Chapter 3

Discrete-Time
Fourier Transform (DTFT)

Chapter 3

Two major topics of this chapter:

- Discrete-Time Fourier Transform
  - **Discrete-Time Fourier Transform (DTFT)**
  - **Basic Properties & Symmetry Relation**
  - **DTFT Theorems**
- Discrete-Time Signals and Systems in Frequency Domain
  - **Spectrum Analysis**
  - **Frequency Response of an LTI Discrete-Time System**
  - **Phase and Group Delay**
  - **The Unwrapped Function**

Chapter 3B

Discrete-Time Signals and
Systems in Frequency Domain

Part B: DTFT Analysis

1. **Spectrum Analysis**
   - **1.1 Energy Density Spectrum**
   - **1.2 Band-limited Discrete-Time Signals**
2. **The Unwrapped Function**
3. **The Frequency Response of an LTI Discrete-time System**
4. **Phase and Group Delay**

1.1 Energy Density Spectrum

- The total energy of a finite-energy sequence $g[n]$ is given by

$$\mathcal{E}_g = \sum_{n=-\infty}^{\infty} |g[n]|^2$$

- From Parseval's relation we observe that

$$\mathcal{E}_g = \sum_{n=-\infty}^{\infty} |g[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|G\left(e^{j\omega}\right)\right|^2 d\omega$$

1.1 Energy Density Spectrum

- The quantity

$$S_{gg}\left(e^{j\omega}\right) = \left|G\left(e^{j\omega}\right)\right|^2$$

is called the energy density spectrum

- The area under this curve in the range $-\pi \le \omega \le \pi$ divided by 2π is the energy of the sequence

1.1 Energy Density Spectrum

- Recall that the auto-correlation sequence $r_{gg}[l]$ of $g[n]$ can be expressed as

$$r_{gg}[l] = \sum_{n=-\infty}^{\infty} g[n]g[-(l-n)] = g[l] * g[-l]$$

- As we know that the DTFT of $g[-l]$ is $G(e^{-j\omega})$, therefore, the DTFT of $g[l] * g[-l]$ is given by $|G(e^{j\omega})|^2$, where we have used the fact that for a real sequence $g[n]$, $G(e^{-j\omega}) = G^*(e^{j\omega})$

1.1 Energy Density Spectrum

- As a result, the energy density spectrum $S_{gg}(e^{j\omega})$ of a real sequence $g[n]$ can be computed by taking the DTFT of its auto-correlation sequence $r_{gg}(l)$, i.e.,

$$S_{gg}\left(e^{j\omega}\right) = \sum_{l=-\infty}^{\infty} r_{gg}[l]e^{-j\omega l}$$

Wiener-Khinchin theorem

Example

- Compute the energy of the sequence

$$h_{LP}[n] = \frac{\sin \omega_c n}{n\pi}, \quad -\infty \le n \le \infty$$

- Here

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_{LP}\left(e^{j\omega}\right)\right|^2 d\omega$$

where $H_{LP}\left(e^{j\omega}\right) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$

- Therefore, Compute the energy of the sequence

$$\sum_{n=-\infty}^{\infty} |h_{LP}(n)|^2 = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} < \infty$$

- Hence, $h_{LP}(n)$ is a finite-energy sequence

- Discrete-time signal is a periodic function of ω with a period 2π.
- A full-band, finite-energy, discrete-time signal has a spectrum occupying the whole frequency range $-\pi \le \omega < \pi$
- A band-limited discrete-time signal has a spectrum that is limited to a portion of the frequency range $-\pi \le \omega < \pi$

- An ideal band-limited signal has a spectrum that is zero outside a finite frequency range ω $0 \le \omega_a \le |\omega| \le \omega_b \le \pi$, that is

$$X(e^{j\omega}) = \begin{cases} 0, & 0 \le |\omega| \le \omega_a \\ 0, & \omega_b < |\omega| < \pi \end{cases}$$

- However, an ideal band-limited signal cannot be generated in practice (*Why?*)

