Lecture 13, Elementary counting; Stirling numbers

Qi Chen, Jingliang Gao

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A counting problem

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Think of the balls as being colored blue and line them up in front of the boxes that they will go into. Then insert a red ball between two consecutive boxes. We end up with a line of $n + k - 1$ balls, $k - 1$ of them red, describing the situation. So the answer to the problem is $\binom{n+k-1}{k-1}$.

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Theorem

The number of solutions of the equation

$$x_1 + x_2 + \cdots + x_k = n \tag{1}$$

in nonnegative integers is $\binom{n+k-1}{k-1}$.

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Proof.

Replace x_i by $y_i \triangleq x_i - 1$. Then $\sum y_i = n - k$. Apply Theorem 13.1. □

Example 13.1

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- Let $x_1 < x_2 < \cdots < x_r$ be such a sequence.
- Define $y_1 \triangleq x_1$, $y_i \triangleq x_i - x_{i-1} - 1$, $2 \leq i \leq r$, $y_{r+1} \triangleq n - x_r + 1$.
- Then the y_i are positive integers and $\sum_{i=1}^{r+1} y_i = n - r + 2$.
- By the Corollary to Theorem 13.1, we see that there are $\binom{n-r+1}{r}$ solutions.

Example 13.2

In how many ways can we arrange r_1 balls of color 1, r_2 balls of color 2, $\cdots$, r_k balls of color k in a sequence of length $n \triangleq r_1 + r_2 + \cdots + r_k$?

Example 13.2

In how many ways can we arrange r_1 balls of color 1, r_2 balls of color 2, $\cdots$, r_k balls of color k in a sequence of length $n \triangleq r_1 + r_2 + \cdots + r_k$?

- If we number the balls 1 to n, then there are $n!$ arrangements.
- Since we ignore the numbering, any permutation of the set of r_i balls of color i, $1 \leq i \leq k$, produces the same arrangement.
- So the answer to the question is the multinomial coefficient $\binom{n}{r_1, \cdots, r_K}$.

Example 13.3

Example

We wish to split $\{1, 2, \cdots, n\}$ in to b_1 subsets of size 1, b_2 subsets of size 2, $\cdots$, b_k subsets of size k. Here $\sum_{i=1}^{k} ib_i = n$.

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- The same argument as used in Example 13.2 applies.
- Furthermore, the subsets of the same cardinality can be permuted among themselves without changing the configuration.
- So the solution is

$$\frac{n!}{b_1!b_2!\cdots b_k!(1!)^{b_1}(2!)^{b_2}\cdots(k!)^{b_k}}.$$

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- Since there are $\binom{n}{i}$ subsets of i. We apparently must calculate $\sum_{i=0}^{n} i\binom{n}{i}$.
- By differentiating $(1+x)^n$, we find

$$\sum_{i=1}^{k} i\binom{n}{i}x^{i-1} = n(1+x)^{n-1}$$

and substitution of $x = 1$ yields the answer $S = n \cdot 2^{n-1}$.

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  and substitution of $x = 1$ yields the answer $S = n \cdot 2^{n-1}$.
- A set A and its complement together contain n elements and there are exactly 2^{n-1} such pairs.

Example 13.5

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Proof.

- One can calculate this sum by determining the coefficient of x^n in $(1+x)^n(1+x)^n$ and using the binomial formula.
- Each side of the above equation just counts (in two ways) the number of ways of selecting n balls from a set consisting of n red balls and n blue balls.

□

Example 13.6

Example

How many sequences $A_1, \cdots, A_k$ are there for which $A_i \subseteq \{1, 2, \cdots, n\}, 1 \leq i \leq k$, and $\cup_{i=1}^{k} A_i = \{1, 2, \cdots, n\}$?

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Solution

- Since we wish to avoid that j, $1 \leq j \leq n$, is not an element of the union of the A_i's, we are tempted to use inclusion-exclusion.
- If we choose i elements from $\{1, 2, \cdots, n\}$ and consider all sequences $A_1, \cdots, A_k$ not containing any of these i elements, then we find $(2^{n-i})^k$ sequences.
- So by Theorem 10.1, the solution to the problem is

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} 2^{(n-i)k} = (2^k - 1)^n.$$

Stirling number of the first kind

Definition

- Let $c(n,k)$ denote the number of permutations $\pi \in S_n$ with exactly k cycles.
- Furthermore define $c(0,0) = 1$ and $c(n,k) = 0$ if $n \leq 0$ or $k \leq 0$, $(n,k) \neq (0,0)$.
- The Stirling numbers of the first kind $s(n,k)$ are defined by

$$s(n,k) \triangleq (-1)^{n-k} c(n,k). \tag{2}$$

Theorem 13.2

Theorem

The numbers $c(n,k)$ satisfy the recurrence relation

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1). \tag{3}$$

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$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1). \qquad (3)$$

Proof.

- If π is a permutation in S_{n-1} with k cycles, then there are $n-1$ positions where we can insert the integer n to produce a permutation $\pi' \in S_n$ with k cycles.
- We can also adjoin (n) as a cycle to any permutation in S_{n-1} with $k-1$ cycles. This accounts for the two terms on the right-hand side of (3).

