Chapter 3

Discrete-Time
Fourier Transform (DTFT)

Chapter 3

Two major topics of this chapter:

- Discrete-Time Fourier Transform
  - Discrete-Time Fourier Transform (DTFT)
  - Basic Properties & Symmetry Relation
  - DTFT Theorems
- Discrete-Time Signals and Systems in Frequency Domain
  - Spectrum Analysis
  - Frequency Response of an LTI Discrete-Time System
  - Phase and Group Delay
  - The Unwrapped Function

Chapter 3A

Discrete-Time
Fourier Transform

Part A: DTFT

1. *The Continuous-Time Fourier Transform*

1.1 Definition **1.2 Energy Density Spectrum**

1.3 Band-limited Continuous-Time Signals

2. The Discrete-Time Fourier Transform

2.1 Definition **2.2 Basic Properties**

2.3 Symmetry Relations **2.4 Convergence Condition**

3. DTFT Computation Using MATLAB

4. DTFT Theorems

Definition

- The CTFT of a continuous-time signal $x_a(t)$ is given by analysis equation

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$$

- Often referred to as the Fourier Spectrum or simply the Spectrum of the continuous-time signal

- The inverse CTFT of a Fourier Transform $X_a(j\Omega)$ is given by synthesis equation

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t} d\Omega$$

- Often referred to as the Fourier integral
- A CTFT pair will be denoted as

$$x_a(t) \xleftrightarrow{\text{CTFT}} X_a(j\Omega)$$

- Ω is **real** and denotes the continuous-time **angular frequency variable** in rad/s
- In general, the CTFT is a **continuous complex function** of Ω in the range $-\infty < \Omega < \infty$
- It can be expressed in the polar form as

$$X_a(j\Omega) = |X_a(j\Omega)| e^{j\theta_a(\Omega)}$$

where

$$\theta_a(\Omega) = \arg\{X_a(j\Omega)\}$$

- The quantity $|X_a(j\Omega)|$ is called the magnitude spectrum and the quantity $\theta_a(\Omega)$ is called the phase spectrum; both spectrums are **real functions** of Ω
- In general, the CTFT $X_a(j\Omega)$ exists if $x_a(t)$ satisfies the Dirichlet Conditions given on the next slide:

Dirichlet Conditions

(a) The signal $x_a(t)$ has a **finite number of discontinuities** and a **finite number of maxima and minima** in any finite interval

(b) The signal is **absolutely integrable**, i.e.

$$\int_{-\infty}^{\infty} |x_a(t)|\, dt < \infty$$

- If the Dirichlet Conditions are satisfied, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t}\, d\Omega$$

converges to $x_a(t)$ at values of t except at values of t where $x_a(t)$ has discontinuities

- It can be shown that if $x_a(t)$ is absolutely integrable, then $|X_a(j\Omega)| < \infty$ proving the existence of the CTFT

Example

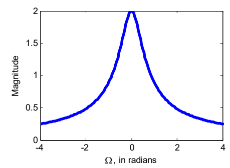
- Find the CTFT of the following signal

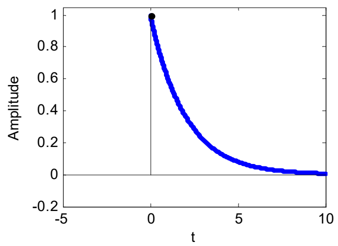
$$x_a(t) = \begin{cases} e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

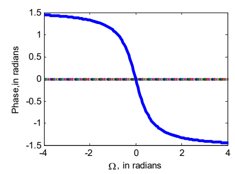
- Solution:

$$X_a(j\Omega) = \int_0^{\infty} e^{-\alpha t} e^{-j\Omega t}\, dt == \frac{1}{\alpha + j\Omega} e^{-(\alpha + j\Omega)t} \Big|_0^{\infty} = \frac{1}{\alpha + j\Omega}$$

$$x_a(t) = \begin{cases} e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \qquad \alpha = 0.5$$







Example

- Find the CTFT of the following signal

$$x_a(t) = \delta(t) \leftrightarrow \Delta(j\Omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt = 1$$

$$x_a(t) = \delta(t - t_0)$$

$$\updownarrow$$

$$X_a(j\Omega) = \int_{-\infty}^{\infty} \delta(t - t_o) e^{-j\Omega t} dt = e^{-j\Omega t_o}$$

