Lecture 7, Flows in networks

Qi Chen, Jingliang Gao

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Transportation network

Definition

By a transportation network, we will mean a finite directed graph D together with two distinguished vertices s and t called the source and the sink, respectively, and which is provided with a function c associating to each edge e a nonnegative real number $c(e)$ called its capacity.

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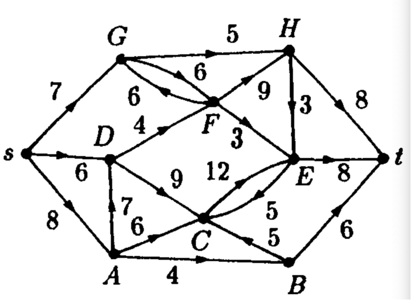

Flow in networks

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A flow in a transportation network is a function f assigning a real number $f(e)$ to each edge e such that:

- $0 \leq f(e) \leq c(e)$ for all edges e (the flow is feasible);
- for each vertex x (not the source or the sink), the sum of the values of f on incoming edges equals the sum of the values of f on outgoing edges (conservation of flow).

The sum of the values of a flow f on the edges leaving the source is called the strength of the flow (denoted by $|f|$).

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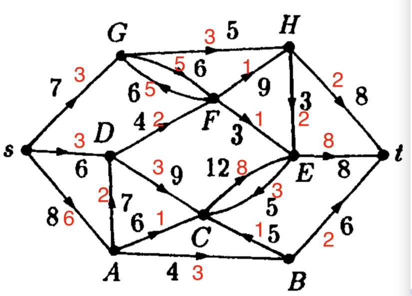

Cut of a network

Definition

By a cut separating s and t (or simply a cut), we mean here a pair (X, Y) of subsets of the vertex set $V := V(D)$ which partition V and such that $s \in X$ and $t \in Y$. We define the capacity $c(X, Y)$ of the cut to be the sum of the capacities of the edges directed from X to Y.

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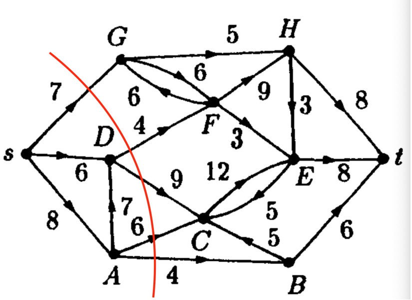

Capacity of any cut is an upper bound for the strengh of any flow

Lemma

$$|f| = f(X,Y) - f(Y,X)$$

where $f(A,B)$ denotes the sum of the value of f on all edges directed from A to B.

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Lemma

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Proof.

For $x \in V$ and $e \in E$, define

$$\phi(x, e) = \begin{cases} -1 & e \text{ is incoming to } x \\ +1 & e \text{ is outgoing from } x \\ 0 & x \text{ is not incident with } e. \end{cases}$$

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The convervation law is equivalenct to

$$\sum_{e \in E} \phi(x, e) f(e) = 0 \quad \text{for } x \neq s, t.$$

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Cont.

$$|f| = \sum_{e \in E} \phi(s,e) f(e) = \sum_{x \in X} \sum_{e \in E} \phi(x,e) f(e)$$
$$= \sum_{e \in E} f(e) \sum_{x \in X} \phi(e,x) = f(X,Y) - f(Y,X).$$

□

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$$
\begin{aligned}
|f| = \sum_{e \in E} \phi(s,e) f(e) &= \sum_{x \in X} \sum_{e \in E} \phi(x,e) f(e) \\
&= \sum_{e \in E} f(e) \sum_{x \in X} \phi(e,x) = f(X,Y) - f(Y,X).
\end{aligned}
$$

□

Remark

This lemma implies that for any flow f,

$$|f| \leq c(X,Y)$$

for any cut (X,Y) with $s \in X$ and $t \in Y$.

Max-flow min-cut theorem

Theorem (Ford and Fulkerson(1956))

In a transportation network, the maximum value of $|f|$ *over all flows* f *is equal to the minimum value of* $c(X, Y)$ *over all cuts* (X, Y).

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In a transportation network, the maximum value of $|f|$ over all flows f is equal to the minimum value of $c(X, Y)$ over all cuts (X, Y).

Proof.

- Fix a flow f.
- We shall say that the sequence $x_0, x_1, \cdots, x_{k-1}, x_k$ of distinct vertices is a special path from x_0 to x_k if for each $i, 1 \leq i \leq k$, either
  1. $e = (x_{i-1}, x_i)$ is an edge with $c(e) - f(e) > 0$; (e is unsaturated)
  2. $e = (x_i, x_{i-1})$ is an edge with $f(e) > 0$.
- Suppose there exists such a special path from s to t. Define α_i as $c(e) - f(e)$ in the first case and as $f(e)$ in the second case and let α be the minimum of these positive numbers α_i.

Max-flow min-cut theorem

Proof (cont.)

- On each edge of type (i) increase the flow value by α, and on each edge of type (ii) decrease the flow by α.
- Clearly the new flow has strength $|f| + \alpha$
- Suppose that no special path from source to sink exists with respect to some flow f_0.
- Let X_0 be the set of vertices x which can be reached from s by a special path, Y_0 the set of remaining vertices. In this way we produce a cut.
- If $x \in X_0$, $y \in Y_0$ and $e = (x, y)$ is an edge, then e must be saturated or we could adjoin y to a special path from s to x to get a special path from s to y, contradicting the definitions of X_0 and Y_0.
- If, on the other hand, $e = (y, x)$ is an edge, then, for a similar reason, $f(e)$ must be 0.