1.2 Band-limited Discrete-Time Signals

- Band-limited signals are classified according to the frequency range where most of the signal's energy is concentrated
- A lowpass, discrete-time signal has a spectrum occupying the frequency range

$$-\pi < -\omega_p \leq |\omega| \leq \omega_p < \pi$$

where ω_p is called the bandwidth of the signal

1.2 Band-limited Discrete-Time Signals

- A highpass, discrete-time signal has a spectrum occupying the frequency range $\omega_p \leq |\omega| < \pi$ where the bandwidth of the signal is $\pi - \omega_p$
- A bandpass, discrete-time signal has a spectrum occupying the frequency range $0 < \omega_L \leq |\omega| \leq \omega_H < \pi$ where $\omega_H - \omega_L$ is the bandwidth
- **A precise definition of the bandwidth depends on applications: 80% of the energy**

2 The Unwrapped Phase Function

- In numerical computation, when the computed phase function is outside the range [-π,π], the phase is computed modulo 2π, to bring the computed value to this range.
- As the result, the phase functions of some sequences exhibit discontinuities of 2π radians in the plot.

2 The Unwrapped Phase Function

- Consider an alternate type of phase function that is a continuous function of ω derived from the original phase function by removing the discontinuities of 2π.
- The process of removing the discontinuities is called "unwrapping the phase," and the new phase function will be denoted as $\theta_c(\omega)$, with the subscript "c" indicating that it is a continuous function of ω.

2 The Unwrapped Phase Function

- The natural logarithm of the Fourier transform $X(e^{j\omega})$ of the sequence $x[n]$ can be expressed as

$$\ln X(e^{j\omega}) = \ln\left|X(e^{j\omega})\right| + j\theta(\omega)$$

- If $\ln X(e^{j\omega})$ exists, then its derivative with respect to ω also exists and is given by

$$\frac{d\ln X(e^{j\omega})}{d\omega} = \frac{1}{X(e^{j\omega})}\left[\frac{dX(e^{j\omega})}{d\omega}\right]$$

$$= \frac{1}{X(e^{j\omega})}\left[\frac{dX_{re}(e^{j\omega})}{d\omega} + j\frac{dX_{im}(e^{j\omega})}{d\omega}\right]$$

2 The Unwrapped Phase Function

- The derivative $\ln X(e^{j\omega})$ of with respect to ω is also given by

$$\frac{d\ln X(e^{j\omega})}{d\omega} = \frac{d\ln\left|X(e^{j\omega})\right|}{d\omega} + j\frac{d\theta(\omega)}{d\omega}$$

- The derivative of $\theta(\omega)$ with respect to ω is given by the imaginary part of the right-hand side

$$\frac{d\theta(\omega)}{d\omega} = \frac{1}{\left|X(e^{j\omega})\right|^2}\left[X_{re}(e^{j\omega})\frac{dX_{im}(e^{j\omega})}{d\omega} - X_{im}(e^{j\omega})\frac{dX_{re}(e^{j\omega})}{d\omega}\right]$$

2 The Unwrapped Phase Function

- The phase function $\theta(\omega)$ can thus be defined unequivocally by its derivative $\frac{d\theta(\omega)}{d\omega}$:

$$\theta(\omega) = \int_0^{\omega}\left[\frac{d\theta(\eta)}{d\eta}\right]d\eta$$

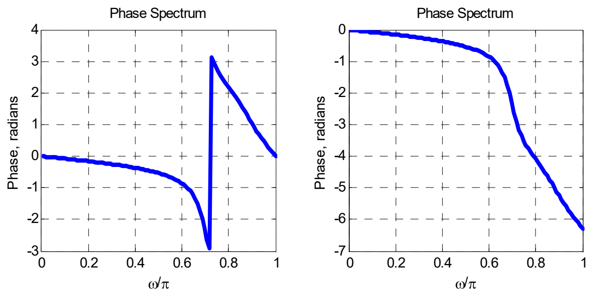
with the constraint $\theta(0) = 0$

- The phase function as defined is called the unwrapped phase function. The unwrapped phase is a continuous function of ω.

2 The Unwrapped Phase Function

- If the above constraint is not satisfied, then the computed phase function will exhibit absolute jumps greater than π.
- For unwrapping the phase, these jumps should be replaced with their 2π complements.
- In Matlab, this can be done using the M-file unwrap.

