Large deviations for perturbed reflected diffusion processes

Lijun Bo, Tusheng Zhang

* Department of Mathematics, Xidian University, Xi'an, P.R. China

b Department of Mathematics, University of Manchester, Manchester, UK

Online publication date: 30 November 2009

To cite this Article Bo, Lijun and Zhang, Tusheng(2009) 'Large deviations for perturbed reflected diffusion processes', Stochastics An International Journal of Probability and Stochastic Processes, 81: 6, 531 — 543

To link to this Article DOI: 10.1080/17442500801981084

URL: http://dx.doi.org/10.1080/17442500801981084

PLEASE SCROLL DOWN FOR ARTICLE

Full terms and conditions of use: http://www.informaworld.com/terms-and-conditions-of-access.pdf

This article may be used for research, teaching and private study purposes. Any substantial or systematic reproduction, re-distribution, re-selling, loan or sub-licensing, systematic supply or distribution in any form to anyone is expressly forbidden.

The publisher does not give any warranty express or implied or make any representation that the contents will be complete or accurate or up to date. The accuracy of any instructions, formulae and drug doses should be independently verified with primary sources. The publisher shall not be liable for any loss, actions, claims, proceedings, demand or costs or damages whatsoever or howsoever caused arising directly or indirectly in connection with or arising out of the use of this material.
Large deviations for perturbed reflected diffusion processes

Lijun Bo and Tusheng Zhang*

"Department of Mathematics, Xidian University, Xi’an 710071, P.R. China; bDepartment of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, UK

(Received 12 December 2006; final version received 3 February 2008)

In this article, we establish a large deviation principle for the solutions of perturbed reflected diffusion processes. The key is to prove a uniform Freidlin–Ventzell estimate of perturbed diffusion processes.

Keywords: large deviations; perturbed diffusion processes; uniform Freidlin–Ventzell estimates

AMS Subject Classification: 60F10; 60J27

1. Introduction

Doney and Zhang [5], obtained the existence and uniqueness of the solutions for the following perturbed diffusion and perturbed reflected diffusion equations:

\[ X_t = x + \int_0^t \sigma(X_s) \, dB_s + \int_0^t b(X_s) \, ds + \alpha \sup_{0 \leq s \leq t} X_s, \quad t \in [0, 1] \]

and

\[ Y_t = y + \int_0^t a(Y_s) \, dB_s + \alpha \sup_{0 \leq s \leq t} Y_s + L_t, \quad t \in [0, 1], \]

where \( \alpha \in (0, 1) \), \( x \in \mathbb{R}, y \in \mathbb{R}_+ \) are deterministic, \( b, \sigma : \mathbb{R} \to \mathbb{R} \) and \( a : \mathbb{R}_+ \to \mathbb{R} \) are bounded Lipschitz continuous function, \( \{L_t, t \in [0, 1]\} \) is non-decreasing with \( L_0 = 0 \) and

\[ \int_0^t \mathbb{1}_{Y_s = 0} \, dL_s = L_t, \quad t \in [0, 1]. \]

We may think of \( \{L_t, t \in [0, 1]\} \) the local time of the semimartingale \( \{Y_t, t \in [0, 1]\} \) at point zero. \( \{B_t, t \in [0, 1]\} \) is a 1D standard Brownian motion on a completed probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \). The perturbed reflected Brownian motion was first introduced by Le Gall and Yor [7,8], and subsequently studied by Carmona et al. [2,3], Perman and Werner [11] and Chaumont and Doney [4]. Consider the small noise perturbations of (1.1) and (1.2):

\[ X^\varepsilon_t = x + \sqrt{\varepsilon} \int_0^t \sigma(X^\varepsilon_s) \, dB_s + \int_0^t b(X^\varepsilon_s) \, ds + \alpha \sup_{0 \leq s \leq t} X^\varepsilon_s, \quad t \in [0, 1], \]

*Corresponding author. Email: tzhang@maths.man.ac.uk
and

\[ Y_t^\varepsilon = y + \sqrt{\varepsilon} \int_0^t a(Y_s^\varepsilon) \, dB_s + \alpha \sup_{0 \leq s \leq t} Y_s^\varepsilon + L_t^\varepsilon, \quad t \in [0, 1]. \quad (1.4) \]

The aim of this paper is to establish a large deviation principle (LDP) for the laws of \( X_s^\varepsilon \) and \( Y_s^\varepsilon \) on the space of continuous functions equipped with uniform topology, respectively. Our approach will be based on a classical result of Azencott [1], which can be stated as follows:

**Proposition 1.1.** Let \((E_i, d_i)(i = 1, 2)\) be two Polish spaces and \( \Phi_i^\varepsilon : \Omega \to E_i, \varepsilon > 0 \) \((i = 1, 2)\) be two families of random variables. Assume that

1. \( \{ \Phi_i^\varepsilon, \varepsilon > 0 \} \) satisfies a LDP with the rate function \( I_1 : E_1 \to [0, \infty] \).
2. There exists a map \( K : \{ I_1 < \infty \} \to E_2 \) such that for every \( a < \infty, K : \{ I_1 \leq a \} \to E_2 \) is continuous.
3. For any \( R, \delta, a > 0 \), there exist \( \rho > 0 \) and \( \varepsilon_0 > 0 \) such that, for any \( h \in E_1 \) satisfying \( I_1(h) \leq a \) and \( \varepsilon \leq \varepsilon_0 \),

\[ \mathbb{P}(d_2(\Phi_2^\varepsilon, K(h)) \geq \delta, d_1(\Phi_1^\varepsilon, h) \leq \rho) \leq \exp \left(-\frac{R}{\varepsilon}\right). \quad (1.5) \]

Then, \( \{ \Phi_2^\varepsilon, \varepsilon > 0 \} \) satisfies a LDP with the rate function

\[ I(g) = \inf \{ I_1(h); K(h) = g \}. \]

The estimate (1.5) is also known as the uniform Freidlin–Ventzell estimates.

