Lecture 6, Dilworth's theorem and extremal set theory

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Partially ordered set

Definition

A partially ordered set (also poset) is a set S with a binary relation $\leq$ (sometimes $\subseteq$ is used) such that:

1. $a \leq a$ for all $a \in S$ (reflexivity),
2. if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity),
3. if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry).

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Definition

If for any a and b in S, either $a \leq b$ or $b \leq a$, then the partial order is called a total order, or a linear order. If $a \leq b$ and $a \neq b$, then we also write $a < b$.

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Definition

If a subset of S is totally ordered, it is called a chain. An antichain is a set of elements that are pairwise incomparable.

Theorem 6.1

Theorem (Dilworth's theorem)

Let P be a partially ordered finite set. The minimum number m of disjoint chains which together contain all elements of P is equal to the maximum number M of elements in an antichain of P.

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Proof.

- It is trivial that $m \geq M$. We prove $m \leq M$ by induction on $|P|$.
- If $|P| = 0$, there is nothing to prove.
- If $|P| > 0$, assume for any P' with $|P'| < |P|$, we have $m' \leq M'$, where m' is the minimum number of disjoint chains which together contain all elements of P and M' is the maximum number of elements in an antichain of P'. Let C be a maximal chain in P.
  1. If every antichain in $P \setminus C$ contains at most $M - 1$ elements, we are done.
  2. assume that $\{a_1, \cdots, a_M\}$ is an antichain in $P \setminus C$.

Theorem 6.1

cont.

- Now define $S^- \triangleq \{x \in P : \exists i[x \leq a_i]\}$, and define S^+ analogously.
- Since C is a maximal chain, the largest element in C is not in S^- and hence by the induction hypothesis, the theorem holds for S^-.
- Hence S^- is the union of M disjoint chains $S_1^-, \cdots, S_M^-$, where $a_i \in S_i^-$.
- Suppose $x \in S_i^-$ and $x > a_i$. Since there is a j with $x \leq a_j$, we would have $a_i < a_j$, a contradiction.
- This shows that a_i is the maximal element of the chain $S_i^-, i = 1, \cdots, M$. We do the same for S^+.
- By combining the chains, we construct a M disjoint chains which together contain all elements of P. This proves $m \leq M$.

Theorem 6.2

Theorem (Dual of Dilworth's theorem)

Let P be a partially ordered set. If P possesses no chain of $m+1$ elements, then P is the union of m antichains.

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Proof.

- For $m = 1$, the theorem is trivial.
- Let $m \geq 2$ and assume that the theorem is true for $m-1$.
- Let P be a partially ordered set that has no chain of $m+1$ elements.
- Let M be the set of maximal elements of P. M is an antichain.
- Suppose $x_1 < x_2 < \cdots < x_m$ were a chain in $P \setminus M$. Then this would also be a maximal chain in P and hence we would have $x_m \in M$, a contradiction. Hence $P \setminus M$ has no chain of m elements.
- By the induction hypothesis, $P \setminus M$ is the union of $m-1$ antichains. This proves the theorem.

Theorem 6.3

Theorem (Sperner's Theorem)

If $A_1, A_2, \cdots, A_m$ are subsets of $N \triangleq \{1, 2, \cdots, n\}$ such that A_i is not a subset of A_j if $i \neq j$, then $m \leq \binom{n}{\lfloor n/2 \rfloor}$.

Proof.

- Consider the poset of subsets of N. $\mathcal{A} \triangleq \{A_1, \cdots, A_m\}$ is an antichain in this poset.
- There are $n!$ maximal chains.
- There are exactly $k!(n-k)!$ maximal chains which contain a given k-subset A of N.
- Now count the number of ordered pairs $(A, \mathcal{C})$ such that $A \in \mathcal{A}$, $\mathcal{C}$ is a maximal chain, and $A \in \mathcal{C}$.
- Since each maximal chain $\mathcal{C}$ contains at most one member of an antichain, this number is at most $n!$.
- If we let α_k denote the number of sets $A \in \mathcal{A}$ with $|A| = k$, then this number is $\sum_{k=0}^{n} \alpha_k k!(n-k)!$.

Theorem 6.3

cont.

- Thus

$$\sum_{k=0}^{n} \alpha_k k!(n-k)! \leq n!$$

or equivalently,

- $$\sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{k}} \leq 1.$$

- Since $\binom{n}{k}$ is maximal for $k = \lfloor n/2 \rfloor$ and $\sum \alpha_k = m$, the result follows.

□

Remark

Equality holds in Theorem 6.3 if we take all $\lfloor n/2 \rfloor$-subsets of N as the antichain.

Symmetric chain

Definition

The poset B_n (with 2^n elements) of the subsets of the n-set N, ordered by inclusion is a boolean lattice of order n. Let $\mathcal{A}_i$ denote the set of all i-subset of N.

Definition

A symmetric chain in B_n is a sequence $P_k, P_{k+1}, \cdots, P_{n-k}$ of vertices such that $P_i \in \mathcal{A}_i$ and $P_i \subseteq P_{i+1}$ for $i = k, k+1, \cdots, n-k-1$.

An algorithm splitting B_n into symmetric chains.

Algorithm

1. Start with B_1. Proceed by induction.
2. If B_n has been split into symmetric chains, then for each such symmetric chain $P_k, \cdots, P_{n-k}$ define two symmetric chains in B_{n+1} , namely
   - $P_{k+1}, \cdots, P_{n-k}$ and
   - $P_k, P_k \cup \{n+1\}, P_{k+1} \cup \{n+1\}, \cdots, P_{n-k} \cup \{n+1\}$.