2 The Unwrapped Phase Function



- Unwrapped phase spectrum of the Fourier transform

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3 The Frequency Response of an LTI Discrete-time System

- An LTI discrete-time system is completely characterized in the time-domain by its **impulse response sequence {*h*[*n*]}**

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3 The Frequency Response of an LTI Discrete-time System

- **Frequency response** – A transform-domain representation of the LTI discrete-time system.
- Such **transform-domain representations** provide *additional insights* into the behavior of such systems.
- It is *easier to design and implement* these systems in the transformed-domain for certain applications.

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3 The Frequency Response of an LTI Discrete-time System

- Most discrete-time signals encountered in practice can be represented as a linear combination of a very large, maybe infinite number of sinusoidal discrete-time signals of different angular frequencies.
- Since a sinusoidal signal can be expressed in terms of an exponential signal, the **response of the LTI system to an exponential input** is of practical interest.

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3.1 Definition

- An important property of an LTI system is that for certain types of input signals, called eigenfunctions, the output signal is the input signal multiplied by a complex constant.
- Consider the following LTI system. We consider one such eigenfunction as the input.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

3.1 Definition

- Its I-O relationship in the time domain is given by the convolution sum.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- If the input is of the form

$$x[n] = e^{j\omega n} \quad -\infty < n < \infty$$

then

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = \underbrace{\left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right)}_{H(e^{j\omega})} e^{j\omega n}$$

3.1 Definition

- Then we can write

$$y[n] = H(e^{j\omega})e^{j\omega n}$$

- It implies for a complex exponential input signal $e^{j\omega n}$, the output of an LTI discrete-time system is also a complex exponential signal of the same frequency multiplied by a complex constant $H(e^{j\omega})$
- Thus $e^{j\omega n}$ is an eigenfunction of the system

3.1 Definition

- The quantity $H(e^{j\omega})$ is called the **frequency response** of the LTI discrete-time system that is a function of the input frequency ω and the system impulse response coefficients $h[n]$.
- $H(e^{j\omega})$ provides the frequency-domain description of the system.

- In this course we shall be concerned with LTI discrete-time systems characterized by linear constant coefficient difference equations of the form:

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

- Applying the DTFT to the difference equation and making use of the linearity and the time-invariance properties, we arrive at the input-output relation in the transform-domain as

$$\left(\sum_{k=0}^{N} d_k e^{-j\omega k}\right) Y(e^{j\omega}) = \left(\sum_{k=0}^{M} p_k e^{-j\omega k}\right) X(e^{j\omega})$$

$$y[n] \xleftrightarrow{\text{DTFT}} x[n]$$

Definition

- The DTFT of the impulse response of an LTI system is called the Frequency Response of this system

$$H(e^{j\omega}) = H(e^{j(\omega+2\pi)}) = H_{re}(e^{j\omega}) + jH_{im}(e^{j\omega})$$

$$= \underbrace{\left|H(e^{j\omega})\right|}_{\text{magnitude response}} \underbrace{e^{j\arg\{H(e^{j\omega})\}}}_{\text{phase response}}$$

- In some cases, the magnitude function is specified in decibels as

$$\mathcal{G}(\omega) = 20\log_{10}\left|H(e^{j\omega})\right| \text{ dB}$$

where $\mathcal{G}(\omega)$ is called the gain function

- The negative of the gain function

$$\mathcal{A}(\omega) = -\mathcal{G}(\omega)$$

is called the attenuation or loss function

3.2 Frequency-Domain Characterization of the LTI Discrete-Time System

- The convolution sum description of the LTI discrete-time system is given by

$$y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- Taking the DTFT of both sides we obtain

$$Y(e^{j\omega})=\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right)e^{-j\omega n}$$