□

Theorem 13.3

Theorem

For $n \geq 0$, we have

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1) \tag{4}$$

and

$$\sum_{k=0}^{n} s(n,k)x^k = (x)_n. \tag{5}$$

where

$$(x)_n \triangleq k(k-1)...(k-n+1) = \frac{k!}{(k-n)!}$$

Theorem 13.3

Proof.

- Write the right-hand side of (4) as

$$F_n(x) = \sum_{k=0}^{n} b(n,k)x^k.$$

- Clearly $b(0,0) = 1$. Define $b(n,k) \triangleq 0$ if $n \leq 0$ or $k \leq 0$, $(n,k) \neq (0,0)$.
- Since

$$\begin{aligned} F_n(x) &= (x+n-1)F_{n-1}(x) \\ &= \sum_{k=1}^{n} b(n-1,k-1)x^k + (n-1)\sum_{k=0}^{n-1} b(b-1,k)x^k, \end{aligned}$$

we see that the numbers $b(n,k)$ satisfy the same recurrence relation as the $c(n,k)$, namely (3).

Theorem 13.3

Proof(cont.)

- Since the numbers are equal if $n \leq 0$ or $k \leq 0$, they are equal for all n and k.
- To prove (5) replace x by $-x$ and use (2).

□

Stirling number of the second kind

Definition

Denote by $P(n, k)$ the set of all partitions of an n-set into k nonempty subsets (blocks). Then

$$S(n, k) \triangleq |P(n, k)| \tag{6}$$

Again we have $S(0, 0) = 1$ and take the numbers to be 0 for all values of the parameters not covered by the previous definition.

Theorem 13.4

Theorem

The Stirling numbers of the second kind satisfy the relation

$$S(n,k) = kS(n-1,k) + S(n-1,k-1). \tag{7}$$

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$$S(n, k) = kS(n-1, k) + S(n-1, k-1). \tag{7}$$

Proof.

A partition of the set $\{1, 2, \cdots, n-1\}$ can be made into a partition of $\{1, 2, \cdots, n\}$

- by adjoining n to one of the blocks
- or by increasing the number of blocks by one by making $\{n\}$ a block.

□

Bell number

Definition (Bell number)

The Bell number $B(n)$ is the total number of partitions of an n-set, i.e.

$$B(n) \triangleq \sum_{k=1}^{n} S(n,k), \quad (n \geq 1). \tag{8}$$

Theorem 13.5

Theorem

For $n \geq 0$ we have

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k. \tag{9}$$

Proof.

- by (6) the number of surjective mappings from an n-set to a k-set is $k!S(n,k)$
- by Example 10.2, we have

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^k\binom{k}{i}(k-i)^n = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n \tag{10}$$

Theorem 13.5

Proof(cont.)

- let x be an integer.
- There are x_n mappings from the n-set $N \triangleq \{1, 2, \cdots, n\}$ to the x-set $\{1, 2, \cdots, x\}$.
- For any k-subset Y of $\{1, 2, \cdots, x\}$, there are $k!S(n, k)$ surjections from N to Y.
- So we find

$$x^n = \sum_{k=0}^{n} \binom{x}{k} k!S(n, k) = \sum_{k=0}^{n} S(n, k)(x)_k.$$

$\square$

Therom 13.6

Theorem

$$\sum_{n \geq k} S(n,k) \frac{x^n}{n!} = \frac{1}{k!}(e^x - 1)^k \quad (k \geq 0)$$

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$$\sum_{n\geq k} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x - 1)^k \quad (k \geq 0)$$

Proof.

- Let $F_k(x)$ denote the sum on the left-hand side.
- By (7) we have

$$F_k'(x) = kF_k(x) + F_{k-1}(x).$$

  The result now follows by induction.
- Since$S(n,1) = 1$, the assertion is true for $k = 1$.
- The induction hypothesis yields a differential equation for F_k, which with the condition $S(k,k) = 1$ has the right-hand side of the assertion as unique solution.

Theorem 13.7

Theorem

$$\sum_{n=k}^{\infty} s(n,k)\frac{z^n}{n!} = \frac{1}{k!}(\log(1+z))^k,$$

Proof.

- Since
$$(1+z)^x = e^{x\log(1+z)} = \sum_{k=0}^{\infty}\frac{1}{k!}(\log(1+z))^k x^k,$$
the right-hand side in the assertion is the coefficient of x^k in the expansion of $(1+z)^x$.
- On the other hand, we have for $|z| < 1$,
$$(1+z)^x = \sum_{0}^{\infty}\binom{x}{n}z^n = \sum_{n=0}^{\infty}\frac{1}{n!}(x)_n z^n$$
$$= \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{r=0}^{n} s(n,r)x^r = \sum_{n=r}^{\infty} s(n,r)\frac{z^n}{n!}.$$

The Relation between two types of Stirling numbers

Proposition

$$\sum_{k=m}^{n} S(n,k)s(k,m) = \delta_{m,n},$$

where

$$\delta_{m,n} = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof.

This follows immediately if we substitute (5) in (9). □