Ch3 Discrete-Time Fourier Transform

3.1 Show that the DTFT of $\mu[n]$ is given by $\frac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{\infty}\pi\delta(\omega+2\pi k)$.

3.2 Determine the DTFT of the two sided signal $y[n]=\alpha^{|n|},\ |\alpha|<1$.

3.3 Determine the DTFT of the causal sequence $x[n]=A\alpha^{n}\cos(\omega_0 n+\phi)\mu[n]$, where $A,\ \alpha,\ \omega_0$ and ϕ are real and $|\alpha|<1$.

3.4 We showed that the inverse DTFT $h_{LP}[n]$ and its DTFT $H_{LP}(e^{j\omega})$ are given respectively by

$$h_{LP}[n]=\frac{\sin\omega_c n}{\pi n},\quad -\infty<n<\infty;\qquad H_{LP}(e^{j\omega})=\begin{cases}1, & 0\le|\omega|\le\omega_c\\ 0, & \omega_c<|\omega|\le\pi\end{cases}$$

Determine and plot the DTFT of $g[n]=\delta[n]-h_{LP}[n],\quad -\infty<n<\infty$.

- (Optional) Determine and plot the DTFT of the cascade of the LTI discrete-time systems with two-sided impulse responses given by $h_1[n]=\delta[n]-\frac{\sin\omega_1 n}{\pi n}$ and $h_2[n]=\frac{\sin\omega_2 n}{\pi n}$, respectively, where $0<\omega_1<\omega_2<\pi$.

- (Optional)Let $X(e^{j\omega})$ denote the DTFT of a real sequence $x[n]$,

  a) Show that if $x[n]$ is even, then it can be computed from $X(e^{j\omega})$ using $x[n]=\frac{1}{\pi}\int_0^{\pi}X(e^{j\omega})\cos(\omega n)d\omega$.

  b) Show that if $x[n]$ is odd, then it can be computed from $X(e^{j\omega})$ using $x[n]=\frac{j}{\pi}\int_0^{\pi}X(e^{j\omega})\sin(\omega n)d\omega$.

3.5 Determine the DTFT of the following sequences:

a) $x_1[n]=n\alpha^{n}\mu[n+1],\ |\alpha|>1$

b) $x_2[n]=\alpha^{n}\mu[-n-1],\ |\alpha|>1$

c) $x_3[n] = \begin{cases} \alpha^{|n|}, & |n| \le M, \\ 0, & \text{otherwise} \end{cases}$

d) $x_4[n] = \alpha^n(\mu[n-1] - \mu[n-4]), \ |\alpha| < 1$

3.6 Determine the DTFT of the following finite-length sequences:

a) $x_1[n] = \begin{cases} 1, & 0 \le n \le N, \\ 0, & \text{otherwise,} \end{cases}$

b) $x_2[n] = \begin{cases} 1 - \dfrac{|n|}{N}, & -N \le n \le N, \\ 0, & \text{otherwise,} \end{cases}$

c) $x_3[n] = \begin{cases} N+1-|n|, & -N \le n \le N, \\ 0, & \text{otherwise,} \end{cases}$

d) $x_4[n] = \begin{cases} \cos(\pi n / 2N), & -N \le n \le N, \\ 0, & \text{otherwise,} \end{cases}$

3.7 Evaluate the inverse DTFT of each of the following DTFTs:

a) $X_1(e^{j\omega}) = \dfrac{e^{j\omega}(1 - e^{j\omega N})}{1 - e^{j\omega}}$

b) $X_2(e^{j\omega}) = 1 + 2\sum_{l=0}^{N} \cos(\omega l)$

c) $X_3(e^{j\omega}) = \dfrac{-\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2}, \ |\alpha| < 1$

d) $X_4(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$

3.8 Evaluate the inverse DTFT of each of the following DTFTs:

a) $X_1(e^{j\omega}) = \sin(4\omega)$

b) $X_2(e^{j\omega}) = \cos(4\omega + 1)$

c) $X_3(e^{j\omega}) = -4 + 3\cos\omega + 4\cos 2\omega$

d) $X_4(e^{j\omega}) = j(-4 + 3\cos\omega + 4\cos 2\omega)\sin(\omega/2)e^{j\omega/2}$