We would like to mention that some works in the literature motivated our present study. Doss and Priouret [6] considered the LDP for the small noise perturbations of reflected diffusions, through checking uniform Freidlin–Ventzell estimates. Further, Millet et al. [9] used the Freidlin–Ventzell estimates to obtain the LDP of a class of anticipating stochastic differential equations. Recently, Mohammed and Zhang [10] studied a LDP for small noise perturbed family of stochastic systems with memory. In this article, the appearance of the terms \( \alpha \sup_{0 \leq s \leq t} X_s^\varepsilon \) and \( \alpha \sup_{0 \leq s \leq t} Y_s^\varepsilon \) make (1.3) and (1.4) different from the equations studied in the literature. To deduce the uniform Freidlin–Ventzell estimates for (1.3) and (1.4), the estimates concerning the perturbed terms \( \alpha \sup_{0 \leq s \leq t} X_s^\varepsilon \) and \( \alpha \sup_{0 \leq s \leq t} Y_s^\varepsilon \) are the key in the paper.

The rest of the paper is organized as follows. Section 2 is for the LDP of perturbed diffusion processes. The large deviation estimates for the perturbed reflected diffusion processes are shown in Section 3.

### 2. LDP for perturbed diffusion processes

In this section, we will give a LDP of the perturbed diffusion process (1.3). Let \( H \) denote the Cameron–Martin space, i.e.,

\[ H = \left\{ h(t) = \int_0^t \dot{h}(s) \, ds : [0, 1] \to \mathbb{R}; \int_0^1 |h(s)|^2 \, ds < +\infty \right\}, \]

and \( \{ \nu_\varepsilon, \varepsilon > 0 \} \) be the probability measure induced by \( X_s^\varepsilon \) on \( C_s([0, 1], \mathbb{R}) \), the space of all continuous functions \( f : [0, 1] \to \mathbb{R} \) such that \( f(0) = x \), equipped with the supremum norm.
topology. For \( h \in C_0([0, 1], \mathbb{R}) \), define \( \tilde{I} : C_0([0, 1], \mathbb{R}) \to [0, \infty) \) by
\[
\tilde{I}(h) = \begin{cases} \frac{1}{2} \int_0^1 |\dot{h}(s)|^2 \, ds, & \text{if } h \in H, \\ +\infty, & \text{otherwise}. \end{cases}
\]

The well-known Schilder theorem states that the laws \( \mu_\varepsilon \) of \( \{\sqrt{\varepsilon} B_t, \ t \in [0, 1]\} \) satisfies a LDP on \( C_0([0, 1], \mathbb{R}) \) with the rate function \( \tilde{I}(\cdot) \), that is,

(a) for any closed subset \( C \subset C_0([0, 1], \mathbb{R}) \),
\[
\lim_{\varepsilon \to 0} \sup \varepsilon \log \mu_\varepsilon(C) \leq - \inf_{g \in C} \tilde{I}(g),
\]

(b) for any open subset \( G \subset C_0([0, 1], \mathbb{R}) \),
\[
\lim_{\varepsilon \to 0} \inf \varepsilon \log \mu_\varepsilon(G) \geq - \inf_{g \in G} \tilde{I}(g).
\]

Let \( F(h) \) be the unique solution of the following deterministic perturbed equation:
\[
F(h)(t) = x + \int_0^t \sigma(F(h)(s))\dot{h}(s) \, ds + \int_0^t b(F(h)(s)) \, ds + \alpha \sup_{0 \leq s \leq t} F(h)(s), \ h \in H, \ t \in [0, 1].
\]

(2.1)

It is not difficult to obtain the existence and uniqueness of the solution to (2.1) by the same arguments of Theorem 2.1 in [5]. Then, we have the following main result:

**THEOREM 2.1.** If \( \alpha \in (0, 1) \), then the family \( \{ \nu_\varepsilon, \varepsilon > 0 \} \) obeys a LDP with the rate function:
\[
I_\varepsilon(g) = \inf_{(h \in H : F(h) = g)} \tilde{I}(h), \ \text{for } g \in C_0([0, 1], \mathbb{R}),
\]

where the inf over the empty set is taken to be \( \infty \).

Before giving the proof of Theorem 2.1, recall the following lemma (e.g. Lemma 2.19 in Stroock [12]).

**LEMMA 2.1.** Let \( Z_t := \int_0^t C(u) \, dB_u + \int_0^t D(u) \, du \) be an Itô process, where \( 0 \leq s < t < \infty \) and \( C, D : [0, \infty) \times \Omega \to \mathbb{R} \) are \( (\mathcal{F}_t)_{t \geq 0} \) progressively measurable random processes. If \( |C(\cdot)| \leq M_1 \) and \( |D(\cdot)| \leq M_2 \), then for \( T > s \) and \( R > 0 \) satisfying \( R > M_2^2(T - s) \), we have
\[
\mathbb{P}\left( \sup_{s \leq t \leq T} |Z_t| \leq R \right) \leq \exp\left( - \frac{(R - M_2(T - s))^2}{2M_1^2(T - s)} \right).
\]

(2.2)

