Closed-Form Expressions for the Exact Symbol Error Probability of 32-Cross-QAM in AWGN and in Slow Nakagami Fading

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***Abstract*— Novel closed-form exact expressions for the average symbol error probability of 32-cross-QAM in an additive white Gaussian noise channel and in a slow, flat Nakagami fading channel with L-branch maximal ratio combining diversity are derived.**

***Index Terms*— Cross-QAM, diversity, fading channels, quadrature amplitude modulation, Nakagami distribution, symbol error probability.**

I. INTRODUCTION

CROSS-QUADRATURE amplitude modulation (cross-QAM) is a preferred QAM signal constellation when the number of symbols, M, is not a power of 4, or equivalently when the number of bits is odd [1, Ch. 4], [2, Ch. 4]. The signal constellation is given the shape of a cross in order to reduce the average energy of the signal set [3]. Cross-QAM is of current interest for employment in adaptive M-ary QAM systems [4] and adaptive orthogonal frequency division multiplexing (OFDM) systems [5]. The signaling constellation of 32-cross-QAM is shown in [3, Fig. 2]. Note that the in-phase and quadrature components cannot be demodulated independently. This fact complicates the implementation and the calculation of the average symbol error probability (SEP) of cross-QAM. In the latter case, it means that the calculation of the average SEP cannot be reduced to a one-dimensional problem by using the independence of the in-phase and quadrature components, as can be done for Gray-coded square QAM and Gray-coded rectangular QAM. Reference [6] derived the SEP of rectangular QAM in Nakagami fading. References [4] - [7, Ch. 4] report using an upper bound on the SEP of 32-cross-QAM. Reference [8] derives the bit error probability of cross-QAM constellations in additive white Gaussian noise (AWGN) and Rayleigh fading. The SEP of 32-cross-QAM in Rayleigh fading can be obtained from the results in [8]. However, [8] does not give an exact closed-form solution for the average SEP of 32-cross-QAM in slow and flat Nakagami fading.

In this letter, we derive exact closed-form expressions for the average SEP of 32-cross-QAM in AWGN using a method simpler than [8]. We then derive exact closed-form expressions for the average SEP of 32-cross-QAM in slow and flat Nakagami fading with maximal ratio combining (MRC) diversity. This useful result has not been presented before.

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II. THE AVERAGE SEP IN AWGN

Fig. 1 shows the three classes of decision region that occur in the 32-cross-QAM constellation. Let P_{1c}, P_{2c} and P_{3c} be the probability of correct decision given the decision region in Fig. 1 (a), Fig. 1 (b) and Fig. 1 (c), respectively. Then,

$$P_{1c} = \left[1 - 2Q\left(\frac{d}{\sigma_n}\right)\right]^2 \tag{1}$$

$$P_{2c} = \left[1 - 2Q\left(\frac{d}{\sigma_n}\right)\right]\left[1 - Q\left(\frac{d}{\sigma_n}\right)\right] \tag{2}$$

$$P_{3c} = 1 - \frac{1}{2}Q\left(\frac{\sqrt{2}d}{\sigma_n}\right) - 2Q\left(\frac{d}{\sigma_n}\right) + \frac{3}{2}Q^2\left(\frac{d}{\sigma_n}\right) \tag{3}$$

where d is half of the distance between two signaling points, σ_n is the standard deviation of the noise and $Q(\cdot)$ is the Gaussian Q-function. Equation (3) follows because

$$P_{3c} = \frac{1}{2}\{Mass(ABC) - Mass(BDE)\} + Mass(FHEG)$$

with

$$Mass(ABC) = \left[1 - Q\left(\frac{\sqrt{2}d}{\sigma_n}\right)\right]^2$$

and

$$Mass(FHEG) = \frac{1}{2}\left[1 - Q\left(\frac{d}{\sigma_n}\right)\right]\left[1 - 2Q\left(\frac{d}{\sigma_n}\right)\right]$$

and [9, eq. (6)]

$$\begin{aligned} & Mass(BDE) \\ = & \frac{1}{4}\left\{\left[1 - 2Q\left(\frac{\sqrt{2}d}{\sigma_n}\right)\right]^2 - \left[1 - 2Q\left(\frac{d}{\sigma_n}\right)\right]^2\right\} \\ = & Q\left(\frac{d}{\sigma_n}\right) - Q\left(\frac{\sqrt{2}d}{\sigma_n}\right) - Q^2\left(\frac{d}{\sigma_n}\right) + Q^2\left(\frac{\sqrt{2}d}{\sigma_n}\right) \end{aligned}$$