3.9 Determine the inverse DTFTs of the following Fourier transforms"

a) $X_1(e^{j\omega}) = \dfrac{1-1.4e^{-j\omega}}{1-1.3e^{-j\omega}+0.4e^{-2j\omega}}$

b) $X_2(e^{j\omega}) = \dfrac{3-2.5e^{-j\omega}}{1-0.25e^{-2j\omega}}$

3.10 Let $X(e^{j\omega})$ denote the DTFT of a real sequence $x[n]$,

a) Express the inverse DTFT $y[n]$ of $Y(e^{j\omega}) = X(e^{j3\omega})$ in terms of $x[n]$;

b) Define $Y(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega/2}) + X(-e^{j\omega/2})\right]$, determine the inverse DTFT $y[n]$ of $Y(e^{j\omega})$.

3.11 The magnitude function $\left|X(e^{j\omega})\right|$ of the discrete-time sequence $x[n]$ is shown in Fig. 1 for a portion of the angular frequency axis. Sketch the magnitude function for the frequency range $-\pi < \omega < \pi$. What type of sequence is $x[n]$?

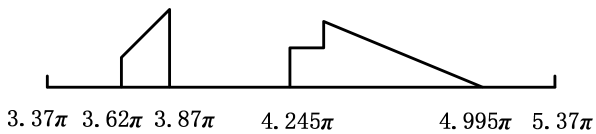

Fig. 1

3.12 Without computing the DTFT, determine which of the following sequences have real-valued DTFTs and which have imaginary-valued DTFTs:

a) $x_1[n] = \begin{cases} n^2, & -N \le n \le N, \\ 0, & \text{otherwise}, \end{cases}$

b) $x_2[n] = \begin{cases} 0, & \text{for } n \text{ even}, \\ \dfrac{2}{\pi n}, & \text{for } n \text{ odd}, \end{cases}$

c) $x_3[n] = \begin{cases} 0, & n = 0, \\ \dfrac{\cos \pi n}{n}, & |n| > 0, \end{cases}$

3.13 Without computing the DTFT, determine which of the DTFT of Fig.2 has an inverse that is an even sequence and which has an inverse that is an odd sequence:

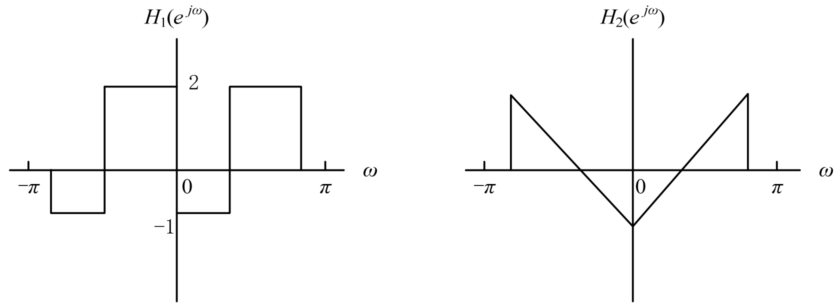

Fig. 2

- (Optional) Let $x[n]$ and $u[n]$ be two real-valued sequences with DTFTs given by $X(e^{j\omega})$ and $U(e^{j\omega})$, respectively. Define a complex-valued $y[n] = x[n] + ju[n]$. Express the $X(e^{j\omega})$ and $U(e^{j\omega})$ in terms of the DTFT $Y(e^{j\omega})$ of $y[n]$. If we wish to force $Y(e^{j\omega})$ to be equal to zero in the frequency range $-\pi \le \omega < 0$, how would we generate $y[n]$ from $x[n]$ and $u[n]$.

- (Optional) Let $X(e^{j\omega})$ denote the DTFT of a complex sequence $x[n]$, determine the DTFT $Y(e^{j\omega})$ of the sequence $y[n] = x[n] \otimes x^*[-n]$ in terms of $X(e^{j\omega})$, and show that it is a real-valued function of ω.