Max-flow min-cut theorem

Proof (cont.)

- In view of Lemma, we have then

$$|f_0| = f_0(X_0, Y_0) - f_0(Y_0, X_0) = c(X_0, Y_0).$$

- Now it is clear that not only can no stronger flow be obtained by our method of special paths, but that no stronger flows exist at all because $|f| \leq c(X_0, Y_0)$ for any flow f. If f_0 is chosen to be a maximum flow, then surely no special paths from s to t exist.
- Note that the constructed cut (X_0, Y_0) is a minimum cut (i.e. a cut of minimum capacity), since $c(X, Y) \geq |f_0|$ for any cut (X, Y).

□

Theorem 7.2

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If all the capacities in a transportation network are integers, then there is a maximum strength flow f for which all values $f(e)$ are integers.

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Problem 7.1

Construct a maximum flow for the transportation network of Fig. 7.1.

Theorem 7.3

Theorem

Let A be a $b \times v$ $(0,1)$-matrix with k ones per row and r ones per column (so $bk = vr$). Let α be a rational number, $0 < \alpha < 1$,such that $k' = \alpha k$ and $r' = \alpha r$ are integers. Then there is a $(0,1)$-matrix A' of size $b \times v$ with k' ones per row and r' ones per column such that entries a'_{ij} of A' are 1 only if the corresponding entries of A are 1, i.e. A' can be obtained from A by changing some ones into zeros.

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Proof.

- Construct a transportation network with vertices s(the source), $x_1, \cdots, x_b$ (corresponding to the rows of A), $y_1, \cdots, y_v$ (corresponding to the columns of A), and t (the sink).
- Edges are (s, x_i) with capacity $k, 1 \leq i \leq b$, (x_i, y_j) with capacity 1 if and only if $a_{ij} = 1$, and (y_j, t) with capacity r, $1 \leq j \leq v$.

Theorem 7.3

Proof (cont.)

- The definition ensures that there is a maximum flow with all edges saturated.
- Change the capacities of the edges from the source to k' and those of the edges to the sink to r'.
- All the capacities are integers and clearly a maximum flow exists for which the flows $f((x_i, y_j))$ are equal to α.
- By Theorem 7.2, there is also a maximum flow f^* for which all the flows are integers, i.e. $f^*((x_i, y_j)) = 0$ or 1.
- From this flow, we immediately find the required matrix A'.

□

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Proof.

- Given a circulation f, consider a circulation g that satisfies

$$\lfloor f(e) \rfloor \leq g(e) \leq \lceil f(e) \rceil \tag{1}$$

  and for which the number of edges e with $g(e)$ an integer is as large as possible subject to (1).
- Let H be the spanning subgraph of D with edge set consisting of those edges of D for which $g(e)$ is not an integer, i.e. for which strict inequality holds both times in (1).

Theorem 7.4

Proof (cont.)

- Conservation of flow implies that no vertex can have degree 1 in H, so if g is not integral, then H contains a polygon.
- Let P be a polygon in H and traverse P with a simple closed path. Let A be the set of edges of P that are forward edges of the path in D, and B the set of edges of P that are backward edges in this path.
- For any constant c, we obtain a new circulation g' by

$$g'(e) \triangleq \begin{cases} g(e) + c & \text{if } e \in A, \\ g(e) - c & \text{if } e \in B, \\ g(e) & \text{if } e \notin E(P) \end{cases}$$

- Now choose

$$c \triangleq \min \left\{ \min_{e \in A} \left(\lceil f(e) \rceil - g(e) \right), \min_{e \in B} \left(g(e) - \lfloor f(e) \rfloor \right). \right\}$$

Then g' still satisfies (1), yet $g'(e)$ is an integer for at least one more edge, contradiction.

Corollary

Let f be an integral circulation on a finite digraph D and d any positive integer. Then f can be written as the sum $g_1 + g_2 + \cdots + g_d$ of integral circulations such that for each index j and each edge e,

$$\lfloor f(e)/d \rfloor \leq g_j(e) \leq \lceil f(e)/d \rceil. \quad (2)$$

Matrix circulation

Definition

From an $m \times n$ matrix A of real numbers a_{ij}, not necessarily nonnegative or integers, we obtain a circulation f on a digraph with $m + n + 2$ vertices and $mn + m + n + 1$ edges.

- There are vertices $x_1, \cdots, x_m$ corresponding to the rows
- vertices $y_1, \cdots, y_n$ corresponding to the columns,
- two others called s and t.
- there is an edge from x_i to y_j with circulation value a_{ij},
- an edge from s to x_i with circulation value equal to the i-th row-sum r_i,
- an edge from y_j to t with circulation value equal to the j-th column-sum k_j
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Multiply f by any scalar α, apply Thm 7.4 to αf, and reinterpret the resulting integral circulation as a matrix, we obtain part (i) of the following theorem. Part (ii) follows from the corollary.

Theorem 7.5

Theorem

1. *Given a matrix A and a real number α, there is an integral matrix B so that the entries of B, the row-sums of B, the column-sums of B, and the sum of all entries of B, are the corresponding values for αA rounded up or down.*
2. *If A is an integral matrix and d any positive integer, then*

$$A = B_1 + B_2 + \cdots + B_d$$

where each B_i is an integral matrix whose entries, row-sums, column- sums, and sum of all entries, are those of $(1/d)A$, rounded up or down.