**Proof of Theorem 2.1.** For \( h \in H \), let \( F(h) \) be defined as in (2.1). First, we prove that for any \( R, \delta, a > 0 \), there exist \( \rho > 0 \) and \( \varepsilon_0 > 0 \) such that, for any \( h \in C_0([0, 1], \mathbb{R}) \) satisfying \( \tilde{I}(h) \leq a \) and \( \varepsilon \leq \varepsilon_0 \),
\[
\mathbb{P}\left( \sup_{0 \leq t \leq 1} |X^\varepsilon_t - F(h)(t)| \geq \delta, \sup_{0 \leq t \leq 1} |\sqrt{\varepsilon} B_t - h(t)| \leq \rho \right) \leq \exp\left( - \frac{R^2}{\varepsilon} \right).
\]

(2.3)
By (1.3) and (2.1),

\[
X_t^\varepsilon - F(h(t)) = \int_0^t \sigma(X_s^\varepsilon)(\sqrt{\varepsilon} \, dB_s - \dot{h}(s) \, ds) + \int_0^t (\sigma(X_s^\varepsilon) - \sigma(F(h(s))) \, h(s) \, ds + \int_0^t (b(X_s^\varepsilon) - b(F(h(s))) \, ds + \alpha \left( \sup_{0 \leq s \leq t} X_s^\varepsilon - \sup_{0 \leq s \leq t} F(h(s)) \right). \tag{2.4}
\]

Consequently,

\[
|X_t^\varepsilon - F(h(t))| \leq \left| \int_0^t \sigma(X_s^\varepsilon)(\sqrt{\varepsilon} \, dB_s - \dot{h}(s) \, ds) \right| + L \int_0^t |X_s^\varepsilon - F(h(s))|(1 + |h(s)|) \, ds + \alpha \sup_{0 \leq s \leq t} |X_s^\varepsilon - F(h(s))|, \tag{2.5}
\]

where \( L > 0 \) is the Lipschitz coefficient, and we also used the fact that

\[
\left| \sup_{0 \leq s \leq t} u(s) - \sup_{0 \leq s \leq t} v(s) \right| \leq \sup_{0 \leq s \leq t} |u(s) - v(s)|,
\]

for any two continuous functions \( u \) and \( v \) on \( \mathbb{R}_+ \). Thus, it follows from (2.5) that, for \( t \in [0, 1] \),

\[
\sup_{0 \leq u \leq t} |X_u^\varepsilon - F(h(u))| \leq \frac{1}{(1 - \alpha)} \sup_{0 \leq u \leq t} \left| \int_0^u \sigma(X_s^\varepsilon)(\sqrt{\varepsilon} \, dB_s - \dot{h}(s) \, ds) \right| + \frac{L}{(1 - \alpha)} \int_0^t \sup_{0 \leq s \leq t} |X_s^\varepsilon - F(h(u))|(1 + |h(s)|) \, ds. \tag{2.6}
\]

By the Gronwall lemma this yields that

\[
\sup_{0 \leq t \leq 1} |X_t^\varepsilon - F(h(t))| \leq \frac{1}{(1 - \alpha)} \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma(X_s^\varepsilon)(\sqrt{\varepsilon} \, dB_s - \dot{h}(s) \, ds) \right| \times \exp \left( \frac{L}{(1 - \alpha)} \int_0^1 (1 + |h(s)|) \, ds \right) \\
\leq C_1 \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma(X_s^\varepsilon)(\sqrt{\varepsilon} \, dB_s - \dot{h}(s) \, ds) \right|, \tag{2.7}
\]

where \( C_1 := 1/(1 - \alpha) \exp((L(1 + \|h\|_H))/(1 - \alpha)) < \infty \) with \( \|h\|_H := \left( \int_0^1 |h(s)|^2 \, ds \right)^{1/2} \) for \( h \in H \). Thus, to prove (2.3), it suffices to prove that, for any \( R, \delta, a > 0 \), there exist
\[ \rho > 0 \text{ and } \varepsilon_0 > 0 \text{ such that, for any } h \in C_0([0, 1], \mathbb{R}) \text{ satisfying } \bar{I}(h) \leq a \text{ and } \varepsilon \leq \varepsilon_0, \]
\[
\mathbb{P}\left( \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma(X_s^\varepsilon)(\sqrt{\varepsilon} dB_s - \dot{h}(s) \, ds \right| \geq \delta, \sup_{0 \leq t \leq 1} |\sqrt{\varepsilon} B_t - h(t)| \leq \rho \right) \leq \exp\left( -\frac{R}{\varepsilon} \right). \tag{2.8}
\]

For \( \varepsilon > 0 \), define a probability measure \( \mathbb{P}^\varepsilon \) on \( \Omega \) by
\[
d\mathbb{P}^\varepsilon = Z_\varepsilon d\mathbb{P} := \exp\left( \frac{1}{\sqrt{\varepsilon}} \int_0^1 \dot{h}(s) \, dB_s - \frac{1}{2\varepsilon} \int_0^1 |\dot{h}(s)|^2 \, ds \right) \, d\mathbb{P}.
\]
Then, by Girsanov theorem, \( \{B_t^\varepsilon := B_t - (1/\sqrt{\varepsilon})\dot{h}(t), t \in [0, 1]\} \) is a standard Brownian motion on \( (\Omega, \mathcal{F}, \mathbb{P}^\varepsilon) \). If we let
\[
A^\varepsilon = \left\{ \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma(X_s^\varepsilon)(\sqrt{\varepsilon} dB_s - \dot{h}(s) \, ds \right| \geq \delta, \sup_{0 \leq t \leq 1} |\sqrt{\varepsilon} B_t^\varepsilon - h(t)| \leq \rho \right\}
\]
then by the Hölder inequality,
\[
\mathbb{P}(A^\varepsilon) = \int_\Omega Z_\varepsilon^{-1} \chi_{A^\varepsilon}(\omega) \mathbb{P}^\varepsilon(\omega) \, d\omega \leq \left( \int_\Omega Z_\varepsilon^{-2}(\omega) \mathbb{P}^\varepsilon(\omega) \, d\omega \right)^{1/2} \left( \mathbb{P}^\varepsilon(A^\varepsilon) \right)^{1/2}.
\]