3.14 Using Parseval's relation, evaluate the following integrals:

a) $\displaystyle\int_0^{\pi} \frac{4}{5 + 4\cos\omega}\,\mathrm{d}\omega$

b) $\displaystyle\int_0^{\pi} \frac{4}{(5 - 4\cos\omega)^2}\,\mathrm{d}\omega$

3.15 Let $X(e^{j\omega})$ denote the DTFT of a length-9 sequence given by

$$x[n] = \{-3,\quad 1,\quad 2,\quad 0,\quad -2,\quad -1,\quad 3\},\quad -3 \le n \le 3$$

Evaluate the following functions of $X(e^{j\omega})$ without computing the transform itself:

a) $X(e^{j0})$

b) $\arg\{X(e^{j\omega})\}$

c) $\displaystyle\int_{-\pi}^{\pi} X(e^{j\omega})\mathrm{d}\omega$

d) $\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 \mathrm{d}\omega$

e) $\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 \mathrm{d}\omega$

f) $j\left.\frac{dX(e^{j\omega})}{d\omega}\right|_{\omega=0}$

3.16 Let $G_1(e^{j\omega})$ denote the DTFT of the sequence $g_1[n]$ shown in the Fig. 3. Express the DTFTs of the remaining sequences in Fig.3 in terms of $G_1(e^{j\omega})$. Do not evaluate $G_1(e^{j\omega})$.

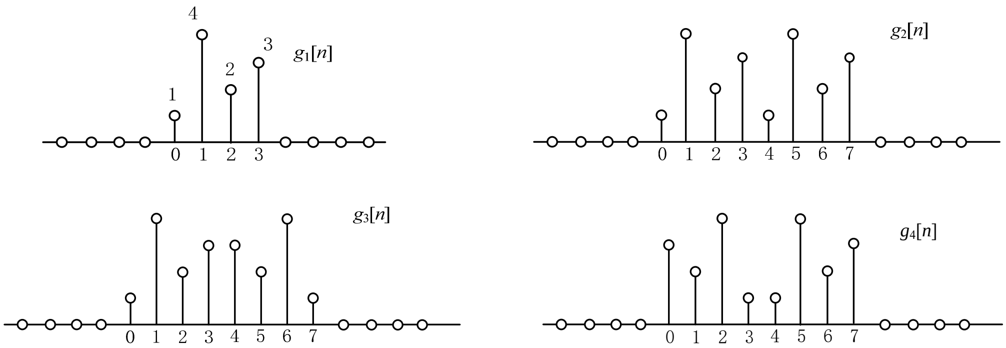

Fig. 3

3.17 Show that the function $u[n] = z^n$, where *z* is a complex constant, is an eigenfunction of an LTI discrete-time system. Is $v[n] = z^n \mu[n]$ with *z* a complex constant also an eigenfunction of an LTI discrete-time system?

3.18 Consider the sequence $g[n] = n(0.4)^n \mu[n]$ with a DTFT $G(e^{j\omega})$. Make use of the symmetry relations given in Table 3.1&3.2 and theorems in Table 3.3, determine without evaluating $G(e^{j\omega})$, the inverse DTFTs of the following functions of $G(e^{j\omega})$:

a) $X_1(e^{j\omega}) = e^{-j4\omega} G(e^{j\omega})$

b) $X_2(e^{j\omega}) = G(e^{j(\omega+0.5\pi)})$

c) $X_3(e^{j\omega}) = \frac{dG(e^{j\omega})}{d\omega}$

d) $X_4(e^{j\omega}) = jG_{im}(e^{j\omega})$

3.19

a) Consider an LTI discrete-time systems with a real and causal impulse response $h[n]$ and a frequency response $H(e^{j\omega})$. Show that $h[n]$ can be determined uniquely from the real part $H_{\text{re}}(e^{j\omega})$ of $H(e^{j\omega})$.

b) The real part of the frequency response of a real and causal LTI discrete-time system is given by $H_{\text{re}}(e^{j\omega}) = 1 + 2\cos\omega + 3\cos 2\omega + 4\cos 3\omega$. Determine the impulse response $h[n]$ of the system.