3.2 Frequency-Domain Characterization of the LTI Discrete-Time System

- Interchanging the summation signs on the right-hand side and rearranging we arrive at

$$Y(e^{j\omega})=\sum_{k=-\infty}^{\infty} h[k]\left(\sum_{n=-\infty}^{\infty} x[n-k]e^{-j\omega n}\right)$$

$$\overset{l=n-k}{=}\sum_{k=-\infty}^{\infty} h[k]\left(\sum_{l=-\infty}^{\infty} x[l]e^{-j\omega(l+k)}\right)$$

$$=\sum_{k=-\infty}^{\infty} h[k]\left(\sum_{l=-\infty}^{\infty} x[l]e^{-j\omega l}\right)e^{-j\omega k}$$

$$=H(e^{j\omega})X(e^{j\omega})$$

3.2 Frequency-Domain Characterization of the LTI Discrete-Time System

- It follows from the previous equation

$$H(e^{j\omega})=Y(e^{j\omega})/X(e^{j\omega})$$

- The output cannot contain sinusoidal components of frequencies that are not present in the input and the system.
- As a result, if the output of a system has new frequency components, then the system is either nonlinear or time-varying or both.

3.2 Frequency-Domain Characterization of the LTI Discrete-Time System

Example

- Convolution Sum Computation Using Fourier Transform
  the input sequence $x[n]=\alpha^n\mu[n]$ with $|\alpha|<1$
  the frequency response of the causal LTI system

$$h[n]=\beta^n\mu[n]$$

with $|\beta|<1$,

$$X(e^{j\omega})=1/(1-\alpha e^{-j\omega}) \qquad H(e^{j\omega})=1/(1-\beta e^{-j\omega})$$

$$Y(e^{j\omega})=X(e^{j\omega})H(e^{j\omega})=\frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$$

3.2 Frequency-Domain Characterization of the LTI Discrete-Time System

$$Y(e^{j\omega}) = \frac{A}{(1-\alpha e^{-j\omega})} + \frac{B}{(1-\beta e^{-j\omega})} = \frac{(A+B)-(A\beta+B\alpha)e^{-j\omega}}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$$

$$A = \frac{\alpha}{\alpha-\beta} \qquad B = -\frac{\beta}{\alpha-\beta}$$

$$y[n] = \frac{\alpha}{\alpha-\beta}\alpha^n \mu[n] - \frac{\beta}{\alpha-\beta}\beta^n \mu[n]$$

$$= \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\mu[n] = \left(\sum_{k=0}^{n} \alpha^k \beta^{n-k}\right)\mu[n]$$

3.3 Frequency Response of Discrete-Time Systems

- Frequency Response of LTI FIR Discrete-Time Systems
- Frequency Response of LTI IIR Discrete-Time Systems

3.3-1 Frequency Response of LTI FIR Discrete-Time Systems

- LTI FIR Discrete-Time System

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k] \qquad N_1 < N_2$$

- The frequency response is

$$H(e^{j\omega}) = \sum_{k=N_1}^{N_2} h[k]e^{-j\omega k}$$

which is seen to be a polynomial in $e^{j\omega}$

3.3-2 Frequency Response of LTI IIR Discrete-Time Systems

- LTI IIR discrete-time systems are characterized by linear constant coefficient difference equations of the form

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

- The frequency response is

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} p_k e^{-j\omega k}}{\sum_{k=0}^{N} d_k e^{-j\omega k}}$$

3.4 Frequency Response Computation using Matlab

- The function **freqz(h,w)** can be used to determine the values of the frequency response vector ***h*** at a set of given frequency points w
- From ***h***, the real and imaginary parts can be computed using the functions **real** and **imag**, and the magnitude and phase functions using the functions **abs** and **angle**

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3.4 Frequency Response Computation using Matlab

Example

- Consider a *moving-average* filter

$$h[n]=\begin{cases}1/M, & 0\le n\le M-1\\ 0, & \text{otherwise}\end{cases}$$

- Its frequency response is given by

$$H\left(e^{j\omega}\right)=\frac{1}{M}\sum_{n=0}^{M-1}e^{-j\omega n}=\frac{1}{M}\frac{1-e^{-jM\omega}}{1-e^{-j\omega}}$$

$$=\frac{1}{M}\frac{\sin\left(M\omega/2\right)}{\sin\left(\omega/2\right)}e^{-j(M-1)\omega/2}$$

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3.4 Frequency Response Computation using Matlab

