pm 2T$,... , where $\Omega_T = 2\pi/T$ is the sampling angular frequency.



*2 Reconstruction Filter Design

- The zero-order hold circuit somewhat attenuates the unwanted replicas centered at multiples of the sampling frequency Ω_T .



*2 Reconstruction Filter Design

- Moreover, it should also compensate for the amplitude distortion, more commonly called droop, caused by the zero-order hold circuit in the band from DC to $\Omega_T / 2$.

*2 Reconstruction Filter Design

- The general specifications for the analog reconstruction filter $H_r(j\Omega)$ can be easily determined if the effect of the droop is neglected.
- If Ω_c denotes the highest frequency of the signal $y_p(t)$ that should be preserved at the output of the reconstruction filter, then the lowest-frequency component present in the residual images in the output of the zero-order hold circuit is of frequency $\Omega_0 = \Omega_T - \Omega_c$

*2 Reconstruction Filter Design

- Therefore, if the system specification calls for a minimum attenuation of A_s dB of all frequency components in the residual images, then the reconstruction filter should provide at least an attenuation of $A_s + 20\log_{10}|H_z(j\Omega_o)|$ dB at Ω_0

*2 Reconstruction Filter Design

- For example, if the normalized value of Ω_c is 0.7π, then the gain of the zero-order hold circuit at 0.7π is -7.2 dB. Now, the lowest normalized frequency of the residual images is given by 1.3π.
- For a minimum attenuation of 50 dB of all signal components in the residual images at the output of the zero-order hold, the reconstruction filter must therefore provide at least an attenuation of 42.8 dB at frequency 1.3π .

*2 Reconstruction Filter Design

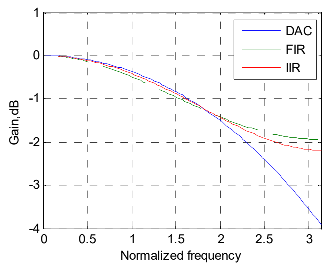
- The droop caused by the zero-order hold circuit can be compensated either before the D/A converter by means of a digital filter or after the zero-order hold circuit by the analog reconstruction filter.
- For the latter approach, we observe that the cascade of the zero-order hold circuit and the analog reconstruction filter must have a frequency response of an ideal reconstruction filter following an ideal D/A converter.

*2 Reconstruction Filter Design

- Alternatively, the effect of the droop can be compensated by including a digital compensation filter $G(z)$ prior to the D/A converter circuit with a modest increase in the digital hardware requirements. The digital compensation filter can be either an FIR or an IIR type.

*2 Reconstruction Filter Design

- The gain responses of the uncompensated and the droop-compensated D/A converters in the baseband.



*2 Reconstruction Filter Design

- Since the above digital compensation filters have a periodic frequency response of period Ω_T , the replicas of the baseband magnitude response outside the baseband need to be suppressed sufficiently to ensure minimal effect from aliasing.