3.20 Evaluate the linear convolution of the following sequences using the DTFT-based method.

a) $x_1[n] = \{1,\ 2,\ 1\},\quad -1 \le n \le 1$

b) $x_2[n] = \{-2,\ 1,\ 0,\ -1,\ 2\},\ 0 \le n \le 4$

3.21 If the input to each of the following discrete-time systems is $x[n] = \cos(\omega_0 n)$, determine the frequencies present in their outputs:

a) $y_1[n] = \sin(\pi n/3)x[n]$

b) $y_2[n] = x^3[n]$

c) $y_3[n] = x[3n]$

3.22 Determine a closed-form expression for the frequency response $H(e^{j\omega})$ of the LTI discrete-time system characterized by an impulse response

$$h[n] = \delta[n] - \alpha\delta[n-R]$$

where $|\alpha| < 1$. What are the maximum and minimum values of its magnitude response? How many peaks and dips of the magnitude response occur in the range $-\pi \le \omega \le \pi$? What are the locations of the dips? Sketch the magnitude and the phase response for $R = 5$.

3.23

a) A noncausal LTI FIR discrete-time system is characterized by an impulse response $h_1[n] = a_1\delta[n-2] + \alpha_2\delta[n-1] + a_3\delta[n] + \alpha_4\delta[n+1]$. For what value of the impulse response samples $\{a_i\},\ 1 \le i \le 4$ will its frequency response $H(e^{j\omega})$ have a zero phase?

b) A causal LTI FIR discrete-time system is characterized by an impulse response $h_2[n]=a_1\delta[n]+\alpha_2\delta[n-1]+a_3\delta[n-2]+\alpha_4\delta[n-3]+\alpha_5\delta[n-4]$. For what value of the impulse response samples $\{a_i\},\ 1\le i\le 5$ will its frequency response $H(e^{j\omega})$ have a linear phase?

- (Optional) Determine the input-output relation of a factor-L up-sampler in the frequency domain.

3.24 The frequency response $H(e^{j\omega})$ of a length-4 FIR filter with a real impulse response has the following specific values: $H(e^{j0})=2$, $H(e^{j\pi/2})=7-j3$ and $H(e^{j\pi})=0$ Determine its impulse response $h[n]$.

- (Optional)

a) Design a length-3 FIR notch filter with a symmetric impulse response $h[n]$ that is $h[n] = h[2-n]$, $0\le n\le 2$, and with a notch frequency at 0.4π and a 0 dB dc gain;

b) Determine the exact expression for the expression for the frequency response of the filter designed, and plot its magnitude and phase responses.

3.25

a) Design a length-5 FIR bandpass filter with a antisymmetric impulse response $h[n]$ that is $h[n] = -h[4-n]$, $0\le n\le 4$ satisfying the following magnitude response values: $\left|H(e^{j0.4\pi})\right|=0.8$, and $\left|H(e^{j0.8\pi})\right|=0.2$;

b) Determine the exact expression for the expression for the frequency response of the filter designed, and plot its magnitude and phase responses.

- (Optional) Consider the two LTI causal digital filters with impulse responses given by

$$h_A[n]=0.3\delta[n]-\delta[n-1]+0.3\delta[n-2], \qquad h_B[n]=0.3\delta[n]+\delta[n-1]+0.3\delta[n-2]$$

a) Sketch the magnitude responses of the two filters and compare their characteristics.

b) Let $h_A[n]$ be the impulse response of a causal digital filter with a frequency response $H_A(e^{j\omega})$. Define another digital filter whose impulse response $h_C[n]$ is given by

$$h_C[n]=(-1)^n h_A[n], \quad \text{for all } n.$$

What is the relation between the frequency response $H_C(e^{j\omega})$ of this new filter and the frequency response $H_A(e^{j\omega})$ of the parent filter?