(Note that there is an error in sign in the second and fourth terms in eq. (6) of [9]). Let P_c be the unconditional probability of correct symbol decision and the average SEP be denoted P_s. One has

$$P_c = \frac{1}{32}(16P_{1c} + 8P_{2c} + 8P_{3c}). \tag{4}$$

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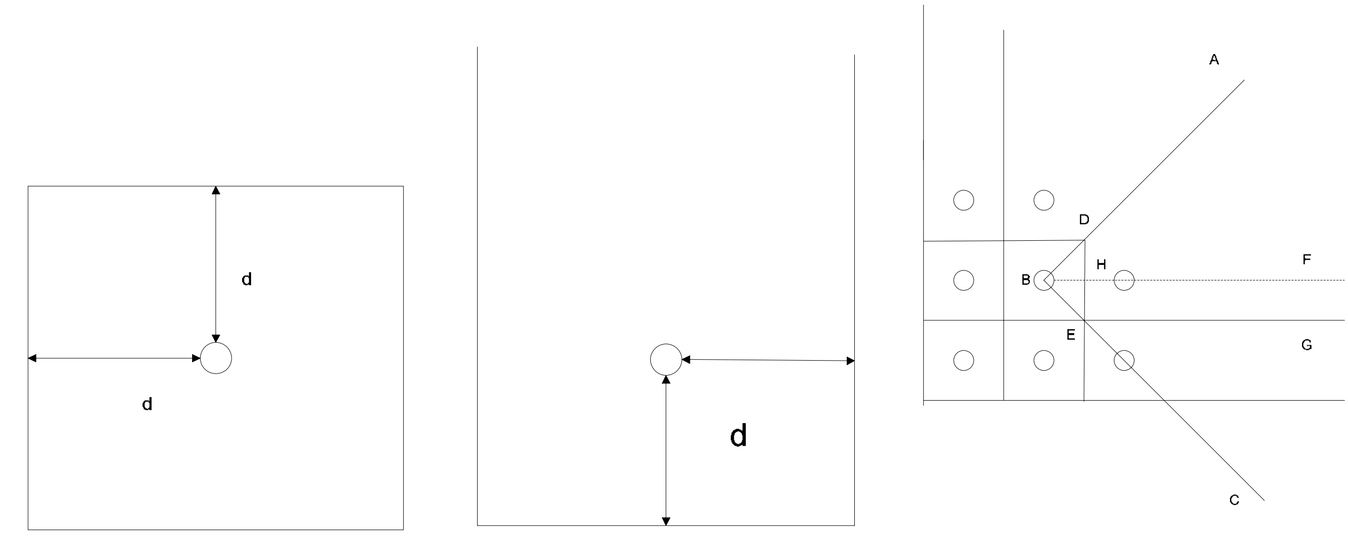

Fig. 1. Illustration of three decision regions for 32-cross-QAM (a) interior point (b) edge point (c) corner point.

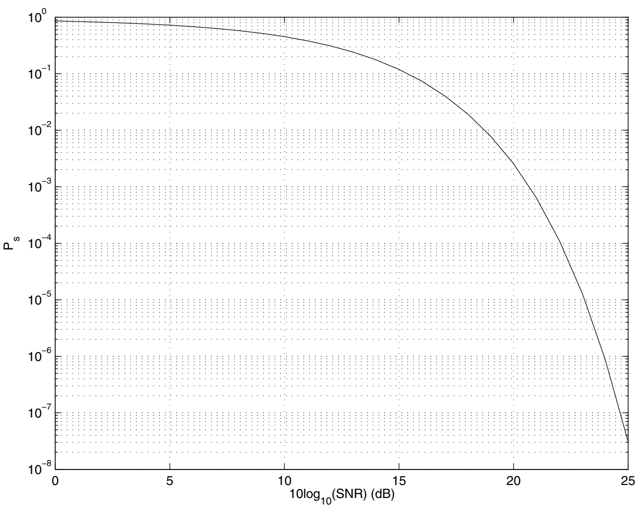

Fig. 2. The SEP of 32-cross-QAM in AWGN as a function of SNR.