Note that \( \{B_t^\varepsilon, t \in [0, 1]\} \) is a standard Brownian motion under \( \mathbb{P}^\varepsilon \), then
\[
\int_\Omega Z_\varepsilon^{-2}(\omega) \mathbb{P}^\varepsilon(\omega) \, d\omega = \mathbb{E}_{\mathbb{P}_\varepsilon}\left[ \exp\left( -\frac{2}{\sqrt{\varepsilon}} \int_0^1 \dot{h}(s) \, dB_s + \frac{1}{\varepsilon} \int_0^1 |\dot{h}(s)|^2 \, ds \right) \right]
\]
\[
= \mathbb{E}_{\mathbb{P}_\varepsilon}\left[ \exp\left( -\frac{2}{\sqrt{\varepsilon}} \int_0^1 \dot{h}(s) \, dB_s - \frac{1}{\varepsilon} \int_0^1 |\dot{h}(s)|^2 \, ds \right) \right]
\]
\[
= \mathbb{E}_{\mathbb{P}_\varepsilon}\left[ \exp\left( -\frac{2}{\sqrt{\varepsilon}} \int_0^1 \dot{h}(s) \, dB_s - \frac{2}{\varepsilon} \int_0^1 |\dot{h}(s)|^2 \, ds \right) \right] \times \exp\left( \frac{1}{\varepsilon} \int_0^1 |\dot{h}(s)|^2 \, ds \right)
\]
\[
= \exp\left( \frac{1}{\varepsilon} \|h\|^2_{\bar{H}} \right).
\]
Therefore, if \( \bar{I}(h) \leq a \), then
\[
\mathbb{P}(A^\varepsilon) \leq \exp\left( \frac{d}{\varepsilon} \right) (\mathbb{P}^\varepsilon(A^\varepsilon))^{1/2}. \tag{2.9}
\]

Note that under the probability measure \( \mathbb{P}^\varepsilon \), \( X^\varepsilon \) satisfies the following stochastic differential equation:
\[
U_t^\varepsilon = x + \sqrt{\varepsilon} \int_0^t \sigma(U_s^\varepsilon) \, dB_s + \int_0^t (b(U_s^\varepsilon) + \sigma(U_s^\varepsilon) \cdot \dot{h}(s)) \, ds + \alpha \sup_{0 \leq s \leq t} U_s^\varepsilon. \tag{2.10}
\]
Therefore,

\[
P^e(A^e) = P^e \left( \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma(U^e_s) \sqrt{e} d B^e_s \right| \geq \delta, \sup_{0 \leq t \leq 1} |\sqrt{e} B^e_t| \leq \rho \right)
\]

\[
= P \left( \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma(U^e_s) \sqrt{e} d B_s \right| \geq \delta, \sup_{0 \leq t \leq 1} |\sqrt{e} B_t| \leq \rho \right).
\]

Thus, in view of (2.9), to prove (2.8), the proof of (2.8) is reduced to show that, for any \( R, \delta, a > 0 \), there exist \( \rho > 0 \) and \( \varepsilon_0 > 0 \) such that, for any \( h \in C_0([0, 1], \mathbb{R}) \) satisfying \( l(h) \leq a \) and \( \varepsilon \leq \varepsilon_0 \),

\[
P \left( \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma(U^e_s) \sqrt{e} d B_s \right| \geq \delta, \sup_{0 \leq t \leq 1} |\sqrt{e} B_t| \leq \rho \right) \leq \exp \left( -\frac{R}{\varepsilon} \right). \tag{2.11}
\]

For \( n \in \mathbb{N} \) fixed, set \( t_k = k/n \) for \( k \in \{0, 1, 2, \ldots, n\} \). Define

\[
U^e_{t_k} := U^e_{t_k}, \quad \text{if } t_k \leq t < t_{k+1}, k \in \{0, 1, \ldots, n-1\}.
\]

Then, for \( \delta_1 > 0 \),

\[
A^e \subset \left\{ \sup_{0 \leq t \leq 1} \left| \int_0^t (\sigma(U^e_s) - \sigma(U^e_{t_k})) \sqrt{e} d B_s \right| \geq \frac{\delta}{2}, \sup_{0 \leq t \leq 1} |U^e_t - U^e_{t_k}| \leq \delta_1 \right\}
\]

\[
\cup \left\{ \sup_{0 \leq t \leq 1} |U^e_t - U^e_{t_k}| > \delta_1 \right\}
\]

\[
\cup \left\{ \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma(U^e_{t_k}) \sqrt{e} d B_s \right| \geq \frac{\delta}{2}, \sup_{0 \leq t \leq 1} |\sqrt{e} B_t| \leq \rho \right\}
\]

\[
:= B^e \cup C^e \cup D^e.
\]

On the set \( \{ \sup_{0 \leq t \leq 1} |U^e_t - U^e_{t_k}| \leq \delta_1 \} \),

\[
\sup_{0 \leq t \leq 1} \varepsilon |\sigma(U^e_s) - \sigma(U^e_{t_k})|^2 \leq L^2 \varepsilon \delta_1^2.
\]

By Lemma 2.1, it follows that

\[
P(B^e) \leq \exp \left( -\frac{\delta^2}{8L^2 \varepsilon \delta_1^2} \right) \leq \exp \left( -\frac{R}{\varepsilon} \right), \tag{2.12}
\]

if \( \delta_1 \leq \delta/2L\sqrt{2R} \).