- The magnitude and phase responses of the moving-average filter are obtained

$$\left|H\left(e^{j\omega}\right)\right|=\left|\frac{1}{M}\frac{\sin\left(M\omega/2\right)}{\sin\left(\omega/2\right)}\right|$$

$$\theta(\omega)=-\frac{(M-1)\omega}{2}+\pi\sum_{k=2}^{\lfloor M/2\rfloor}\mu\left(\omega-\frac{2\pi k}{M}\right)$$

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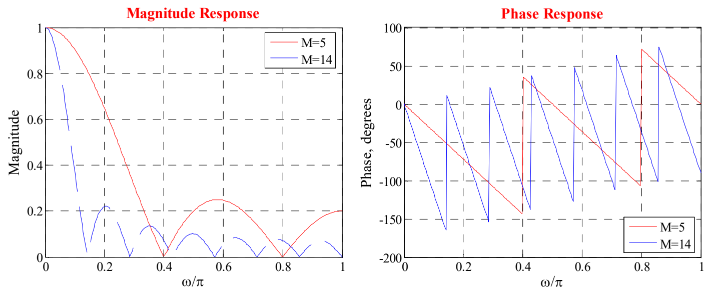
3.4 Frequency Response Computation using Matlab



- Program 4_3 can be used to generate the magnitude and gain responses of an *M*-point moving average filter as shown in the next slide

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- One application of an LTI discrete-time system is to pass certain frequency components in an input sequence without any distortion (if possible) and to block other frequency components.
- Such systems are called *digital filters* and one of the main subjects of discussion in this course.

- The key to the filtering process is

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

- It expresses an arbitrary input as a linear weighted sum of an infinite number of exponential sequences, or equivalently, as a linear weighted sum of sinusoidal sequences.

- By appropriately choosing the values of the magnitude function $|H(e^{j\omega})|$ of the LTI digital filter at frequencies corresponding to the frequencies of the sinusoidal components of the input, some of these components can be selectively heavily attenuated or filtered with respect to the others.

- To understand the mechanism behind the design of frequency-selective filters, consider a real-coefficient LTI discrete-time system characterized by a magnitude function.

$$|H(e^{j\omega})| \cong \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

- We apply an input

  $x[n] = A\cos\omega_1 n + B\cos\omega_2 n, \quad 0 < \omega_1 < \omega_c < \omega_2 < \pi$

  to this system
- Because of *linearity*, the output of this system is of the form

$$y[n] = A\left|H(e^{j\omega_1})\right|\cos(\omega_1 n + \theta(\omega_1))$$
$$+B\left|H(e^{j\omega_2})\right|\cos(\omega_2 n + \theta(\omega_2))$$

Eigen-function, conjugate-symmetric for real *h*[*n*]

- As $\left|H(e^{j\omega_1})\right| \cong 1 \quad \left|H(e^{j\omega_2})\right| \cong 0$

  the output reduces to

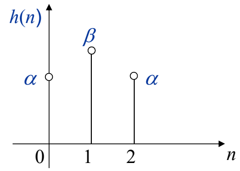
$$y[n] \cong A\left|H(e^{j\omega_1})\right|\cos(\omega_1 n + \theta(\omega_1))$$

- Thus, the system acts like a lowpass filter
- In the following example, we consider the design of a very simple digital filter.

Example

- Design of a high pass digital filter

  The input $x[n] = [\cos(0.1n) + \cos(0.4n)] \cdot u[n]$ which consists of two frequency components 0.1 *rad/sample* and 0.4 *rad/sample*.
- For simplicity, assume the filter to be an FIR filter of length-3 with an impulse response:



- Note that $h[n]$ is a **linear phase FIR filter** which will be discussed in the latter chapters
- The frequency response of this filter is given by

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-2j\omega}$$
$$= (2\alpha\cos\omega + \beta)e^{-j\omega}$$

3.5 The Concept of Filtering

- The magnitude and phase functions are

$$\left|H(e^{j\omega})\right| = \left|2\alpha\cos\omega + \beta\right| \qquad \theta(\omega) = -\omega$$