Combining (1) - (3) with (4), using $P_s = 1 - P_c$, and defining $32E_A = 4\times 2d^2 + 8\times 10d^2 + 4\times 18d^2 + 8\times 26d^2 + 8\times 34d^2 = 640d^2$ or $E_A = 20d^2$ from [3, Fig. 2], one has

$$P_s = \frac{1}{32}\left[4Q\left(\sqrt{\frac{E_A}{10\sigma_n^2}}\right) + 104Q\left(\sqrt{\frac{E_A}{20\sigma_n^2}}\right) - 92Q^2\left(\sqrt{\frac{E_A}{20\sigma_n^2}}\right)\right] \quad (5)$$

where E_A is the average total energy and $E_A/(2\sigma_n^2) = SNR$ is the signal-to-noise ratio. The result (5) can also be derived from the bit error probability in [8, eq. (30)]. However, the present method calculates the SER directly for the case of 32-cross-QAM and is simpler. Once obtained, (5) can be used to derive results for MRC diversity in Nakagami fading. Fig. 2 shows P_s as a function of SNR, computed using (5).

III. The Average SEP in Nakagami Fading

The instantaneous SNR per symbol of L independent and identically distributed Nakagami-m fading channels with MRC, denoted as γ, has the probability density function (PDF) [10, eq. (13)]

$$f(\gamma) = \left(\frac{m}{\gamma_s}\right)^{Lm} \frac{\gamma^{Lm-1}}{\Gamma(Lm)} e^{-\frac{m}{\gamma_s}\gamma}, \quad \gamma > 0$$

where m is the Nakagami m parameter, L is the number of diversity branches and γ_s is the mean fading power on each channel [7]. We set $\gamma = \frac{E_A}{2\sigma_n^2}$. Then, the average SEP of 32-cross-QAM in slow, flat Nakagami-m fading with MRC diversity is given by

$$P_e = \int_0^\infty P_s(\gamma) f(\gamma) d\gamma \quad (6)$$

where $P_s(\gamma)$ is given by (5). One has that [11, eq. (5.18)]

$$I_1(a) = \int_0^\infty Q(a\sqrt{\gamma}) f(\gamma) d\gamma = \frac{1}{2}\left[1 - \sqrt{\frac{a^2\gamma_s}{2m + a^2\gamma_s}} \sum_{k=0}^{Lm-1} \frac{\binom{2k}{k}}{4^k(1 + \frac{a^2\gamma_s}{2m})^k}\right] \quad (7)$$

and [11, eq. (5.30)]

$$I_2(a) = \int_0^\infty Q^2(a\sqrt{\gamma}) f(\gamma) d\gamma = \frac{1}{4} - \frac{\alpha}{\pi}\left\{\left(\frac{\pi}{2} - \arctan\alpha\right) \cdot \sum_{k=0}^{Lm-1} \frac{\binom{2k}{k}}{4^k(1+c)^k} - \sum_{k=1}^{Lm-1}\sum_{i=1}^{k} \frac{T_{ik}\sin(\arctan\alpha)}{(1+c)^k}\left[\cos(\arctan\alpha)\right]^{2(k-i)+1}\right\} \quad (8)$$

where $c = \frac{a^2\gamma_s}{2m}$, $\alpha = \sqrt{\frac{c}{1+c}}$, $T_{ik} = \frac{\binom{2k}{k}}{\binom{2(k-i)}{(k-i)} 4^i [2(k-i)+1]}$ and Lm is an integer. Note that the value of the m parameter is