On the other hand, on the set \( \{ \sup_{0 \leq t \leq 1} |\sqrt{e} B_t| \leq \rho \} \), for any \( t \in [0, 1] \),

\[
\left| \int_0^t \sigma(U^e_{t_k}) \sqrt{e} d B_s \right| = \sqrt{e} \sum_{j=0}^{n-1} \sigma(U^e_{t_j})(B_{t_{j+1}} - B_{t_j})
\]

\[
\leq 2 \sup_{0 \leq t \leq 1} |\sqrt{e} B_t| \sum_{j \in \{0, \ldots, n-1\}; t_j \leq t} |\sigma(U^e_{t_j})|
\]

\[
\leq 2nM \rho,
\]
where $M > 0$ is a common bound of $b$ and $\sigma$. Therefore, if $\rho < \delta/4nM$, then

$$D^e = \emptyset.$$ 

To treat $C^e$, we note that for $t \in [t_k, t_{k+1})$, $k \in \{0, 1, \ldots, n-1\}$,

$$|U_t^e - U_t^{e,n}| \leq \left| \int_{t_k}^{t} \sigma(U_s^e) \sqrt{\epsilon} dB_s + \int_{t_k}^{t} (b(U_s^e) + \sigma(U_s^e) \hat{h}(s)) \, ds \right| + \alpha \left( \sup_{0 \leq s \leq t} U_s^e - \sup_{0 \leq s \leq t} U_s^e \right).$$

(2.13)

Consider two possible cases:

- The case I: $\sup_{0 \leq s \leq t} U_s^e = \sup_{0 \leq s \leq t} U_s^e$.

We have

$$|U_t^e - U_t^{e,n}| \leq \left| \int_{t_k}^{t} \sigma(U_s^e) \sqrt{\epsilon} dB_s + \int_{t_k}^{t} (b(U_s^e) + \sigma(U_s^e) \hat{h}(s)) \, ds \right|.$$  

(2.14)

- The case II: $\sup_{0 \leq s \leq t} U_s^e > \sup_{0 \leq s \leq t} U_s^e$.

Then, there exists $u \in (t_k, t]$ such that $U_u^e = \sup_{0 \leq s \leq t} U_s^e$, which yields that

$$|U_t^e - U_{t_k}| \leq \left| \int_{t_k}^{t} \sigma(U_s^e) \sqrt{\epsilon} dB_s + \int_{t_k}^{t} (b(U_s^e) + \sigma(U_s^e) \hat{h}(s)) \, ds \right| + \alpha \sup_{t_k \leq s < t_{k+1}} |U_s^e - U_{t_k}|.$$ 

This implies

$$\sup_{t_k \leq s < t_{k+1}} |U_s^e - U_s^{e,n}| = \sup_{t_k \leq s < t_{k+1}} |U_s^e - U_t^e|$$

$$\leq C_2 \sup_{t_k \leq s < t_{k+1}} \left| \int_{t_k}^{t} \sigma(U_s^e) \sqrt{\epsilon} dB_s + \int_{t_k}^{t} (b(U_s^e) + \sigma(U_s^e) \hat{h}(s)) \, ds \right|,$$

(2.15)

where $C_2 := 1/(1-\alpha)$.

Since $\sup_{t_k \leq s < t_{k+1}} \epsilon |\sigma(U_s^e)|^2 \leq \epsilon M^2$, $\sup_{t_k \leq s < t_{k+1}} |b(U_s^e)| \leq M$ and

$$\sup_{t_k \leq s < t_{k+1}} \left| \int_{t_k}^{t} \sigma(U_s^e) \hat{h}(s) \, ds \right| \leq M \sqrt{\frac{2a}{n}},$$

Therefore,
it follows from (2.15) and Lemma 2.1 that for \( n \geq n_1 := \lfloor 2MC_2/\delta_1 \rfloor + 1 \vee n_2 := \lfloor 8C_2^2 M^2 \delta_1^2 \rfloor + 1 \),

\[
\mathbb{P} \left( \sup_{t_k \leq t < t_{k+1}} |U_t^\varepsilon - U_{t_{k+1}}^\varepsilon| > \delta_1 \right) \leq \mathbb{P} \left( \sup_{t_k \leq t < t_{k+1}} \left| \int_{t_k}^{t} \sigma(U_s^\varepsilon) \sqrt{\varepsilon} \, dB_s + \int_{t_k}^{t} b(U_s^\varepsilon) \, ds \right| > \frac{\delta_1}{2C_2} \right) \\
+ \mathbb{P} \left( \sup_{t_k \leq t < t_{k+1}} \left| \int_{t_k}^{t} \sigma(U_s^\varepsilon) h(s) \, ds \right| > \frac{\delta_1}{2C_2} \right) \\
\leq \exp \left( -n \frac{\delta_1^2}{8C_2^2 M^2 \varepsilon} \right). \tag{2.16}
\]

Therefore,

\[
\mathbb{P}(C^\varepsilon) = \mathbb{P} \left( \max_{k \in \{0, 1, \ldots, n-1\}} \sup_{t_k \leq t < t_{k+1}} |U_t^\varepsilon - U_{t_{k+1}}^\varepsilon| > \delta_1 \right) \\
\leq n \exp \left( -n \frac{\delta_1^2}{8C_2^2 M^2 \varepsilon} \right). \tag{2.17}
\]

Now, given \( R > 0 \), choose first \( \delta_1 > 0 \) such that (2.12) holds. Then, choose \( n \) large enough so that

\[
n \exp \left( -n \frac{\delta_1^2}{8C_2^2 M^2 \varepsilon} \right) \leq \exp \left( -\frac{R}{\varepsilon} \right).
\]

Finally, for the fixed \( n \in \mathbb{N} \), there exists \( \rho > 0 \) such that \( D^\varepsilon = 0 \). Combining above inequalities proves (2.11).