- The total energy $\mathcal{E}_x$ of a finite energy continuous-time complex signal $x_a(t)$ is given by

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x_a(t)|^2 dt = \int_{-\infty}^{\infty} x_a(t) x_a^*(t) dt$$

- The above expression can be rewritten as

$$\mathcal{E}_x = \int_{-\infty}^{\infty} x_a(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) e^{-j\Omega t} d\Omega \right] dt$$

- Interchanging the order of the integrations, we get

$$\mathcal{E}_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) \left[\int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt \right] d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) X_a(j\Omega) d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 d\Omega$$

- Hence

$$\int_{-\infty}^{\infty} |x_a(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 d\Omega$$

- The above relation is more commonly known as the Parseval's relation for finite energy continuous-time signals

1.2 Energy Density Spectrum

- The quantity $|X_a(j\Omega)|^2$ is called the energy density spectrum of $x_a(t)$ and usually denoted as

$$S_{xx}(\Omega) = \left|X_a(j\Omega)\right|^2$$

- The energy over a specified range of frequencies $\Omega_a \leq \Omega \leq \Omega_b$ can be computed using

$$\mathcal{E}_{x,r} = \frac{1}{2\pi}\int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega)d\Omega$$

1.3 Band-limited Continuous-Time Signals

- A full-band, finite-energy, continuous-time signal has a spectrum occupying the whole frequency range $-\infty<\Omega<\infty$
- A band-limited continuous-time signal has a spectrum that is limited to a portion of the frequency range $-\infty<\Omega<\infty$

1.3 Band-limited Continuous-Time Signals

- An ideal band-limited signal has a spectrum that is zero outside a finite frequency range $\Omega_a \leq |\Omega| \leq \Omega_b$, that is

$$X_a(j\Omega) = \begin{cases} 0, & 0 \leq |\Omega| \leq \Omega_a \\ 0, & \Omega_b \leq |\Omega| \leq \infty \end{cases}$$

- However, an ideal band-limited signal cannot be generated in practice (*Why?*)

1.3 Band-limited Continuous-Time Signals

- Band-limited signals are classified according to the frequency range where most of the signal's is concentrated
- A lowpass, continuous-time signal has a spectrum occupying the frequency range $|\Omega| \leq \Omega_p < \infty$ where Ω_p is called the bandwidth of the signal

1.3 Band-limited Continuous-Time Signals

- A highpass, continuous-time signal has a spectrum occupying the frequency range $0<\Omega_p \le |\Omega| < \infty$ where the bandwidth of the signal is from Ω_p to ∞
- A bandpass, continuous-time signal has a spectrum occupying the frequency range $0<\Omega_L \le |\Omega| \le \Omega_H < \infty$ where $\Omega_H - \Omega_L$ is the bandwidth
- **A precise definition of the bandwidth depends on applications.**

2.1 Definition of DTFT

Definition

- The discrete-time Fourier transform (DTFT) $X(e^{j\omega})$ of a sequence $x[n]$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- In general, $X(e^{j\omega})$ is a continuous complex function of the real variable ω

2.1 Definition of DTFT

- From the definition:

$$X(e^{j(\omega+2k\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2k\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j2k\pi n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j\omega})$$

- It should be noted that DTFT is a periodic function of ω with a period 2π

2.1 Definition of DTFT

Example

- Determine the DTFT of the unit sample sequence $\{\delta[n]\}$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = \delta[0] = 1$$

- Consider the causal sequence $x[n]=\alpha^n u[n]$ $|\alpha|<1$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \frac{1}{1-\alpha e^{-j\omega}}$$

2.1 Definition of DTFT

- The Inverse discrete-time Fourier transform (IDTFT) of $X(e^{j\omega})$ is given by

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

Proof

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

- It represents the Fourier series expansion of the periodic function $X(e^{j\omega})$.
- $x[n]$ can be computed from $X(e^{j\omega})$ using the Fourier integral.