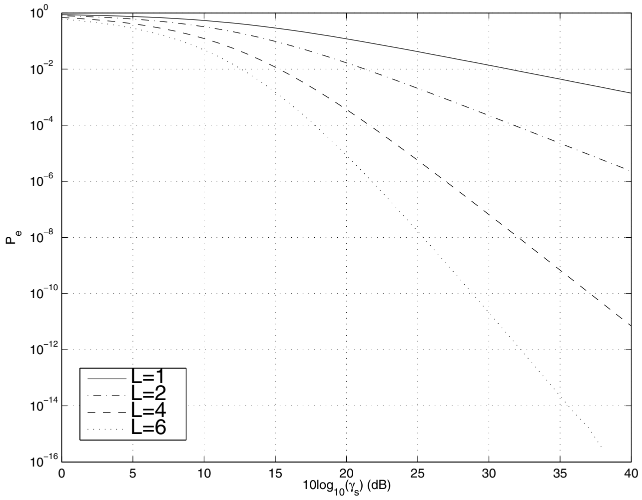

Fig. 3. The average SEP of 32-cross-QAM in Rayleigh fading as a function of the average fading SNR for different diversity orders.

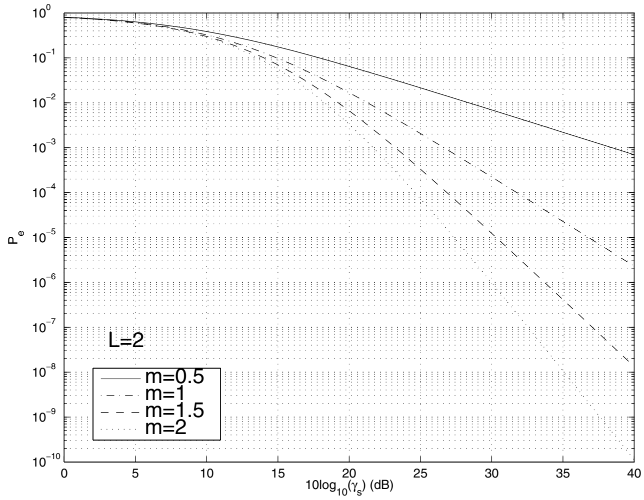

Fig. 4. The average SEP of 32-cross-QAM in Nakagami fading as a function of the average fading SNR for different fading parameters when $L = 2$.

restricted in this case. Using (7) and (8) in (6) gives

$$P_e = \frac{1}{32}\left[4I_1(\frac{1}{\sqrt{5}}) + 104I_1(\frac{1}{\sqrt{10}}) - 92I_2(\frac{1}{\sqrt{10}})\right]. \quad (9)$$

Equation (9) is a new closed-form expression for the exact average SEP of 32-cross-QAM in slow, flat Nakagami fading. When $\gamma_s = 0$, $P_e = \frac{31}{32}$, as expected. Fig. 3 shows P_e versus $10\log_{10}\gamma_s$ for different diversity orders when $m = 1$ and the fading is Rayleigh, computed using (9). Fig. 4 shows P_e versus $10\log_{10}\gamma_s$ for different Nakagami fading parameters when $L = 2$, again, computed using (9). When $m = 1$ and $L = 1$, (9) is simplified to

$$\begin{aligned} P_e &= \frac{1}{32}\Bigg[31 - 52\sqrt{\frac{\gamma_s}{20+\gamma_s}} - 2\sqrt{\frac{\gamma_s}{10+\gamma_s}} \\ &\quad + \frac{92}{\pi}\sqrt{\frac{\gamma_s}{20+\gamma_s}}\arctan\left(\sqrt{\frac{20+\gamma_s}{\gamma_s}}\right)\Bigg] \end{aligned} \quad (10)$$

which gives the average SEP of 32-cross-QAM in a single slow and flat Rayleigh fading channel, an important special case. This result can also be obtained by using [8, eq. (32)] and transforming the bit error probability into SEP. The results in Figs. 2 - 4 are obtained using the analytical results in (5) and (9). They are plotted using MATLAB.

IV. Conclusion

Previously, only upper bounds to the average symbol error probability of 32-cross-QAM and the bit error probability of 32-cross-QAM in additive white Gaussian noise and Rayleigh fading have been reported in the literature. In this letter, exact closed-form solutions for the SEP of 32-cross-QAM in AWGN and in flat Nakagami fading have been derived.

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