Next, we prove the map \( F : H \rightarrow C([0, 1], \mathbb{R}) \) is continuous on \( \{ h : \tilde{I}(h) \leq a \} \) for \( a \in [0, \infty) \). Let \( h_n, h \in \{ h \in C_0([0, 1], \mathbb{R}) : \tilde{I}(h) \leq a \} \) with \( h_n \rightarrow h \) in \( C_0([0, 1], \mathbb{R}) \). Since \( \{ h : \tilde{I}(h) \leq a \} \) is weakly compact in \( H \), we conclude that \( h_n \) also weakly converges to \( h \) in \( H \). By the Lipschitz continuity,

\[
\sup_{0 \leq s \leq t} |F(h_n)(s) - F(h)(s)| \leq \xi_\alpha(t) + \alpha \sup_{0 \leq s \leq t} |F(h_n)(s) - F(h)(s)| \\
+ L \int_0^t \sup_{0 \leq u \leq s} |F(h_n)(u) - F(h(u))| (1 + |\dot{h}_n(s)|) \, ds,
\]

where

\[
\xi_\alpha(t) = \sup_{0 \leq s \leq t} \left| \int_0^s \sigma(F(h)(u)) (\dot{h}_n(u) - h(u)) \, du \right|. \tag{2.18}
\]
Applying the Gronwall’s lemma,
\[
\sup_{0 \leq t \leq 1} |F(h_n(t)) - F(h(t))| \leq \frac{2M}{1 - \alpha} \xi_\alpha(1) \times \exp \left( \frac{L(1 + \sqrt{2\alpha h_n})}{1 - \alpha} \right)
\]
\[
\leq \frac{2M}{1 - \alpha} \exp \left( \frac{L(1 + \sqrt{2\alpha})}{1 - \alpha} \right) \xi_\alpha(1).
\]
Since \( h_n \to h \) weakly in \( H \), for every \( s \in [0, 1] \),
\[
\int_0^s \sigma(F(h(u))(h_n(u) - h(u))) \, du \to 0, \quad \text{as } n \to \infty.
\]
On the other hand, it is easy to see that there exists a constant \( C_a > 0 \) such that
\[
\left| \int_s^{t'} \sigma(F(h(u))(\dot{h_n(u)} - \dot{h(u)})) \, du \right| \leq C_a |t - s|^{1/2}.
\]
Therefore, we conclude that \( \xi_\alpha(1) \to 0 \), as \( n \to \infty \). Consequently,
\[
\sup_{0 \leq t \leq 1} |F(h_n(t)) - F(h(t))| \to 0, \quad \text{as } n \to \infty,
\]
which proves the continuity of the mapping \( F \). Now, letting \( \Phi^e_t = \sqrt{\epsilon} B \), and \( K(\cdot) = F(\cdot) \).

Theorem 2.1 follows from Proposition 1.1. \( \square \)

3. LDP for perturbed reflected diffusion processes

In this section, we will prove the LDP for the solution of the perturbed reflected diffusion equation (1.4).

For \( y \geq 0 \) and \( f \in C_y([0, 1], \mathbb{R}) \), define two operators \( \Gamma : C_y([0, 1], \mathbb{R}) \to C_y([0, 1], \mathbb{R}_+) \) and \( K : C_y([0, 1], \mathbb{R}) \to C_y([0, 1], \mathbb{R}_+) \) by
\[
\Gamma f = f + \tilde{f} \quad \text{and} \quad K f = \tilde{f}, \quad \text{where} \quad \tilde{f}(t) := \inf_{s \leq t} (f(s) \wedge 0), \quad t \in [0, 1].
\]

By the reflection principle, the solution \( Y^e \) of (1.4) is given by
\[
Y^e_t = (\Gamma Z^e)(t) \quad \text{and} \quad L^e_t = (KZ^e)(t), \quad t \in [0, 1],
\]
where \( Z^e \) is a solution of the following stochastic equation:
\[
Z^e_t = y + \sqrt{\epsilon} \int_0^t \sigma((\Gamma Z^e)(s)) \, dB_s + \alpha \sup_{0 \leq s \leq t} (\Gamma Z^e)(s), \quad t \in [0, 1].
\]

For \( h \in H \), let \( G(h) \) be the unique solution of the following equation:
\[
G(h)(t) = y + \int_0^t \sigma(G(h)(s)) \dot{h}_s \, ds + \alpha \sup_{0 \leq s \leq t} G(h)(s) + \eta_t, \quad t \in [0, 1],
\]
where \( G(h) \) is continuous, non-negative and \( \eta \) is an increasing continuous function satisfying \( \eta_t = \int_0^t \chi_{(G(h)(s) = 0)} \, d\eta_s \). The existence and uniqueness of the solution to (3.2) might be obtained by Theorem 3.1 and Theorems 4.2 and 4.3 in [5].
Similar as (3.1), $G(h)$ can also be written as

$$G(h)(t) = (\Gamma V(h))(t) \quad \text{and} \quad \eta_t = (KV(h))(t), \quad t \in [0, 1],$$