- In order to block the low-frequency component, the magnitude function at $\omega = 0.1$ should be equal to zero
- Likewise, to pass the high-frequency component, the magnitude function at $\omega = 0.4$ should be equal to one

3.5 The Concept of Filtering

- Thus, the two conditions that must be satisfied are

$$2\alpha\cos 0.1 + \beta = 0 \qquad 2\alpha\cos 0.4 + \beta = 1$$

- Solving the above two equations we get

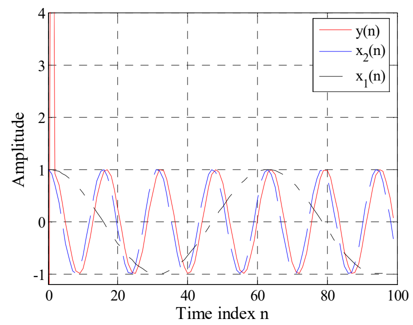
$$\alpha = -6.76195 \qquad \beta = 13.456335$$

- Thus the output-input relation of the FIR filter is given by

$$y[n] = -6.76195x[n] + 13.456335x[n-1]$$
$$-6.76195x[n-2]$$

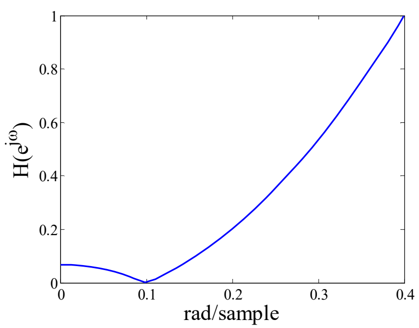
3.5 The Concept of Filtering

- Figure below shows the plots generated by running **program 4_4**



3.5 The Concept of Filtering

- Figure below shows the frequency response of this highpass filter



4 Definition of Phase and Group delays

- They are two important additional parameters that characterize the form of the output response $y[n]$ of an LTI discrete-time system excited by an input signal $x[n]$ composed of a weighted linear combination of sinusoidal sequences.

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4.1 Definition of Phase Delay

- The output $h[n]$ of a frequency-selective LTI discrete-time system with a frequency response $H(e^{j\omega})$ exhibits some *delay* relative to the input caused by the *nonzero phase response* of the system

$$\theta(\omega) = \arg\left\{H(e^{j\omega})\right\}$$

- For an input

$$x[n] = A\cos(\omega_0 n + \phi) \quad -\infty < n < \infty$$

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4.1 Definition of Phase Delay

The output is **Proof**

$$y[n] = A\left|H(e^{j\omega_0})\right|\cos(\omega_0 n + \theta(\omega_0) + \phi)$$

- Thus, the output *lags in phase* by $\theta(\omega_0)$ radians
- Rewriting the above equation we get

$$y[n] = A\left|H(e^{j\omega})\right|\cos\left(\omega_0\left(n + \frac{\theta(\omega_0)}{\omega_0}\right) + \phi\right)$$

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4.1 Definition of Phase delays

- Phase delay $\boxed{\tau_p(\omega_0) = -\frac{\theta(\omega_0)}{\omega_0}}$
- The phase delay of s discrete-time signal leads to a phase shift in the output of the system with respect to the input, with the amount of shift depending on the frequency ω
- The output $y[n]$ will not be a delayed replica of the input $x[n]$ unless the phase delay is an integer.

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4.1 Definition of Phase delays

- Now consider the case when the input signal contains many sinusoidal components with different frequencies that are not harmonically related
- In this case, each component of the input will go through different phase delays when processed by a frequency-selective LTI discrete-time system
- Then, the output signal, in general, will not look like the input signal.