Remark

A proof of Hall's theorem by Dilworth theorem

Proof.

- Consider the bipartite graph G on $X \cup Y$. Let $|X| = n, |Y| = n' \geq n$.
- Introduce a partial order by defining $x_i < y_j$ if and only if there is an edge from vertex x_i to vertex y_j.
- Suppose that the largest antichain contains s elements. Let this antichain be $\{x_1, \cdots, x_h, y_1, \cdots, y_k\}$, where $h + k = s$.
- Since $\Gamma(\{x_1, \cdots, x_h\}) \subseteq Y \setminus \{y_1, \cdots, y_k\}$, we have $h \leq n' - k$. Hence $s \leq n'$.
- The partially ordered set is the union of s disjoint chains. This will consist of a matching of size a, the remaining $n - a$ elements of X, and the remaining $n' - a$ elements of Y.
- Therefore $n + n' - a = s \leq n'$, i.e. $a \geq n$, which means that we have a complete matching.

□

Theorem 6.4

Theorem (Erdős-Ko-Rado)

Let $\mathcal{A} = \{A_1, \cdots, A_m\}$ be a collection of m distinct k-subsets of $\{1, 2, \cdots, n\}$, where $k \leq n/2$, with the property that any two of the subsets have a nonempty intersection. Then $m \leq \binom{n-1}{k-1}$.

Proof.

- Place the integers 1 to n on a circle and consider the family $\mathcal{F} \triangleq \{F_1, \cdots, F_n\}$ of all consecutive k-tuples on the circle, i.e. F_i denotes $\{i, i+1, \cdots, i+k-1\}$ where the integers should be taken mod n.
- Observe that $|\mathcal{A} \cap \mathcal{F}| \leq k$ because if some F_i equals A_j, then at most one of the sets $\{l, l+1, \cdots, l+k-1\}$, $\{l-k, \cdots, l-1\}(i < l < i+k)$ is in $\mathcal{A}$.
- The same assertion holds for the collection F^π obtained from F by applying a permutation π to $\{1, 2, \cdots, n\}$.

Theorem 6.4

cont.

- Theorefore
$$\Sigma \triangleq \sum_{\pi \in S_n} |\mathcal{A} \cap \mathcal{F}^{\pi}| \leq k \cdot n!$$
- Count this sum by fixing $A_j \in \mathcal{A}, F_i \in \mathcal{F}$ and observing that there are $k!(n-k)!$ permutations π such that $F_i^{\pi} = A_j$.
- Hence $\Sigma = m \cdot n \cdot k!(n-k)!$ which implies
$$m \cdot n \cdot k!(n-k) \leq k \cdot n!$$
or $m \leq \binom{n-1}{k-1}$. The theorem is proved.

□

Theorem 6.5

Theorem

Let $A = \{A_1, \cdots, A_m\}$ *be a collection of* m *subsets of* $N \triangleq \{1, 2, \cdots, n\}$ *such that* $A_i \not\subseteq A_j$ *and* $A_i \cap A_j \neq \emptyset$ *if* $i \neq j$ *and* $|A_i| \leq k \leq n/2$ *for all* i*. Then* $m \leq \binom{n-1}{k-1}$.

Theorem 6.5

Proof.

1. If all the subsets have size k, then we are done by Theorem 6.4.
2. Let $A_1, \ldots, A_s$ be the subsets with the smallest cardinality, say $l \leq \frac{n}{2} - 1$.
   - Consider all the $(l+1)$-subsets B_j of N that contain one or more of the sets $A_i, 1 \leq i \leq s$.
   - Clearly, none of these is in A. Each of the sets $A_i, 1 \leq i \leq s$, is in exactly $n - l$ of the B_j's and each B_j contains at most $l + 1 \leq n - l$ of the A_i's.
   - So by Theorem 5.1, we can pick s distinct sets, say $B_1, \cdots, B_s$, such that $A_i \subseteq B_i$.
   - If we replace $A_1, \cdots, A_s$ by $B_1, \cdots, B_s$, then the new collection $\mathcal{A}'$ satisfies the conditions of the theorem and the subsets of smallest cardinality now all have size $> l$.
   - By induction, we can reduce to case 1.

□

Theorem 6.6

Theorem

Let $A = \{A_1, \cdots, A_m\}$ be a collection of m subsets of $\{1, 2, \cdots, n\}$ such that $A_i \cap A_j \neq \emptyset$ if $i \neq j$ and $|A_i| \leq k \leq n/2$ for all i. Then

$$\sum_{i=1}^{m} \frac{1}{\binom{n-1}{|A_i|-1}} \leq 1.$$

Proof.

- Let π be a permutation of $\{1, 2, \cdots, n\}$ placed on a circle and let us say that $A_i \in \pi$ if the elements of A_i occur consecutively somewhere on that circle.
- By the same argument as in the proof of Theorem 6.4 we see that if $A_i \in \pi$, then $A_j \in \pi$ for at most $|A_i|$ values of j.
- Now define

$$f(\pi, i) \triangleq \begin{cases} \frac{1}{|A_i|}, & \text{if } A_i \in \pi \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6.6

cont.

- By the argument above $\sum_{\pi \in S_n} \sum_{i=1}^{m} f(\pi, i) \leq n!$.
- Changing the order of summation we have to count, for a fixed A_i, the number of permutations π placed on a circle such that $A_i \in \pi$. The number is $n \cdot |A_i|!(n - |A_i|)!$.
- So we have

$$\sum_{i=1}^{m} \frac{1}{A_i} \cdot n \cdot |A_i|!(n - |A_i|!) \leq n!$$

which yields the result.

□