(3.3)

where $V(h)$ satisfies

$$V(h)(t) = y + \int_0^t \sigma((\Gamma V(h))(s))h_s \, ds + \alpha \sup_{0 \leq s \leq t} (\Gamma V(h))(s), \quad t \in [0, 1].$$

Let $\nu^1_\varepsilon$ be the law of $Y^\varepsilon_1$ on $C_\gamma([0, 1], \mathbb{R}_+)$. Our main result in this section is

**Theorem 3.1.** If $\alpha \in (0, 1/2)$, then the family $\{\nu^1_\varepsilon, \varepsilon > 0\}$ obeys a LDP with the rate function:

$$I_y(g) = \inf_{\{h \in H: G(h) = g\}} \tilde{I}(h), \quad \text{for} \quad g \in C_\gamma([0, 1], \mathbb{R}_+),$$

where the inf over the empty set is taken to be $\infty$.

By Proposition 1.1, Theorem 3.1 is the consequence of the following two propositions.

**Proposition 3.1.** If $\alpha \in (0, 1/2)$, then for any $R, \delta, a > 0$, there exist $\rho > 0$ and $\varepsilon_0 > 0$ such that, for any $h \in C_0([0, 1], \mathbb{R})$ satisfying $\bar{I}(h) \leq a$ and $\varepsilon \leq \varepsilon_0$,

$$\mathbb{P}\left( \sup_{0 \leq t \leq 1} \left| Y^\varepsilon_t - G(h)(t) \right| + \sup_{0 \leq t \leq 1} \left| L^\varepsilon_t - \eta_t \right| \geq \delta, \sup_{0 \leq t \leq 1} \left| \sqrt{\varepsilon} B_t - h(t) \right| \leq \rho \right) \leq \exp\left( -\frac{R}{\varepsilon} \right).$$

**Proof.** The proof is similar to that of Theorem 2.1. We only highlight the differences. Note that for $f_1, f_2 \in C_\gamma([0, 1], \mathbb{R})$ and $t \in [0, 1]$,

$$\sup_{0 \leq s \leq t} |(\Gamma f_1)(s) - (\Gamma f_2)(s)| \leq 2 \sup_{0 \leq s \leq t} |f_1(s) - f_2(s)|.$$  

(3.4)

Then, by (3.1) and (3.3),

$$\sup_{0 \leq t \leq 1} \left| Y^\varepsilon_t - G(h)(t) \right| + \sup_{0 \leq t \leq 1} \left| L^\varepsilon_t - \eta_t \right| \leq 3 \sup_{0 \leq t \leq 1} \left| Z^\varepsilon_t - V(h)(t) \right|. $$

Therefore, the proof of Proposition 3.1 is reduced to show that for any $R, \delta, a > 0$, there exist $\rho > 0$ and $\varepsilon_0 > 0$ such that, for any $h \in C_0([0, 1], \mathbb{R})$ satisfying $\bar{I}(h) \leq a$ and $\varepsilon \leq \varepsilon_0$,

$$\mathbb{P}\left( \sup_{0 \leq t \leq 1} \left| Z^\varepsilon_t - V(h)(t) \right| \geq \delta, \sup_{0 \leq t \leq 1} \left| \sqrt{\varepsilon} B_t - h(t) \right| \leq \rho \right) \leq \exp\left( -\frac{R}{\varepsilon} \right).$$  

(3.5)

Using (3.4) and the Gronwall’s lemma,

$$\sup_{0 \leq t \leq 1} \left| Z^\varepsilon_t - V(h)(t) \right| \leq C_3 \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma((\Gamma Z^\varepsilon)(s))\sqrt{\varepsilon} dB_s \right|, $$

where $C_3$ is a positive constant.
where

\[ C_3 := \exp \left( \frac{2|\gamma|}{1-2\alpha} H \| h \|_H \right). \]

Thus, as in Section 2, by the Girsanov’s theorem, to prove (3.5), it suffices to prove

\[
P \left( \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma((\Gamma \tilde{Z}^e)(s)) \sqrt{\varepsilon} \, dB_s \right| \geq \delta, \sup_{0 \leq t \leq 1} \left| \sqrt{\varepsilon} B_t \right| \leq \rho \right) \leq \exp \left( - \frac{R}{\varepsilon} \right),
\]

where \( \tilde{Z}^e \) satisfies

\[
\tilde{Z}^e_t = y + \sqrt{\varepsilon} \int_0^t \sigma((\Gamma \tilde{Z}^e)(s)) \, dB_s + \int_0^t \sigma((\Gamma \tilde{Z}^e)(s))h(s) \, ds + \alpha \sup_{0 \leq s \leq t} (\Gamma \tilde{Z}^e)(s), \quad t \in [0, 1].
\]