4.2 Definition of Group delays

- To develop the necessary expression, consider a discrete-time signal $x[n]$ obtained by a double-sideband suppressed carrier (DSB-SC) modulation with a carrier frequency ω_c of a low-frequency sinusoidal signal of frequency ω_0

$$x[n] = A\cos(\omega_0 n)\cos(\omega_c n)$$
$$= \frac{A}{2}\cos(\omega_l n) + \frac{A}{2}\cos(\omega_u n)$$
$$\omega_l = \omega_c - \omega_0 \quad \omega_u = \omega_c + \omega_0$$

4.2 Definition of Group delays

- Let the input be processed by an LTI discrete-time system with a frequency response $H(e^{j\omega})$ satisfying the condition

$$\left|H(e^{j\omega})\right| \cong 1 \quad \text{for} \quad \omega_l \le |\omega| \le \omega_u$$

- The output $y[n]$ is then given by

$$y[n] = \frac{A}{2}\cos\left(\omega_l n + \theta(\omega_l)\right) + \frac{A}{2}\cos\left(\omega_u n + \theta(\omega_u)\right)$$
$$= A\cos\left(\omega_c n + \frac{\theta(\omega_u)+\theta(\omega_l)}{2}\right)\cos\left(\omega_0 n + \frac{\theta(\omega_u)-\theta(\omega_l)}{2}\right)$$

4.2 Definition of Group delays

- Note: The output is also in the form of a modulated carrier signal with the same carrier frequency ω_c and the same modulation frequency ω_0 as the input.
- However, the two components have different phase lags relative to their corresponding components in the input

4.2 Definition of Group delays

- Now consider the case when the modulated input is a narrow band signal with the frequencies ω_l and ω_u very close to the carrier frequency ω_c, i.e. ω_0 is very small
- In the neighborhood of ω_c we can express the phase response $\theta(\omega)$ as

$$\theta(\omega) \cong \theta(\omega_c) + \left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c} \cdot (\omega - \omega_c)$$

by making a Taylor's series expansion and keeping only the first two terms

4.2 Definition of Group delays

- Using the above formula, we now evaluate the time delays of the carrier and the modulating components:
- In the case of the carrier signal we have

$$-\frac{\theta(\omega_u)+\theta(\omega_l)}{2\omega_c} \cong -\frac{\theta(\omega_c)}{\omega_c}$$

which is seen to be the same as the phase delay if only the carrier signal is passed through the system

4.2 Definition of Group delays

- In the case of the modulating component we have

$$-\frac{\theta(\omega_u)-\theta(\omega_l)}{2\omega_0} = -\frac{\theta(\omega_u)-\theta(\omega_l)}{\omega_u-\omega_l} \cong -\left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c}$$

- The parameter $\boxed{\tau_g(\omega_c) = -\left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c}}$

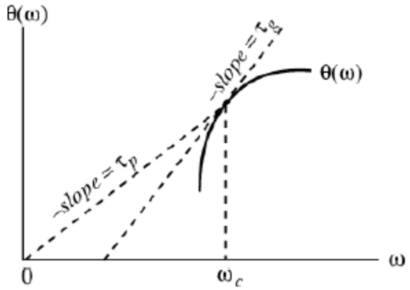
is called the group delay or envelope delay caused by the system at $\omega = \omega_c$

4.2 Definition of Group delays

- The **group delay** is a measure of the linearity of the phase function as a function of the frequency
- It is the time delay between the waveforms of underlying continuous-time signals whose sampled versions, sampled at $t = nT$, are precisely the input and the output discrete-time signals
- If the phase function and the angular frequency ω are in *radians per second*, then the group delay is in *seconds*

4.3 Phase and Group delays

- Figure below illustrates the evaluation of the phase delay and the group delay

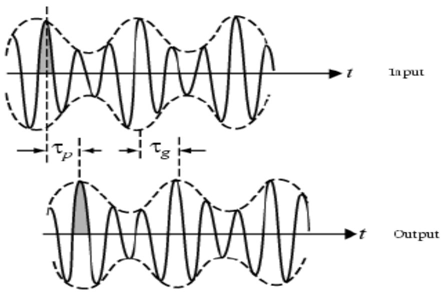


4.3 Definition of Phase and Group delays

- As can be seen, the group delay $\tau_g(\omega_0)$ is the negative of the slope of the phase function at a frequency ω_0 ,
- Whereas the phase delay $\tau_p(\omega_0)$ is the negative of the slope of the straight line from the origin to the point $[\omega_0, \theta(\omega_0)]$ on the phase function plot.