2.2 Basic Properties

- In general, $X(e^{j\omega})$ is a complex function of the real variable ω and can be written as

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$$

$$X_{re}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{j\omega})\}$$

$$X_{im}(e^{j\omega}) = \frac{1}{2j}\{X(e^{j\omega}) - X^*(e^{j\omega})\}$$

- $X_{re}(e^{j\omega})$ and $X_{im}(e^{j\omega})$ are, respectively, the real and imaginary parts of $X(e^{j\omega})$, and are real functions of ω

2.2 Basic Properties

- $X(e^{j\omega})$ can alternately be expressed as

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$

where

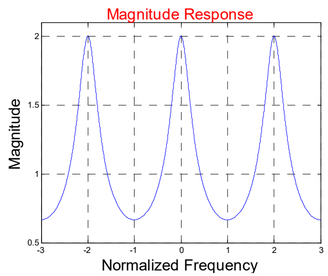
$$\theta(\omega) = \arg\left\{X(e^{j\omega})\right\}$$

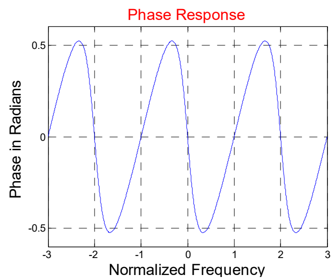
  - $\left|X(e^{j\omega})\right|$ is called the magnitude function
  - $\theta(\omega)$ is called the phase function
- Both quantities are again real functions of ω

2.1 Definition of DTFT

Simulation Results $x[n] = 0.5^n \mu[n]$

- The magnitude and phase of the DTFT $X(e^{j\omega})=1/(1-0.5e^{-j\omega})$ are shown below





2.2 Basic Properties

- In many applications, the DTFT is called the Fourier spectrum
- Likewise, $|X(e^{j\omega})|$ and $\theta(\omega)$ are called the magnitude and phase spectra

2.2 Basic Properties

- Note that, for any integer k

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$
$$= \left|X(e^{j\omega})\right| e^{j(\theta(\omega)+2k\pi)}$$

- $\theta(\omega)$ is also a a periodic function of ω with a period 2π

The phase function $\theta(\omega)$ cannot be uniquely specified for any DTFT

2.2 Basic Properties

- Unless otherwise stated, we shall assume that the phase function $\theta(\omega)$ is restricted to the following range of values:

$$-\pi \leq \theta(\omega) \leq \pi$$

called the principal value

2.2 Basic Properties

- The relations between rectangular and polar forms of $X(e^{j\omega})$ are given below:

$$X_{\text{re}}(e^{j\omega}) = \left|X(e^{j\omega})\right| \cos\theta(\omega)$$
$$X_{\text{im}}(e^{j\omega}) = \left|X(e^{j\omega})\right| \sin\theta(\omega)$$
$$\left|X(e^{j\omega})\right|^2 = X(e^{j\omega})X^*(e^{j\omega}) = X_{\text{re}}^2(e^{j\omega}) + X_{\text{im}}^2(e^{j\omega})$$
$$\tan\theta(\omega) = \frac{X_{\text{im}}(e^{j\omega})}{X_{\text{re}}(e^{j\omega})}$$

Complex Sequences

- For a given sequence $x[n]$ with a Fourier transform $X(e^{j\omega})$, the Fourier transforms of its time-reversed sequence $x[-n]$ and the complex conjugate sequence $x^*[n]$ are

$$x[-n] \leftrightarrow \sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n} = \sum_{m=\infty}^{-\infty} x[m]e^{j\omega m} = X(e^{-j\omega})$$

$$x^*[n] \leftrightarrow \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} = \left(\sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}\right)^* = X^*\left(e^{-j\omega}\right)$$

$$x^*[-n] \leftrightarrow \sum_{n=-\infty}^{\infty} x^*[-n]e^{-j\omega n} = \left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right)^* = X^*\left(e^{j\omega}\right)$$

Complex Sequences

- A Fourier transform $X(e^{j\omega})$ is defined to be a conjugate-symmetric function of ω if

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

- The Fourier transform $X(e^{j\omega})$ is a conjugate-antisymmetric function of ω if

$$X(e^{j\omega}) = -X^*(e^{-j\omega})$$

Recall

- A complex sequence $x[n]$ can be rewritten as

$$x[n] = x_{od}[n] + x_{ev}[n] \qquad x[n] = x_{re}[n] + jx_{im}[n]$$

$$x[n] = x_{cs}[n] + x_{ca}[n]$$

- An Fourier transform $X(e^{j\omega})$ can be rewritten as

$$X(e^{j\omega}) = X_{ev}(e^{j\omega}) + X_{od}(e^{j\omega})$$

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$$

$$X(e^{j\omega}) = X_{cs}(e^{j\omega}) + X_{ca}(e^{j\omega})$$

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_{cs}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\mathrm{Im}\{x[n]\}$	$X_{ca}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{cs}[n]$	$X_{re}(e^{j\omega})$
$x_{ca}[n]$	$jX_{im}(e^{j\omega})$

2.3 Symmetry Relations

Real Sequences

- The real part $X_{re}(e^{j\omega})$ and imaginary part $X_{im}(e^{j\omega})$ of the Fourier transform of a real sequence are, respectively, even and odd functions of ω.