3.26 Show that the group delay $\tau(\omega)$ of an LTI discrete-time system characterized by a frequency response $H(e^{j\omega})$ can be expressed as

$$\tau(\omega) = \mathrm{Re}\left\{ \frac{j\dfrac{dH(e^{j\omega})}{d\omega}}{H(e^{j\omega})} \right\}$$

3.27 The frequency response of an LTI FIR discrete-time system is given by $H(e^{j\omega}) = \alpha_0 + \alpha_1 e^{-j\omega} + \alpha_2 e^{-2j\omega} + \alpha_3 e^{-3j\omega} + \alpha_4 e^{-4j\omega}$. For what relations between the coefficients $\alpha_i,\ 0 \le i \le 4$ will $H(e^{j\omega})$ have a constant group delay?

3.28 Determine the expressions for the group delay of each of the LTI systems whose frequency responses are given below:

a) $H_a(e^{j\omega}) = \alpha + \beta e^{-j\omega}$

b) $H_b(e^{j\omega}) = \dfrac{\alpha + \beta e^{-j\omega}}{1 + \gamma e^{-j\omega}},\quad |\gamma| < 1$

- (Optional) Using the equation in question above, determine the group delay of the LTI discrete-time systems with frequency responses given below:

  a) $H_a(e^{j\omega}) = 2 - 0.5e^{-j\omega}$

  b) $H_b(e^{j\omega}) = \dfrac{0.3 + 0.7e^{-j\omega}}{1 + 0.5e^{-j\omega}},\quad |\gamma| < 1$

- (Optional) Let $H(e^{j\omega})$ denote the frequency response of an LTI discrete-time system with an impulse response $h[n]$. Let $G(e^{j\omega})$ denote the Fourier transform of the sequence $nh[n]$. Show that the group delay of the LTI system can be computed using

$$\tau(\omega) = \frac{H_{\mathrm{re}}(e^{j\omega})G_{\mathrm{re}}(e^{j\omega}) + H_{\mathrm{im}}(e^{j\omega})G_{\mathrm{im}}(e^{j\omega})}{\left|H(e^{j\omega})\right|^2}$$

  where $H_{\mathrm{re}}(e^{j\omega})$ and $H_{\mathrm{im}}(e^{j\omega})$ denote the real and imaginary parts of $H(e^{j\omega})$, respectively, and $G_{\mathrm{re}}(e^{j\omega})$ and $G_{\mathrm{im}}(e^{j\omega})$ denote the real and imaginary parts of $G(e^{j\omega})$,

- Which one of the following function of ω can be the DTFT of a discrete-time sequence? Justify your answers.

  a) $X_1(e^{j\omega}) = 2\cos(0.4\omega)$

b) $X_2(e^{j\omega}) = 3\cos(0.75\omega) + 4\cos(0.25\omega)$

c) $X_3(e^{j\omega}) = \cos(0.2\omega) + 3\sin(2\omega)$

- Let $\{x[n]\},\ 0 \le n \le N-1$, be a length-N sequence with a DTFT given by $X(e^{j\omega})$

c) Let $\{x_a[n]\}$, be a length-*M* sequence obtained by zero-padding $\{x[n]\}$ with *M*-*N* zeros at the end, i.e.,

$$\{x_a[n]\} = \begin{cases} x[n], & \text{for } 0 \le n \le N-1 \\ 0, & \text{for } N \le n \le M-1 \end{cases}$$

with a DTFT given by $X_a(e^{j\omega})$. What is the relation between $X(e^{j\omega})$ and $X_a(e^{j\omega})$.

d) Let $\{x_b[n]\}$, be a length-*M* sequence obtained by zero-padding $\{x[n]\}$ with *M*-*N* zeros at the beginning, i.e.,

$$\{x_a[n]\} = \begin{cases} 0, & \text{for } 0 \le n \le M-N-1 \\ x[n], & \text{for } M-N \le n \le M-1 \end{cases}$$

with a DTFT given by $X_b(e^{j\omega})$. What is the relation between $X(e^{j\omega})$ and $X_b(e^{j\omega})$.