4.3 Phase and Group delays

- Figure below shows the waveform of an amplitude-modulated input and the output generated by an LTI system



4.3 Phase and Group delays

Example

- The phase function of the FIR Filter

$$y[n] = \alpha x[n] + \beta x[n-1] + \alpha x[n-2]$$

is $\theta(\omega) = -\omega$

- Hence its group delay is given by $\tau_g(\omega) = 1$

Example

- Consider an LTI continuous-time system with a frequency response $H_a(j\Omega)=|H_a(j\Omega)|e^{j\theta_a(\Omega)}$ and excited by a narrow-band amplitude modulated continuous-time signal given by

$$x_a(t)=a(t)\cos(\Omega_c t)$$

where $a(t)$ is a lowpass modulating signal with a band-limited continuous-time Fourier transform given by $|A(j\Omega)|=0, \quad |\Omega|>\Omega_0$

- We assume that in the frequency range

$$\Omega_c-\Omega_0<|\Omega|<\Omega_c+\Omega_0$$

the frequency response of the continuous-time system has a constant magnitude and a linear phase; that is,

$$|H_a(j\Omega)|=|H_a(j\Omega_c)|$$

$$\theta_a(\Omega)=\theta_a(\Omega_c)-(\Omega-\Omega_c)\left.\frac{d\theta_a(\Omega)}{d\Omega}\right|_{\Omega=\Omega_c}$$

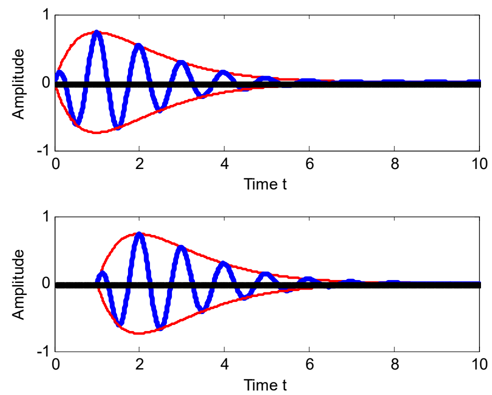
$$=-\Omega_c\tau_p(\Omega_c)+(\Omega-\Omega_c)\tau_g(\Omega_c)$$

- The continuous-time Fourier transform of the input signal $x_a(t)$ is of the form

$$X_a(j\Omega)=\frac{1}{2}\left\{A\left[j(\Omega+\Omega_c)\right]+A\left[j(\Omega-\Omega_c)\right]\right\}$$

$y_a(t)$ of the LTI continuous-time system is given by

$$y_a(t)=a\left[t-\tau_g(\Omega_c)\right]\cos\Omega_c\left[t-\tau_p(\Omega_c)\right]$$



4.3 Phase and Group delays

- The group delay $\tau_g(\Omega_c)$ is precisely the delay of the envelope $a(t)$ of the input signal $x_a(t)$, whereas the phase delay $\tau_p(\Omega_c)$ is the delay of the carrier signal.
- The carrier component at the output is delayed by the phase delay and the envelope of the output is delayed by the group delay relative to the waveform of the continuous-time input signal in the previous slide

4.3 Phase and Group delays

- The waveform of the underlying continuous time output shows distortion when the group delay of the LTI system is not constant over the bandwidth of the modulated signal
- In the case of LTI systems with a wide-band frequency response, the two delays do not have any physical meanings.

4.3 Phase and Group delays

- If the distortion is unacceptable, a delay equalizer is usually cascaded with the LTI system so that the overall group delay of the cascade is approximately linear over the band of interest.
- To keep the magnitude response of the parent LTI system unchanged, the equalizer must have a constant magnitude response at all frequencies

4.4 Phase and Group delay Computation Using Matlab

- Phase delay and group delay can be computed using the function **`phasedelay, grpdelay`** respectively
- Figures in the next slide shows the phase delay and group delay of the DTFT

$$H(e^{j\omega}) = \frac{0.1367(1-e^{-j2\omega})}{1-0.5335e^{-j\omega}+0.7265e^{-j2\omega}}$$

4.4 Phase and Group delay Computation Using Matlab

$$H(e^{j\omega}) = \frac{0.136728736(1-e^{j\omega})}{1-0.53353098e^{-j\omega}+0.726542528e^{-j2\omega}}$$