- $|X(e^{j\omega})|$ is an even function of ω. $\theta(\omega)$ is an odd function of ω.

2.3 Symmetry Relations (Real sequences)

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$
$x_{ev}[n]$	$X_{re}(e^{j\omega})$
$x_{od}[n]$	$jX_{im}(e^{j\omega})$
Symmetry relatin	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$ $X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

2.4 Convergence Condition

- The Fourier transform $X(e^{j\omega})$ of $x[n]$ is said to exist if the series

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

converges in some sense.

- Uniform convergence
- Mean-square convergence

2.4 Convergence Condition

- If $x[n]$ is an absolutely summable sequence, i.e., if

$$\sum_{n=\infty}^{\infty} |x[n]| < \infty$$

Then

$$|X(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right| < \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- Thus, the absolute summability of $x[n]$ is a *sufficient condition* for the existence of the DTFT

Example

- The sequence $x[n] = \alpha^n \mu[n]$ for $|\alpha| < 1$ is absolutely summable as

$$\sum_{n=-\infty}^{\infty} |\alpha^n \mu[n]| = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1-|\alpha|} < \infty$$

and its DTFT $X(e^{j\omega})$ therefore converges to $1/(1-\alpha e^{-j\omega})$ uniformly.

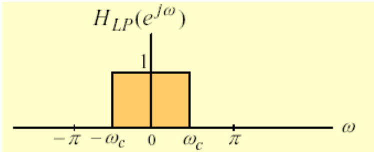
2.4 Convergence Condition

Example

- Consider the DTFT

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

shown below



2.4 Convergence Condition

Example

- The inverse DTFT of $H_{LP}(e^{j\omega})$ is given by

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right) = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- $h_{LP}[n]$ is a **finite-energy sequence,** but it is **not absolutely summable.**

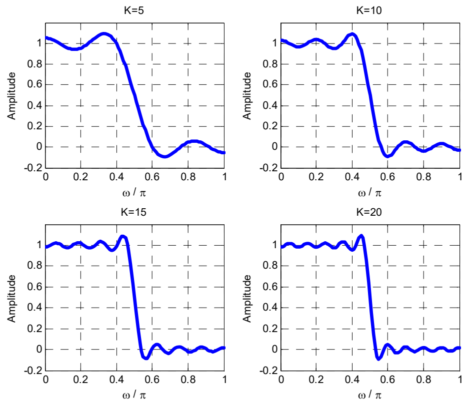
2.4 Convergence Condition

- The mean-square convergence property of the sequence $h_{LP}[n]$ can be further illustrated by examining the plot of the function

$$H_{LP,K}(e^{j\omega}) = \sum_{n=-K}^{-K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

for various values of K as shown next

2.4 Convergence Condition



2.4 Convergence Condition

- As can be seen from these plots, there are ripples in the plot of $H_{LP,K}(e^{j\omega})$ around both sides of the point $\omega = \omega_c$
- The number of ripples increases as K increases with the height of the largest ripple remaining the same for all values of K.

2.4 Convergence Condition

- As K goes to infinity, the condition

$$\lim_{K\to\infty} \int_{-\pi}^{\pi} \left| H_{LP}(e^{j\omega}) - H_{LP,K}(e^{j\omega}) \right|^2 d\omega = 0$$

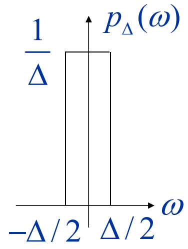
  holds indicating the convergence of $H_{LP,K}(e^{j\omega})$ to $H_{LP}(e^{j\omega})$
- The oscillatory behavior of $H_{LP,K}(e^{j\omega})$ approximating $H_{LP}(e^{j\omega})$ in the mean square sense at a point of discontinuity is known as the Gibbs phenomenon.