For a function \( f \), put \( \Gamma(f)^{(m)}(t) = \Gamma(f)(t_k) \), if \( t_k \leq t < t_{k+1} \), \( k \in \{0, 1, \ldots, n-1\} \). Then,

\[
\bar{A}^e := \left\{ \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma((\Gamma \tilde{Z}^e)(s)) \sqrt{\varepsilon} \, dB_s \right| \geq \delta, \sup_{0 \leq t \leq 1} \left| \sqrt{\varepsilon} B_t \right| \leq \rho \right\}
\subset \left\{ \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma((\Gamma \tilde{Z}^e)(s)) - \sigma((\Gamma \tilde{Z}^{(m)})(s)) \sqrt{\varepsilon} \, dB_s \right| \geq \frac{\delta}{2} \right\}
\cup \left\{ \sup_{0 \leq t \leq 1} |\Gamma(\tilde{Z}^e)(t) - \Gamma(\tilde{Z}^{(m)})(t)| \leq \delta_1 \right\}
\cup \left\{ \sup_{0 \leq t \leq 1} |\Gamma(\tilde{Z}^e)(t) - \Gamma(\tilde{Z}^{(m)})(t)| > \delta_1 \right\}
\cup \left\{ \sup_{0 \leq t \leq 1} \left| \int_0^t \sigma((\Gamma \tilde{Z}^{(m)})(s)) \sqrt{\varepsilon} \, dB_s \right| \geq \frac{\delta}{2}, \sup_{0 \leq t \leq 1} \left| \sqrt{\varepsilon} B_t \right| \leq \rho \right\}
:= \bar{B}^e \cup \bar{C}^e \cup \bar{D}^e.
\]

Let us just sketch the proof of

\[
P(\bar{C}^e) \leq \exp \left( - \frac{R}{\varepsilon} \right). \quad (3.6)
\]

For \( t \in [t_k, t_{k+1}) \), it is easy to see that

\[
\sup_{t \leq s \leq t} |\Gamma(\tilde{Z}^e)(s) - \Gamma(\tilde{Z}^{(m)})(s)| = \sup_{t \leq s \leq t} |\Gamma(\tilde{Z}^e)(s) - \Gamma(\tilde{Z}^e)(t_k)|
\leq 2 \sup_{t \leq s \leq t} |\tilde{Z}^e(s) - \tilde{Z}^e(t_k)|. \quad (3.7)
\]
But

\[
|Z^e_i - Z^e_{t_i}| \leq \left| \int_{t_i}^{t_{i+1}} \sigma((\Gamma Z^e)(s))\sqrt{\varepsilon} \, dB_s + \int_{t_i}^{t_{i+1}} \sigma((\Gamma Z^e)(s))\dot{h}(s) \, ds \right|
\]

\[
+ \alpha \left| \sup_{0 \leq s \leq t} (\Gamma Z^e)(s) - \sup_{0 \leq s \leq t} (\Gamma Z^e)(s) \right|
\]

\[
\leq \left| \int_{t_i}^{t_{i+1}} \sigma((\Gamma Z^e)(s))\sqrt{\varepsilon} \, dB_s + \int_{t_i}^{t_{i+1}} \sigma((\Gamma Z^e)(s))\dot{h}(s) \, ds \right|
\]

\[
+ \alpha \sup_{t_i \leq s \leq t} |\Gamma(Z^e)(s) - \Gamma(Z^e)(t_i)|
\]

\[
\leq \left| \int_{t_i}^{t_{i+1}} \sigma((\Gamma Z^e)(s))\sqrt{\varepsilon} \, dB_s + \int_{t_i}^{t_{i+1}} \sigma((\Gamma Z^e)(s))\dot{h}(s) \, ds \right|
\]

\[
+ 2\alpha \sup_{t_i \leq s \leq t} |Z^e(s) - Z^e(t_i)|. \quad (3.8)
\]

Therefore,

\[
\sup_{t_i \leq s \leq t} |Z^e_i - Z^e_{t_i}| \leq \frac{1}{1 - 2\alpha} \sup_{0 \leq s \leq t \leq t_{i+1}} \left| \int_{t_i}^{t_{i+1}} \sigma((\Gamma Z^e)(s))\sqrt{\varepsilon} \, dB_s + \int_{t_i}^{t_{i+1}} \sigma((\Gamma Z^e)(s))\dot{h}(s) \, ds \right|.
\]

The rest of the proof is the same as the proof of (2.17) of Theorem 2.1. We omit it. \(\square\)

**Proposition 3.2.** If \(\alpha \in (0, 1/2)\), then \(G : H \to C_0([0, 1], \mathbb{R}_+)\) and \(\eta : H \to C_0([0, 1], \mathbb{R}_+)\) defined by (3.3) are continuous on the compact set \(\{h \in C_0([0, 1], \mathbb{R}) : \tilde{I}(h) \leq a\}\) for any \(a \in [0, \infty)\).

**Proof.** By (3.4), it suffices to prove that the conclusion holds for \(V : H \to C_0([0, 1], \mathbb{R})\). Let \(h_n, h \in \{h \in C_0([0, 1], \mathbb{R}) : \tilde{I}(h) \leq a\}\) with \(h_n \to h\) in \(C_0([0, 1], \mathbb{R})\). Then, we have

\[
\sup_{0 \leq s \leq t} |V(h_n)(t) - V(h)(t)| \leq \frac{2L}{1 - 2\alpha} \int_0^t \sup_{0 \leq s' \leq s} |V(h_n)(s') - V(h)(s')| \, ds + \frac{1}{1 - 2\alpha} \xi_n(t),
\]

where

\[
\xi_n(t) := \sup_{0 \leq s \leq t} \left| \int_0^s \sigma((\Gamma V(h))(s))\dot{h}_n(s) - \dot{h}(s) \, ds \right|.
\]

The rest of the proof is similar to the last part of the proof of Theorem 2.1. \(\square\)

**Acknowledgement**

We thank Professor Yongjin Wang for his stimulating discussions. The research of the first author was supported by the Keygrant Project of Chinese Ministry of Education (No. 309009).

**References**