2.4 Convergence Condition

- The Fourier transform can also be defined for a certain class of sequences that are neither absolutely summable nor square-summable.
  - the unit step sequence
  - the sinusoidal sequence
  - the complex exponential sequence
- For this type of sequences, DTFT representation is possible using the Dirac delta function $\delta(\omega)$

- A Dirac delta function $\delta(\omega)$ is a function of ω with infinite height, zero width, and unit area
- It is the limiting form of a unit area pulse function $p_\Delta(\omega)$ as Δ goes to zero satisfying

$$\lim_{\Delta\to 0}\int_{-\infty}^{\infty} p_\Delta(\omega)d\omega = \int_{-\infty}^{\infty}\delta(\omega)d\omega$$



- Consider the complex exponential sequence

$$x[n] = e^{j\omega_0 n}$$

- Its DTFT is given by

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2k\pi)$$

where $\delta(\omega)$ is an impulse function of ω and

$$-\pi \le \omega_0 \le \pi$$

- The function

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2k\pi)$$

is a periodic function of ω with a period 2π and is called a periodic impulse train

- To verify that $X(e^{j\omega})$ given above is indeed the DTFT of $x[n] = e^{j\omega_0 n}$, we compute the inverse DTFT of $X(e^{j\omega})$

2.4 Convergence Condition

- Thus

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2k\pi) e^{j\omega_0 n} d\omega$$

$$= \int_{-\pi}^{\pi} 2\pi\delta(\omega - \omega_0) e^{j\omega_0 n} d\omega = e^{j\omega_0 n}$$

where we have used the sampling property of the impulse function $\delta(\omega)$

Commonly Used DTFT Pairs

Some Common Discrete-Time Fourier Transform Pairs

Sequence	Transform
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$1(\forall n)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$\frac{\sin(\omega_0 n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_0 \\ 0, \omega_0 < \lvert\omega\rvert < \pi \end{cases}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
$\cos(\omega_0 n + \phi)$	$\pi \sum_{k=-\infty}^{\infty} \left[e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)\right]$
$x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{Otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$

3 DTFT Computation Using MATLAB

- The Signal Processing Toolbox in Matlab includes a number of M-files to aid in the DTFT-based analysis of discrete-time signals.
- Specifically, the functions that can be used are freqz, abs, angle, and unwrap.
- In addition, the built-in Matlab functions real and imag are also useful in some applications.

3 DTFT Computation Using MATLAB

- The function `freqz` can be used to compute the values of the DTFT of a sequence, described as a rational function in the form of

$$X\left(e^{j\omega}\right) = \frac{p_0 + p_1 e^{-j\omega} + \cdots + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + \cdots + d_N e^{-j\omega N}}$$

at a prescribed set of discrete frequency points $\omega = \omega_l$.

3 DTFT Computation Using MATLAB

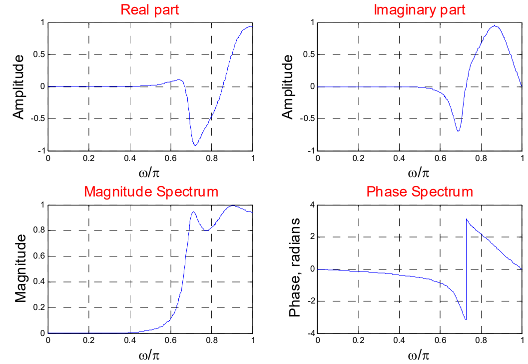
- For example, the statement

$$H = freqz(num, den, \omega)$$

returns the frequency response values as a vector ***H*** of a DTFT defined in terms of the vectors ***num*** and ***den*** containing the coefficients $\{p_i\}$ and $\{d_i\}$, respectively at a prescribed set of frequencies between 0 and 2π given by the vector ω.

- For example p=[0.008 −0.033 0.05 −0.033 0.008]
  d=[1 2.37 2.7 1.6 0.41]

3 DTFT Computation Using MATLAB



- Exercise： Program 3_1.m

4.1 DTFT Theorems

- Linearity
- Time-Reversal

$$g[-n] \Leftrightarrow G\left(e^{-j\omega}\right)$$

- Shifting (in time and in frequency domain)

$$g[n-n_0] \Leftrightarrow e^{-j\omega n_0} G\left(e^{j\omega}\right)$$

$$e^{j\omega_0 n} g[n] \Leftrightarrow G\left(e^{j(\omega-\omega_0)}\right)$$

4.1 DTFT Theorems

- Differentiation

$$ng[n] \Leftrightarrow j\frac{dG\left(e^{j\omega}\right)}{d\omega}$$

- Convolution (in time and in frequency domain)

$$\boxed{g[n] * h[n] \Leftrightarrow G\left(e^{j\omega}\right) \cdot H\left(e^{j\omega}\right)}$$

$$\boxed{g[n]h[n] \Leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} G\left(e^{j\theta}\right) H\left(e^{j(\omega-\theta)}\right) d\theta}$$

4.1 DTFT Theorems

- Area Theorem (simple but useful)

$$x[0]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{j\omega}\right)d\omega \qquad X\left(e^{j0}\right)=\sum_{n=-\infty}^{\infty}x[n]$$

- Parseval's Theorem

$$\sum_{n=-\infty}^{\infty}g[n]h^{*}[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}G\left(e^{j\omega}\right)H^{*}\left(e^{j\omega}\right)d\omega$$

Corollary——Energy is preserved

$$\sum_{n=-\infty}^{\infty}\left|x[n]\right|^{2}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X\left(e^{j\omega}\right)\right|^{2}d\omega$$

Type of Property	Sequence	DTFT
	$g(n)$ $h(n)$	$G(e^{j\omega})$ $H(e^{j\omega})$
Linearity	$\alpha g(n)+\beta h(n)$	$\alpha G(e^{j\omega})+\beta H(e^{j\omega})$
Time-shifting	$g(n-n_0)$	$e^{-j\omega n_0}G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_0 n}g(n)$	$G(e^{j(\omega-\omega_0)})$
Differentiation in frequency	$ng(n)$	$j\frac{dG(e^{j\omega})}{d\omega}$
Convolution	$g(n)*h(n)$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g(n)h(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi}G(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$
Parseval's relation	$\sum_{n=-\infty}^{\infty}g(n)h^{*}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}G(e^{j\omega})H^{*}(e^{j\omega})d\omega$	

4.1 DTFT Theorems

Example

Determine the DFT $Y(e^{j\omega})$ of $y[n]=(n+1)a^{n}u[n]$ $(|a|<1)$

Step 1: Let $x[n]=a^{n}u[n]$. Therefore

$$y[n]=nx[n]+x[n]$$

Step 2: Calculate the DTFT $X(e^{j\omega})$

$$X(e^{j\omega})=\frac{1}{1-ae^{-j\omega}}$$

4.1 DTFT Theorems

Step 3: Calculate the DTFT of $nx[n]$

$$j\frac{dX(e^{j\omega})}{d\omega}=j\frac{-aje^{-j\omega}}{\left(1-ae^{-j\omega}\right)^{2}}=\frac{ae^{-j\omega}}{\left(1-ae^{-j\omega}\right)^{2}}$$

Step 4: Calculate the DTFT $Y(e^{j\omega})$ of $y[n]$

$$Y\left(e^{j\omega}\right)=\frac{ae^{-j\omega}}{\left(1-ae^{-j\omega}\right)^{2}}+\frac{1}{1-ae^{-j\omega}}=\frac{1}{\left(1-ae^{-j\omega}\right)^{2}}$$

Example

- Determine the DTFT $V(e^{j\omega})$ of the sequence $v[n]$ defined by

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

Solution: Using the time-shifting property, we observe that the DTFT of $\delta[n-1]$ is $e^{-j\omega}$ and the DTFT of $v[n-1]$ is $e^{-j\omega}V(e^{j\omega})$

- Using the linearity property we then obtain the frequency-domain representation of

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

as

$$d_0 V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = p_0 + p_1 e^{-j\omega}$$

- Solving the above equation we get

$$V(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega}}{d_0 + d_1 e^{-j\omega}}$$

- According to the convolution theorem

$$y[n] = x[n] * h[n] \Leftrightarrow Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) H\left(e^{j\omega}\right)$$

- An implication of this result is that the linear convolution $y[n]$ of the sequences $x[n]$ and $h[n]$ can be performed as follows:

Step 1: Compute the DTFTs $X(e^{j\omega})$ and $H(e^{j\omega})$ of the sequences $x[n]$ and $h[n]$, respectively.

Step 2: Form the DTFT $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

Step 3: Compute the IDTFT $y[n]$ of $Y(e^{j\omega})$